Field statistics in some generalized Jaynes-Cummings models

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Photon statistics in some fully quantized models of the interaction of a two-level atom with a single-mode radiation field have been studied using the operator equations of motion. Expressions for the photon number distribution and the mean photon number are presented for various initial conditions. It is found that the mean photon number may show decays and revivals of coherence similar to those of the atomic inversion in the coherent-state Jaynes-Cummings model. Application of these models to the study of multiphoton laser, absorption, and emission processes is also discussed.

I. INTRODUCTION

The Jaynes-Cummings model[1] of a two-level atom interacting with a quantized single-mode electromagnetic field is at the core of many problems in quantum optics, NMR, and quantum electronics. The importance of this model lies in that it is perhaps the simplest solvable model that describes the essential physics of radiation-matter interaction. Recent studies of this model by Eberly *et al.*[2] involving an electromagnetic field initially in a coherent state have revealed periodic collapse and revival of atomic coherence which clearly are a manifestation of the role of quantum mechanics in the coherence and fluctuation properties of radiation-matter systems. We shall refer to the model studied by Eberly *et al.* as the standard Jaynes-Cummings model. In a series of papers Sukumar and Buck[3] have proposed two exactly solvable generalizations of the Jaynes-Cummings model, one involving intensity dependent coupling and the other involving multiphoton interaction between the field and atom. These models also exhibit periodic decay and revival of atomic coherence. The emphasis, however, has been on atomic dynamics. The behavior of the field statistics seems to have been studied only for the standard model.[4–6] In this paper we discuss the dynamics of various generalized Jaynes-Cummings models. We also show that these models can be used to study multiple atom scattering of radiation and multiphoton emission, absorption, and laser processes.

In our discussion we shall follow mainly the notation of Ref. 2 and assume the rotating wave approximation (RWA) to hold. The Hamiltonian for the standard model can be expressed in terms of the inversion, raising and lowering operators of the two-level atom denoted by $\hat{\sigma}_3,\hat{\sigma}_\pm$ and the annihilation and creation operators $\hat{a},\hat{a}^\dagger$ of the radiation field as

$$\hat{H}=\frac{\hbar\omega_0}{2}\hat{\sigma}_3+\hbar\omega(\hat{a}^\dagger\hat{a})+\hbar\lambda(\hat{\sigma}_+\hat{a}+\hat{\sigma}_-\hat{a}^\dagger)\ . \tag{1}$$

Here ω_0 is the transition frequency of the atom and ω is the mode frequency. λ is the coupling strength for radiation-atom interaction. The $\hat{\sigma}$'s are 2×2 Pauli matrices

$$\hat{\sigma}_3=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad \hat{\sigma}_+=\begin{bmatrix}0&1\\0&0\end{bmatrix},\quad \hat{\sigma}_-=\begin{bmatrix}0&0\\1&0\end{bmatrix}. \tag{2}$$

The $\hat{\sigma}$'s and the $\hat{a}$'s obey the following commutation relations:

$$[\hat{\sigma}_3,\hat{\sigma}_\pm]=\pm2\hat{\sigma}_\pm\ ,\quad [\hat{\sigma}_+,\hat{\sigma}_-]=\hat{\sigma}_3\ ,\quad [\hat{a},\hat{a}^\dagger]=1\ . \tag{3}$$

We now proceed to discuss photon statistics in the two generalized models separately.

II. INTENSITY DEPENDENT COUPLING MODEL

In this model the coupling strength is assumed to depend on the number operator. The Hamiltonian for this model is obtained from Eq. (1) to be

$$\hat{H}=\frac{\hbar\omega_0}{2}\hat{\sigma}_3+\hbar\omega\hat{a}^\dagger\hat{a}+\hbar\lambda(\hat{\sigma}_+\hat{R}+\hat{\sigma}_-\hat{R}^\dagger)\ , \tag{4}$$

where $\hat{R}=\hat{a}\sqrt{(\hat{n})}$, $\hat{R}^\dagger=\sqrt{(\hat{n})}\hat{a}^\dagger$, $\hat{n}=\hat{a}^\dagger\hat{a}$. It is easy to show that

$$[\hat{R},\hat{n}]=\hat{R}\ ,\quad [\hat{R}^\dagger,\hat{n}]=-\hat{R}^\dagger\ ,$$
$$[\hat{R},\hat{R}^\dagger]=2\hat{n}+1\ . \tag{5}$$

The Hamiltonian in (4) can be rewritten as

$$\hat{H}=\hbar\omega\hat{N}+\hbar\hat{C}\ , \tag{6}$$

where

$$\hat{N}=(\hat{a}^\dagger\hat{a}+\tfrac{1}{2}\hat{\sigma}_3)\ ,\quad \hat{C}=\frac{\Delta}{2}\hat{\sigma}_3+\lambda(\hat{\sigma}_+\hat{R}+\hat{\sigma}_-\hat{R}^\dagger)\ ,$$
$$\Delta=\omega_0-\omega\ . \tag{7}$$

