Operation of a two-mode laser in a three-level atomic system with a common upper level

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The quantum theory of the three-energy-level two-mode laser in which all the levels are being pumped is investigated and the master equation of the laser operation is obtained. It contains two variables, n_1, the photon number of the first mode, and n_2, the photon number of the second mode. There appear four terms in the master equation which represent two-photon processes between the two modes, the absorption of a photon from one mode, and emission of a photon to the other. They are absent in the single-mode case and in the two-mode case with pumping only to the upper level. The master equation can be represented by a diagram of probability in two dimensions which can be extended to infinity, and each arrow of it represents a term on the right-hand side of the master equation. By summing each variable (n_1 or n_2) of the master equation, two equations can be obtained. Each equation contains only one variable, and can be expressed by a diagram of probability flow in one dimension. By considering the correspondence between macroscopic equilibrium and microscopic detailed balance the equations of motion in the steady state are obtained. By introducing a parameter H, the equation of motion is simplified and a formal solution can be deduced which is dependent on H. It appears that the parameter H cannot be fully determined, but some properties of it can be deduced which prove that the introduction of H is reasonable. The operation characteristics of the laser under different conditions are discussed on the basis of the formal solution: among them are the threshold condition, the condition for one-mode operation, and the change of the photon-statistical distribution. We especially discuss the laser output power curves under two-mode operation above threshold. The curve of one mode, which oscillates first, shows a bending-down phenomenon as the excitation increases when the other mode goes beyond its threshold as demonstrated experimentally by Otsuka. This phenomenon is explained qualitatively and is attributed to the two-photon processes between the two modes.

I. INTRODUCTION

In recent years, Scully and Lamb's laser theory of the two-level atomic system was extended to that of the multimode laser of the multilevel atomic system which reflects the active medium more faithfully.[1–6] Since there appeared more complicated phenomena in multimode operation than in the single-mode case,[4] there is even greater attention being focused on this extension. In this paper the master equation of the two-mode laser in a three-level atomic system with a common upper level, in which all levels are being pumped, is obtained. Using the principle of detailed balance the steady-state-motion equations are deduced. By introducing a parameter H the equations are simplified and a formal solution is obtained. The operation characteristics in different cases are discussed. It is shown that the unusual bending-down phenomena of the power-output curve under multimode operation is due to the two-photon processes between the two modes.

II. EQUATIONS OF MOTION

We consider a three-level atomic system as shown in Fig. 1. R_a, R_b, and R_c are the pumping rates for $|a\rangle$, $|b\rangle$, and $|c\rangle$, respectively. The Wigner-Weisskopf theorem is assumed to prevail and γ is the decay constant. [Here the same γ is used for different levels. If differences exist on the value of the γ's their effect can be included by adjusting the value of A_i's and (B_i/A_i)'s. Therefore, there is no influence on our results.] An excited atom in state $|a\rangle$ can make a transition to state $|b\rangle$ (or $|c\rangle$) by emitting a photon into mode 1 (or 2) with frequency Ω_1 (or Ω_2). The transition

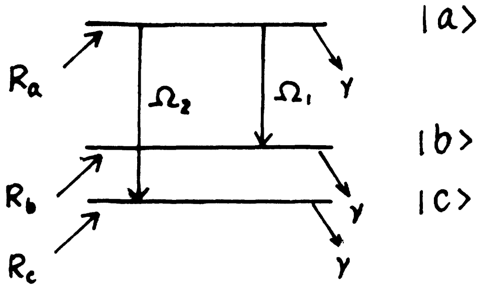


FIG. 1. Three-level atomic system.

between $|b\rangle$ and $|c\rangle$ is forbidden.

The Hamiltonian of the atom-plus-field system is

$$\begin{aligned} &H=H_A+H_F+V\ ,\\ &H_A=\sum_i \omega_i A_i^\dagger A_i\ ,\\ &H_F=\Omega_1(a_1^\dagger a_1+\tfrac{1}{2})+\Omega_2(a_2^\dagger a_2+\tfrac{1}{2})\ ,\\ &V=-\frac{1}{\hbar}e\vec{r}(t)\cdot\vec{E}(\vec{r},t)\ , \end{aligned} \tag{1}$$

where $\hbar\omega_i$ is the energy eigenvalue associated with the atomic level, Ω_1 and Ω_2 are the frequencies of modes 1 and 2, respectively, a_1 (a_2), $a_1^\dagger$ ($a_2^\dagger$) are the corresponding annihilation and creation operators of the field, and $A_i^\dagger$ (A_i), that of the atomic levels.

Supposing the field is linearly polarized and propagates along the Z axis, we have

$$V=g_1a_1A_a^\dagger A_b+g_2a_2A_a^\dagger A_c+\text{H.c.}\ , \tag{2}$$

where the rotating-wave approximation has been made and the transitions between $|a\rangle$ and $|b\rangle$ are connected with photons of Ω_1 only, and those between $|a\rangle$ and $|c\rangle$ are connected with photons of Ω_2 only. g_i is the atom-field coupling constant. H.c. stands for Hermitian conjugate.

Let ρ_f represent the density matrix of field, $\rho_{n_1,n_2;m_1,m_2}$ its element. As the three levels are all being pumped, and there exists cavity losses, we can write

$$\dot{\rho}_f=\dot{\rho}_f^{(a)}+\dot{\rho}_f^{(b)}+\dot{\rho}_f^{(c)}+\dot{\rho}_f^{\,l}\ , \tag{3}$$

where $\dot{\rho}_f^{(a)}$ stands for the change caused by the pumping for the level α and $\dot{\rho}_f^{\,l}$ caused by the cavity losses. Following the procedure of Sargent *et al.*,[7] they can be determined.