Using the commutation relations in Eqs. (3) and (5) it can be shown that both $\hat{N}$ and $\hat{C}$ are constants of motion, i.e.,

$$[\hat{H},\hat{N}]=0=[\hat{H},\hat{C}]\ ,\quad [\hat{N},\hat{C}]=0\ . \tag{8}$$

This allows the time evolution operator $\hat{U}(t,0)$ to be written as

$$\hat{U}(t,0)=e^{-(i\hat{H}t/\hbar)}=e^{-i\omega\hat{N}t}e^{-i\hat{C}t}\ . \tag{9}$$

If we make use of the matrix representation for the $\hat{\sigma}$'s and recall that they satisfy the properties

$$\hat{\sigma}_3^2=\hat{1}\ ,\quad \hat{\sigma}_+^2=0=\hat{\sigma}_-^2\ ,\quad \hat{\sigma}_\pm\hat{\sigma}_\mp=\tfrac{1}{2}(1\pm\hat{\sigma}_3)\ , \tag{10}$$

the time evolution operator can be expressed as a 2×2 matrix in the following form:

$$\hat{U}(t,0)=e^{-i\omega\hat{n}t}\begin{bmatrix} e^{-(i\omega t/2}\left[\cos(\lambda t\sqrt{\hat{\mu}+2\hat{n}+1})-\dfrac{i\delta}{2}\,\dfrac{\sin(\lambda t\sqrt{\hat{\mu}+2\hat{n}+1})}{\sqrt{\hat{\mu}+2\hat{n}+1}}\right] & \dfrac{-ie^{-i\omega t/2}\hat{R}\sin(\lambda t\sqrt{\hat{\mu}})}{\sqrt{\hat{\mu}}} \\ -ie^{i\omega t/2}\hat{R}^\dagger\,\dfrac{\sin\lambda t\sqrt{\hat{\mu}+2\hat{n}+1}}{\sqrt{\hat{\mu}+2\hat{n}+1}} & e^{i\omega t/2}\left[\cos\lambda t\sqrt{\hat{\mu}}+\dfrac{i\delta}{2}\,\dfrac{\sin\lambda t\sqrt{\hat{\mu}}}{\sqrt{\hat{\mu}}}\right]\end{bmatrix}, \tag{11}$$

where

$$\hat{\mu}=\frac{\delta^2}{4}+\hat{n}^2\ ,\quad \delta=\frac{\omega_0-\omega}{\lambda}=\frac{\Delta}{\lambda}\ ,\quad \hat{n}=\hat{a}^\dagger\hat{a}\ . \tag{12}$$

The time evolution of any operator is now determined by applying the transformation (12) to its value at the initial time $t=0$. In particular, the density operator $\hat{\rho}(t)$ will be given by

$$\hat{\rho}(t)=\hat{U}(t,0)\hat{\rho}(0)\hat{U}^\dagger(t,0)\ , \tag{13}$$

in terms of its value at time $t=0$. The density matrix $\hat{\rho}_F(t)$ of the radiation field and the probability $p(n,t)$ of finding n photons in the radiation field are found from Eq. (13) to be

$$\hat{\rho}_F(t)=\mathrm{Tr}_A[\hat{U}(t,0)\hat{\rho}(0)\hat{U}^\dagger(t,0)]\ , \tag{14a}$$

$$p(n,t)=\langle n\,|\hat{\rho}_F(t)|\,n\rangle\ . \tag{14b}$$

Using Eqs. (11)–(14) we can now discuss photon statistics for a given initial state of the system. Some cases of interest will now be treated.

We first consider the case when the field is initially in a coherent state $|\alpha\rangle$ with mean photon number $\bar{n}=|\alpha|^2$. The density matrix of the field will be

$$\hat{\rho}_F(0)=\sum_{m,m'}e^{-|\alpha|^2}\frac{\alpha^m\alpha^{*m'}}{\sqrt{m!m'!}}|m\rangle\langle m'|\ . \tag{15}$$

If the atom initially starts in the lower state,

$$\hat{\rho}(0)=\begin{bmatrix}0 & 0\\ 0 & \hat{\rho}_F(0)\end{bmatrix}. \tag{16}$$

From Eqs. (11), (14)–(16) we find

$$p_\delta^C(n,t)=(n+1)^2\frac{\sin^2\lambda t[\delta^2/4+(n+1)^2]^{1/2}}{\delta^2/4+(n+1)^2}\frac{(\bar{n})^{n+1}}{(n+1)!}e^{-\bar{n}}+\left[\cos^2\lambda t(\delta^2/4+n^2)^{1/2}+\frac{\delta^2}{4}\frac{\sin^2\lambda t(\delta^2/4+n^2)^{1/2}}{\delta^2/4+n^2}\right]\frac{(\bar{n})^n}{n!}e^{-\bar{n}}, \tag{17}$$

which for exact resonance between the atom and the field leads to

$$p_0^C(n,t)=\left[\frac{\bar{n}}{n+1}\sin^2\lambda t(n+1)+\cos^2\lambda tn\right]\frac{(\bar{n})^n}{n!}e^{-\bar{n}}. \tag{18}$$

The mean number of photons in the optical field can be calculated from Eq. (18) or from

$$\langle\hat{n}(t)\rangle=\mathrm{Tr}[\hat{\rho}_F(t)\hat{a}^\dagger\hat{a}], \tag{19}$$

and is found to be

$$\langle\hat{n}(t)\rangle_0^C=\bar{n}-\tfrac{1}{2}\left[1-e^{-\bar{n}}\sum_{n=0}^{\infty}\frac{(\bar{n})^n}{n!}\cos 2\lambda nt\right]=\bar{n}-\tfrac{1}{2}+\tfrac{1}{2}\exp(-2\bar{n}\sin^2\lambda t)\cos(\bar{n}\sin 2\lambda t). \tag{20}$$

A comparison of this equation with the expression for the inversion,

$$\langle\hat{\sigma}_3(t)\rangle_0^C=-\exp(-2\bar{n}\sin^2\lambda t)\cos(\bar{n}\sin 2\lambda t), \tag{21}$$

shows that the mean photon number will undergo periodic collapse and revival similar to those of the inversion $\langle\hat{\sigma}_3(t)\rangle$, but the two will be out of phase. This result is a direct consequence of the fact that the "excitation number" $\hat{N}$ is a constant of motion. In fact, from Eqs. (20) and (21) one can explicitly verify that $\hat{N}$ is indeed a constant of the motion. This conclusion also holds for the standard model discussed by Eberly *et al.* The expressions for the collapse and revival times can be found in the papers of Sukumar and Buck.[3]

In the case of nonzero detuning $\langle\hat{n}(t)\rangle$ cannot be expressed in a compact form and one obtains

$$\langle\hat{n}(t)\rangle_\delta^C=\bar{n}-\tfrac{1}{2}+\tfrac{1}{2}\langle\hat{\sigma}_3\rangle_\delta^C=\bar{n}-\tfrac{1}{2}+\tfrac{1}{2}e^{-\bar{n}}\sum_{n=0}^{\infty}\frac{(\bar{n})^n}{n!}\frac{\delta^2/4+n^2\cos[2\lambda t(\delta^2/4+n^2)^{1/2}]}{\delta^2/4+n^2}. \tag{22}$$

The sum in Eq. (22) is similar to the series for the inversion in the standard model.[2] This similarity suggests that one can use a saddle-point method[2] to derive an approximate analytic expression for $\langle\hat{n}(t)\rangle$. Thus, for δ small compared with $\bar{n}$ and $\bar{n}\gtrsim 20$ we find

$$\langle\hat{n}(t)\rangle_\delta^C\simeq\bar{n}-\frac{1}{2}+\frac{1}{2}\left[\frac{\delta^2/4}{\delta^2/4+\bar{n}^2}+\frac{\bar{n}^2}{\delta^2/4+\bar{n}^2}\frac{e^{-\Psi(t)}\cos\Phi(t)}{\left[1+\frac{\lambda^2t^2\delta^2}{4\bar{n}^2(1+\delta^2/4\bar{n}^2)^3}\right]^{1/4}}\right], \tag{23a}$$

where

$$\Psi(t)=2\bar{n}\left[1+\frac{\tau^2\delta^4}{16\bar{n}^4(1+\delta^2/4\bar{n}^2)^2}\right]^{-1}\sin^2(\tfrac{1}{2}\tau), \tag{23b}$$

$$\Phi(t)=\bar{n}\left[1+\frac{\delta^2}{4\bar{n}^2}\right]-\bar{n}\tau+\bar{n}\sin\tau+\frac{1}{2}\tan^{-1}\left[\frac{\tau\delta^2}{4\bar{n}^2(1+\delta^2/4\bar{n}^2)}\right], \tag{23c}$$

$$\tau=2\lambda t\left[1+\frac{\delta^2}{4\bar{n}^2}\right]^{-1/2}. \tag{23d}$$

The collapse and revival times T_C and T_R are estimated to be

$$T_C=\frac{(1+\delta^2/4\bar{n}^2)^{1/2}}{2\lambda}\ , \quad T_R=\frac{\pi}{\lambda}(1+\frac{\delta^2}{4\bar{n}^2})^{1/2}\ , \tag{24}$$

respectively. From Eqs. (23) it follows that $\langle \hat{n}(t)\rangle$ undergoes a series of collapses and revivals. However, the revivals are not complete. It turns out that the behavior of $\langle \hat{n}(t)\rangle$, and therefore also of $\langle \hat{\sigma}_3(t)\rangle$, with nonzero detuning is closer to its behavior in the standard model than to their on-resonance behavior in the present model. It is easy to check that by putting $\delta=0$ in Eqs. (22)–(24) we recover the results of Refs. 3.

We now consider the case when the atom is initially in the ground state and the field mode is in a thermal state represented by the density matrix

$$\hat{\rho}_F(0)=\sum_{n=0}^{\infty}\frac{(\bar{n})^n}{(1+\bar{n})^{n+1}}\,|\,n\rangle\langle n\,|\ . \tag{25}$$

It follows from Eqs. (11), (12), (14), and (16) that

$$p_\delta^T(n,t)=\frac{(n+1)^2\sin^2\lambda t[\delta^2/4+(n+1)^2]^{1/2}}{\delta^2/4+(n+1)^2}\frac{(\bar{n})^{n+1}}{(1+\bar{n})^{n+2}} + \left[\cos^2\lambda t(\delta^2/4+n^2)^{1/2}+\delta^2/4\frac{\sin^2\lambda t(\delta^2/4+n^2)^{1/2}}{\delta^2/4+n^2}\right]\frac{(\bar{n})^n}{(1+\bar{n})^{n+1}}\ , \tag{26}$$

and

$$\langle \hat{n}(t)\rangle_\delta^T=\bar{n}-\tfrac{1}{2}+\tfrac{1}{2}(1+\bar{n})^{-1}\sum_{n=0}^{\infty}\left[\frac{\delta^2/4+n^2\cos 2\lambda t(\delta^2/4+n^2)^{1/2}}{\delta^2/4+n^2}\right]\left[\frac{\bar{n}}{1+\bar{n}}\right]^n . \tag{27}$$

It does not seem possible to sum this series analytically in general. For the special case of zero detuning ($\delta=0$) we find

$$\langle \hat{n}(t)\rangle_0^T=\bar{n}-\frac{1}{2}+\frac{1-\dfrac{\bar{n}}{1+\bar{n}}\cos 2\lambda t}{2(1+\bar{n})\left[1+\left[\dfrac{\bar{n}}{1+\bar{n}}\right]^2-\dfrac{2\bar{n}}{1+\bar{n}}\cos 2\lambda t\right]}\ . \tag{28}$$

From Eq. (28) we conclude that $\langle \hat{n}(t)\rangle$ will oscillate with period $T=\Pi\lambda^{-1}$. Based on this one might conclude that in the case of nonzero detuning $\langle \hat{n}(t)\rangle$ may exhibit revivals and decays. Unfortunately, we have been unable to prove it analytically. It is, however, clear that revivals and collapses are not peculiar to coherent-state initial conditions. So far in our discussion we have considered only single-photon absorption processes. From Eqs. (20), (23), and (27) we find that the maximum amplitude of the modulation of $\langle \hat{n}(t)\rangle$ is unity which simply reflects the fact that the atom absorbs and emits only a single photon. In Sec. III we consider another generalization of the Jaynes-Cummings model that involves the m-photon absorption or emission process.

III. MULTIPHOTON INTERACTION MODEL

The transition between the upper and lower levels of the atom may involve $m\,(\geq 2)$ photons if the energy separation between the level is close to the energy of m quanta of the electromagnetic field. In the RWA the Hamiltonian for a single-mode electromagnetic field interacting with a two-level atom via an m-photon process is

$$\hat{H}=\hbar\omega\left[\hat{a}^\dagger\hat{a}+\frac{m}{2}\hat{\sigma}_3\right]+\frac{\hbar\Delta}{2}\hat{\sigma}_3+\hbar\lambda(\hat{\sigma}_+\hat{a}^m+\hat{\sigma}_-\hat{a}^{\dagger m}),\ \ \Delta=\omega_0-m\omega\ . \tag{29}$$

The rotating wave approximation is reliable only if $|\Delta| << \omega_0,\omega$. Clearly one has to be careful in multiphoton processes because for large m detuning, $|\Delta|$ may be of the same order as ω. We shall, however, restrict ourselves to those cases when $|\Delta| << \omega$. Again defining the operators

$$\hat{N}=\hat{a}^\dagger\hat{a}+\frac{m}{2}\hat{\sigma}_3\ , \quad \hat{C}=\frac{\Delta}{2}\hat{\sigma}_3+\lambda(\hat{\sigma}_+\hat{a}^m+\hat{\sigma}_-\hat{a}^{\dagger m})\ , \tag{30}$$

we find that both $\hat{N}$ and $\hat{C}$ are constants of motion, i.e.,

$$[\hat{H},\hat{N}]=0=[\hat{H},\hat{C}]\ , \quad [\hat{N},\hat{C}]=0\ . \tag{31}$$

These commutation relations are easily established by using the following commutation rules for the $\hat{a}$'s and $\hat{n}=\hat{a}^\dagger\hat{a}$:

$$[\hat{a},\hat{a}^{\dagger m}]=m\hat{a}^{\dagger m-1}\ , \quad [\hat{a}^\dagger,\hat{a}^m]=-m\hat{a}^{m-1}\ , \tag{32a}$$

$$[\hat{n},\hat{a}^{\dagger m}]=m\hat{a}^{\dagger m}\ , \quad [\hat{n},\hat{a}^m]=-m\hat{a}^m\ , \tag{32b}$$

and the commutation relations Eqs. (2) and (3) for the $\hat{\sigma}$'s.

The time evolution operator $\hat{U}(t,0)$ can be expressed as a product of $e^{-i\omega\hat{N}t}$ and $e^{-i\hat{C}t}$ [Eq. (9)]. Using the properties of $\hat{\sigma}_3$, it can be shown that $\hat{U}(t,0)$ has the following matrix representation in the atomic Hilbert space :

$$\hat{U}(t,0)=e^{-i\omega\hat{n}t}\begin{bmatrix} e^{-im\omega t/2}\left[\cos\lambda t\sqrt{\hat{\nu}}-\frac{i\delta}{2}\frac{\sin\lambda t\sqrt{\hat{\nu}}}{\sqrt{\hat{\nu}}}\right] & -ie^{-im\omega t/2}\frac{\hat{a}^m\sin\lambda t\sqrt{\hat{\nu}'}}{\sqrt{\hat{\nu}'}} \\ -ie^{im\omega t/2}\hat{a}^{\dagger m}\frac{\sin\lambda t\sqrt{\hat{\nu}}}{\sqrt{\hat{\nu}}} & e^{im\omega t/2}\left[\cos\lambda t\sqrt{\hat{\nu}'}+\frac{i\delta}{2}\frac{\sin\lambda t\sqrt{\hat{\nu}'}}{\sqrt{\hat{\nu}'}}\right]\end{bmatrix}\ , \tag{33a}$$

where

$$\delta=(\omega_0-m\omega)\lambda^{-1}=\Delta\lambda^{-1}\ , \quad \hat{\nu}=\frac{\delta^2}{4}+\hat{a}^m\hat{a}^{\dagger m}\ , \quad \hat{\nu}'=\frac{\delta^2}{4}+\hat{a}^{\dagger m}\hat{a}^m\ . \tag{33b}$$

Equations (33) allow us to discuss the time dependence of any operator or state given initial conditions. In our discussion we shall also use the following identities:

$$\hat{a}^m\hat{a}^{\dagger m}=\frac{(\hat{n}+m)!}{\hat{n}!}\ , \quad \hat{a}^{\dagger m}\hat{a}^m=\frac{\hat{n}!}{(\hat{n}-m)!}\ , \tag{34}$$

which are easily established using Eqs. (32). We are now in a position to consider various initial conditions of the system and discuss its time evolution. As before we shall be concerned mainly with the field statistics.

Let the atom start in the lower state and the field in the coherent state so that the initial conditions are given by Eqs. (15) and (16). Then using Eqs. (14), (33), and (34) we find that the probability of finding n photons in the field is given by

$$p^C_{\delta,m}(n,t)=e^{-\bar{n}}\frac{(\bar{n})^{n+m}}{n!}\frac{\sin^2\left[\lambda t\left[\delta^2/4+\frac{(n+m)!}{n!}\right]^{1/2}\right]}{\delta^2/4+\frac{(n+m)!}{n!}}$$
$$+e^{-\bar{n}}\frac{(\bar{n})^n}{n!}\left[\cos^2\left[\lambda t\left[\delta^2/4+\frac{n!}{(n-m)!}\right]^{1/2}\right]+\frac{\delta^2}{4}\frac{\sin^2\lambda t\left[\delta^2/4+\frac{n!}{(n-m)!}\right]^{1/2}}{\delta^2/4+\frac{n!}{(n-m)!}}\right]\ , \tag{35}$$

and using Eqs. (14), (19), and (33) the mean number of photons is found to be

$$\langle \hat{n}(t)\rangle^{C}_{\delta,m}=\bar{n}-m(\bar{n})^{m}\sum_{r=0}^{\infty}e^{-\bar{n}}\frac{(\bar{n})^{r}}{r!}\frac{\sin^{2}\left[\lambda t\left[\delta^{2}/4+\frac{(r+m)!}{r!}\right]^{1/2}\right]}{\delta^{2}/4+\frac{(r+m)!}{r!}} . \tag{36}$$

This equation describes the behavior of the mean photon number as a function of time. For $m=1$ we find

$$\langle \hat{n}(t)\rangle^{C}_{\delta,1}=\bar{n}-\tfrac{1}{2}+\tfrac{1}{2}\sum_{r=0}^{\infty}e^{-\bar{n}}\frac{(\bar{n})^{r}}{r!}\frac{\delta^{2}/4+r\cos[2\lambda t(\delta^{2}/4+r)^{1/2}]}{\delta^{2}/4+r} , \tag{37}$$

which is the expression for the mean photon number in the standard model. A comparison of Eq. (37) with the series for the inversion in the standard model shows that for $\bar{n}>>\delta,1$

$$\langle \hat{n}(t)\rangle^{C}_{\delta,1}\simeq\bar{n}-\frac{1}{2}+\frac{1}{2}\left[\frac{\delta^{2}/4}{\delta^{2}/4+\bar{n}}+\frac{\bar{n}}{\delta^{2}/4+\bar{n}}\left[1+\frac{\bar{n}^{2}\lambda^{2}t^{2}}{4(\delta^{2}/4+\bar{n})^{3}}\right]^{-1/4}e^{-\Psi}\cos\Phi\right] , \tag{38a}$$

where

$$\Psi(t)=2\bar{n}\left[1+\frac{\bar{n}^{2}\lambda^{2}t^{2}}{4(\delta^{2}/4+\bar{n})^{3}}\right]^{-1}\sin^{2}\left[\frac{\lambda t}{(\delta^{2}/4+\bar{n})^{1/2}}\right] , \tag{38b}$$

$$\Phi(t)=\lambda t(\delta^{2}/4+\bar{n})^{1/2}+\bar{n}\sin\frac{\lambda t}{(\delta^{2}/4+\bar{n})^{1/2}}-\frac{\lambda\bar{n}t}{(\delta^{2}/4+\bar{n})^{1/2}}-\frac{1}{2}\tan^{-1}\frac{\bar{n}\lambda t}{2(\delta^{2}/4+\bar{n})^{3/2}} . \tag{38c}$$

For the general m-photon case we note that the behavior of the series in Eq. (36) can be quite different depending on whether $\bar{n}$ is large or small compared with m. For simplicity we consider the on-resonance case ($\delta=0$). Then for $\bar{n}$ small compared with m, only the tail of the Poisson distribution contributes to the sum in Eq. (36) and the behavior of $\langle \hat{n}(t)\rangle$ will be similar to that of the initial chaotic conditions for the field. In the opposite limit $\bar{n}>>m$ one can drive the following approximate expression:

$$\langle \hat{n}(t)\rangle^{C}_{o,m}\simeq\bar{n}-\frac{m}{2}+\frac{m}{2}\left[1+\frac{\lambda^{2}t^{2}(\bar{n})^{m-2}[m(m-2)]^{2}}{4}\right]^{-1/4}e^{-\Psi}\cos\Phi , \tag{39a}$$

where

$$\Psi(t)=2\bar{n}\left[1+\frac{\lambda^{2}t^{2}(\bar{n})^{m-2}[m(m-2)]^{2}}{4}\right]^{-1}\sin^{2}\left[\frac{m\lambda t(\bar{n})^{(m-2)/2}}{2}\right] , \tag{39b}$$

$$\Phi(t)=2\lambda t(\bar{n})^{m/2}-m\lambda t(\bar{n})^{m/2}+\bar{n}\sin[m\lambda t(\bar{n})^{m/2}]-\frac{1}{2}\tan^{-1}\left[\frac{\lambda t m(m-2)(\bar{n})^{(m-4)/2}}{2}\right] . \tag{39c}$$

We have derived this formula under a similar kind of assumption that was used in arriving at Eqs. (38). However, in the m-photon case Eqs. (39) should be interpreted, strictly speaking, as "local solutions" near revival times

$$t_R=\frac{2k\pi}{m\lambda(\bar{n})^{(m-2)/2}} . \tag{40}$$

The reason is that in the derivation of Eqs. (39) it is not sufficient to assume $(t=kt_R+t_k)t_k<<1$ but $mt_k<<1$, and even though $\bar{n}>>m$, m could still be a large number. Thus for larger deviations from $t=kt_R$, one might expect departures from the behavior given by Eqs. (39).

The case of an initial chaotic field interacting with the atom initially in the ground state can be treated in a straightforward manner also. For the probability

$$p^T_{\delta,m}(n,t)=\frac{(\bar{n})^{n+m}}{(1+\bar{n})^{n+m+2}}\left\{\frac{\sin^2\lambda t\left[\delta^2/4+\frac{(n+m)!}{n!}\right]^{1/2}}{\delta^2/4+\frac{(n+m)!}{n!}}\right\}\frac{(n+m)!}{n!}$$

$$+\left\{\cos^2\lambda t\left[\delta^2/4+\frac{n!}{(n-m)!}\right]^{1/2}+\frac{\delta^2}{4}\frac{\sin^2\lambda t\left[\delta^2/4+\frac{n!}{(n-m)!}\right]^{1/2}}{\delta^2/4+\frac{n!}{(n-m)!}}\right\}\frac{(\bar{n})^n}{(1+\bar{n})^{n+1}}\ , \tag{41}$$

and for the mean photon number

$$\langle\hat{n}(t)\rangle^T_{\delta,m}=\bar{n}-\frac{m(\bar{n})^m}{(1+\bar{n})^{m+1}}\sum_{r=0}^{\infty}\frac{(\bar{n})^r}{(1+\bar{n})^r}\frac{(n+m)!}{n!}\frac{\sin^2\left[\lambda t\left[\delta^2/4+\frac{(n+m)!}{n!}\right]^{1/2}\right]}{\delta^2/4+\frac{(n+m)!}{n!}}\ . \tag{42}$$

This completes our discussion of the field statistics. So far we have restricted our considerations to the interaction of a single-mode electromagnetic field with a single atom. In Sec. IV we show how simple generalizations of the results developed so far allow us to discuss various multiphoton processes in multiatom systems when atomic decays are present.

IV. APPLICATION TO MULTIPHOTON PROCESSES IN MULTIATOM SYSTEMS WITH DECAYS

We consider a collection of N identical two-level atoms noninteracting with each other except via the action of the electromagnetic field. For simplicity we assume the field to be a single-mode radiation field. So far we have ignored the effect of atomic decays. To take into account atomic decays we assume that for the duration of its lifetime t, each atom interacts with the field in a manner discussed in Secs. II and III. The atomic lifetimes will be distributed according to the probability law

$$\mathscr{P}(t)=\gamma e^{-\gamma t}\ , \tag{43}$$

with mean lifetime given by γ^{-1}. In the following we shall be interested in the behavior of the density matrix of the field alone. The state of the field, after it has interacted with a large number N of atoms whose lifetimes are distributed according to Eq. (43), will be described by

$$\hat{\rho}_F(\gamma^{-1})=\gamma\int_0^\infty e^{-\gamma t}\hat{\rho}_F(t)dt=\mathrm{Tr}_A\gamma\int_0^\infty e^{-\gamma t}[\hat{U}(t,0)\hat{\rho}(0)\hat{U}^\dagger(t,0)]dt\ , \tag{44}$$

where $\hat{\rho}_F(t)$ is calculated according to Eqs. (14) with the time evolution operator $\hat{U}(t,0)$ given by Eqs. (33). If the initial density matrix of the field is $\hat{\rho}_F(0)$, then from Eq. (44) the average change in the density matrix after a time γ^{-1} will be $\hat{\rho}_F(t)-\hat{\rho}_F(0)$, and we can define a coarse-grained rate of change of $\hat{\rho}_F$ due to the interaction of N homogeneously broadened atoms by

$$\frac{d\hat{\rho}_F}{dt}=N\gamma[\hat{\rho}_F(\gamma^{-1})-\hat{\rho}_F(0)]\ . \tag{45}$$

We now proceed to discuss various applications of Eq. (45).

A. *m*-photon laser

The gain mechanism in the m-photon laser is the introduction of atoms in the upper state. The initial state-of-the-field atom system will be described by

$$\hat{\rho}(0)=\begin{bmatrix}\hat{\rho}_F(0) & 0\\ 0 & 0\end{bmatrix}. \tag{46}$$

Taking the matrix element of Eq. (45) in the Fock state $|n\rangle$ we find that the coarse-grained rate of change of $P(n,t)\equiv\langle n\,|\hat{\rho}_F\,|\,n\rangle$, the probability of finding n photons will be

$$\left[\frac{dP(n,t)}{dt}\right]_{\text{gain}}=N\gamma^2\int_0^\infty e^{-\gamma t}\left\{-\frac{(n+m)!\left[1-\cos 2\lambda t\left[\delta^2/4+\frac{(n+m)!}{n!}\right]^{1/2}\right]}{2(n!)\left[\delta^2/4+\frac{(n+m)!}{n!}\right]}P(n,t)+\frac{n!\left[1-\cos 2\lambda t\left[\delta^2/4+\frac{n!}{(n-m)!}\right]^{1/2}\right]}{2(n-m)!\left[\delta^2/4+\frac{n!}{(n-m)!}\right]}P(n-m,t)\right\}$$

$$=-\frac{A\frac{(n+m)!}{n!}}{\left[\frac{\Delta}{\gamma}\right]^2+\left[\frac{B}{A}\frac{(n+m)!}{n!}+1\right]}P(n,t)+\frac{A\frac{n!}{(n-m)!}}{\left[\frac{\Delta}{\gamma}\right]^2+\left[\frac{B}{A}\frac{n!}{(n-m)!}+1\right]}P(n-m,t)\,, \tag{47a}$$

where

$$A=\frac{2N\lambda^2}{\gamma}\,,\quad \frac{B}{A}=\frac{4\lambda^2}{\gamma^2}\,,\quad \Delta=\omega_0-m\omega=\lambda\delta\,. \tag{47b}$$

To simulate losses we can introduce a fictitious set of atoms in the lower state which will absorb the laser radiation. Two kinds of models have been used in the literature. In the m-photon loss model, which is somewhat artificial, one introduces a set of atoms that absorb m photons at a time. This model leads to the following equation for the losses:

$$\left[\frac{dP(n,t)}{dt}\right]_{\text{loss}}=C\frac{(n+m)!}{n!}P(n+m,t)-C\frac{n!}{(n-m)!}P(n,t)\,, \tag{48}$$

when saturation is ignored. C is the loss coefficient of the Scully-Lamb theory[7] of an optical maser. Adding Eqs. (47) and (48) we obtain the equation of motion for $P(n,t)$. The m-photon loss model is appealing since it ensures detailed balance in the steady state. In practice, however, losses are due to the escape of photons from the cavity, and there is no reason why photons should escape in groups of m. A realistic model of the losses is to consider a set of atoms that absorb a single photon from the field at a time. This model gives

$$\left[\frac{dP(n,t)}{dt}\right]_{\text{loss}}=C(n+1)P(n+1,t)-CnP(n,t)\,. \tag{49}$$

Finally, adding Eqs. (47) and (49) we obtain the equation for the photon probability distribution $P(n,t)$ in an m-photon laser,

$$\frac{dP(n,t)}{dt}=-\frac{A\frac{(n+m)!}{n!}}{\left[\frac{\Delta}{\gamma}\right]^2+1+\frac{B}{A}\frac{(n+m)!}{n!}}P(n,t)+C(n+1)P(n+1,t)+\frac{A\frac{n!}{(n-m)!}}{\left[\frac{\Delta}{\gamma}\right]^2+1+\frac{B}{A}\frac{n!}{(n-m)!}}P(n-m,t)-CnP(n,t)\,. \tag{50}$$

For $\Delta=0$ we obtain the equation derived by McNeil and Walls[8] and for $m=1$ we obtain the Scully-Lamb master equation for the single-mode one-photon laser in homogeneously broadened medium.

We can also discuss the case of an m-photon laser in an inhomogeneously broadened medium by considering a distribution of detuning Δ given by

$$W(\Delta)=\frac{1}{\sqrt{\pi}\Delta_0}e^{-(\Delta/\Delta_0)^2} . \tag{51}$$

We must then average Eq. (45) over atomic detunings. In the limit of large inhomogeneous broadening $\Delta_0/\gamma >> 1+(4\lambda^2/\gamma^2)(n+m)/n!$ we obtain the following equation of motion for $P(n,t)$:

$$\frac{dP(n,t)}{dt}=-\frac{\tilde{A}\frac{(n+m)!}{n!}}{\left[1+\frac{\tilde{B}}{\tilde{A}}\frac{(n+m)!}{n!}\right]^{1/2}}P(n,t)+C(n+1)P(n+1,t)$$
$$+\frac{\tilde{A}\frac{n!}{(n-m)!}}{\left[1+\frac{\tilde{B}}{\tilde{A}}\frac{(n!)}{(n-m)!}\right]^{1/2}}P(n-m,t)CnP(n,t) , \tag{52a}$$

where

$$\tilde{A}=\frac{2N\lambda^2\sqrt{\pi}}{\Delta_0} , \quad \tilde{B}=\tilde{A}\frac{4\lambda^2}{\gamma^2} . \tag{52b}$$

Thus, we have been able to derive the master equations for both homogeneous and inhomogeneous media including the effect of detuning. We shall, however, not attempt to discuss the solutions of these here. The behavior of off-diagonal elements can be described similarly. We next consider the problem of photon statistics in a multiphoton absorption or emission process.

B. m-photon absorption and emission

Here, as in the case of the laser problem, we find it convenient to work with the matrix elements of the density matrix $\hat{\rho}_F$ in the Fock state representation. We shall illustrate the method by considering the diagonal elements $\langle n|\hat{\rho}_F|n\rangle=P(n,t)$. The behavior of other matrix elements can be discussed similarly. In the m-photon absorption process in the presence of N atoms the initial density matrix of the atom field system is

$$\hat{\rho}(0)=\begin{bmatrix}0 & 0\\ 0 & \hat{\rho}_F(0)\end{bmatrix} . \tag{53}$$

Substituting this in Eq. (45) and using Eqs. (14), (33), and (44) we find that $P(n,t)$ satisfies the following equation:

$$\frac{dP(n,t)}{dt}=\frac{-A\frac{n!}{(n-m)!}}{1+\frac{B}{A}\frac{n!}{(n-m)!}}P(n,t)+\frac{A\frac{(n+m)!}{n!}}{1+\frac{B}{A}\frac{(n+m)!}{n!}}P(n+m,t) . \tag{54}$$

Here we have put $\Delta=0$ for simplicity and the coefficients A, B are defined by Eq. (47b).

In a similar fashion the m-photon emission process is described by an initial density matrix given by Eq. (46). Again considering the on-resonance case we find that $P(n,t)$ obeys the following equation in the m-photon emission process :

$$\frac{dP(n,t)}{dt}=-\frac{A\frac{(n+m)!}{n!}}{1+\frac{B}{A}\frac{(n+m)!}{n!}}P(n,t)+\frac{A\frac{n!}{(n-m)!}}{1+\frac{B}{A}\frac{n!}{(n-m)!}}P(n-m,t) . \tag{55}$$

Here A and B are again defined by Eq. (47b) and we have restricted our considerations to a set of homogeneously broadened atoms.

Finally, if the initial state of the atoms is a superposition of the lower and upper states with probabilities K_1 and K_2, from Eqs. (54) and (55) the equation of motion obeyed by $P(n,t)$ will be

$$\frac{dP(n,t)}{dt}=-K_1\frac{A\frac{n!}{(n-m)!}}{1+\frac{B}{A}\frac{n!}{(n-m)!}}P(n,t)-K_2\frac{A\frac{(n+m)!}{n!}}{1+\frac{B}{A}\frac{(n+m)!}{n!}}P(n,t)$$
$$+K_1\frac{A\frac{(n+m)!}{n!}}{1+\frac{B}{A}\frac{(n+m)!}{n!}}P(n+m,t)+K_2\frac{A\frac{n!}{(n-m)!}}{1+\frac{B}{A}\frac{n!}{(n-m)!}}P(n-m,t)\ . \qquad (56)$$

If we ignore the saturation terms in Eq. (56) we find

$$\frac{dP(n,t)}{dt}=-K_1\frac{An!}{(n-m)!}P(n,t)-K_2A\frac{(n+m)!}{n!}P(n,t)$$
$$+K_1\frac{A(n+m)!}{n!}P(n+m,t)+K_2A\frac{n!}{(n-m)!}P(n-m,t)\ . \qquad (57)$$

Equation (56) is the same as the equation derived by Agarwal[9] using a different method. Solutions of Eq. (57) were discussed by him and by Zubairy and Yeh.[10] Equation (56) includes saturation effects also and is derived here for the first time. We shall not, however, discuss here the consequences of Eq. (56) and other problems in m-photon processes that can be formulated based on the approach presented in this paper.

V. SUMMARY

We have discussed the behavior of photon statistics in some generalized Jaynes-Cummings models. It is shown that the mean photon number may show revivals and decays similar to those of the atomic inversion under certain circumstances. We have also indicated how these models can be used to study the behavior of the photon statistics of multiphoton absorption, emission, and laser processes.

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