Let $\rho_{\alpha,n_1,n_2;\beta,m_1,m_2}$ be the element of the total system including both the atoms and field. The change in $\rho_{n_1,n_2;m_1,m_2}$ due to one atom in the state $|\alpha\rangle$ interacting for a very short time τ, is given by

$$\delta^\tau\rho^{(\alpha)}_{n_1,n_2;m_1,m_2}(t)=\sum_\beta \rho^{(\alpha)}_{\beta,n_1,n_2;\beta,m_1,m_2}(t+\tau)-\rho_{n_1,n_2;m_1,m_2}(t)\ . \tag{4}$$

Considering the atom decay which is described by a probability distribution $P(\tau)=\gamma e^{-\gamma\tau}$, the average change of field per atom in $\Delta t >> 1/\gamma$ is given by

$$\delta\rho^{(\alpha)}_{n_1n_2;m_1,m_2}(t)=\int_0^\infty P(\tau)\delta^\tau\rho^{(\alpha)}_{n_1,n_2;m_1,m_2}(t)d\tau\ . \tag{5}$$

As the change of field is very slow, by virtue of the adiabatic approximation, the upper limit of the integral may be extended to infinity. If the injection rate of atoms in state $|\alpha\rangle$ is R_α, we have

$$\begin{aligned}\dot{\rho}^{(\alpha)}_{n_1,n_2;m_1,m_2}(t)&=\frac{\Delta\rho^{(\alpha)}_{n_1,n_2;m_1,m_2}(t)}{\Delta t}=R_\alpha\delta\rho^{(\alpha)}_{n_1,n_2;m_1,m_2}(t)\\ &=R_\alpha\int_0^\infty d\tau\,\gamma e^{-\gamma\tau}\left[\sum_\beta \rho^{(\alpha)}_{\beta,n_1,n_2;\beta,m_1,m_2}(t+\tau)-\rho_{n_1,n_2;m_1,m_2}(t)\right]\ .\end{aligned} \tag{6}$$

The calculation of Eq. (6) is rather tedious. We examine the contributions of different atoms by parts.

A. Calculation of $\dot{\rho}_f^{(a)}$

The atom is initially in state $|a\rangle$. Let $|\psi_{Af}\rangle$ be the atom-field-state vector and $|\psi_f\rangle$ the field-state vector

$$|\psi_{Af}(t)\rangle = |\psi_f\rangle\,|a\rangle = \left[\sum_{n_1,n_2} F_{n_1,n_2}(t)\,|n_1,n_2\rangle\right]|a\rangle\ . \tag{7}$$

After time interval τ, the state vector develops into

$$|\psi_{Af}(t+\tau)\rangle = \sum_{n_1,n_2}[a_{n_1,n_2}(t+\tau)\,|a\rangle\,|n_1,n_2\rangle + b_{n_1,n_2}(t+\tau)\,|b\rangle\,|n_1+1,n_2\rangle + c_{n_1,n_2+1}(t+\tau)\,|c\rangle\,|n_1,n_2+1\rangle]\ . \tag{8}$$

From Eq. (7), we have

$$a_{n_1,n_2}(t)=F_{n_1,n_2}(t)\ ,$$
$$b_{n_1+1,n_2}(t)=c_{n_1,n_2+1}(t)=0\ . \tag{9}$$

From Eqs. (A5)–(A7) of Appendix A we obtain

$$a_{n_1,n_2}(t+\tau)=F_{n_1,n_2}(t)\cos\{[|g_1|^2(n_1+1)+|g_2|^2(n_2+1)]^{1/2}\tau\}\ , \tag{10}$$

$$b_{n_1+1,n_2}(t+\tau)=\frac{-iF_{n_1,n_2}(t)g_1^*\sqrt{n_1+1}}{[|g_1|^2(n_1+1)+|g_2|^2(n_2+1)]^{1/2}}\sin\{[|g_1|^2(n_1+1)+|g_2|^2(n_2+1)]^{1/2}\tau\}\ , \tag{11}$$

$$c_{n_1,n_2+1}(t+\tau)=\frac{-iF_{n_1,n_2}(t)g_2^*\sqrt{n_2+1}}{[|g_1|^2(n_1+1)+|g_2|^2(n_2+1)]^{1/2}}\sin\{[|g_1|^2(n_1+1)+|g_2|^2(n_2+1)]^{1/2}\tau\}\ , \tag{12}$$

$$\begin{aligned}\rho^{(a)}_{a,n_1,n_2;a,m_1,m_2}(t+\tau)&=\sum_\psi P_\psi a_{n_1,n_2}(t+\tau)a^*_{m_1,m_2}(t+\tau)\\&=\sum_\psi P_\psi F_{n_1,n_2}(t)F^*_{n_1,n_2}(t)\cos\{[|g_1|^2(n_1+1)+|g_2|^2(n_2+1)]^{1/2}\tau\}\\&\quad\times\cos\{[|g_1|^2(m_1+1)+|g_2|^2(m_2+1)]^{1/2}\tau\}\\&=\rho_{n_1,n_2;m_1,m_2}(t)\cos\{[|g_1|^2(n_1+1)+|g_2|^2(n_2+1)]^{1/2}\tau\}\\&\quad\times\cos\{[|g_1|^2(m_1+1)+|g_2|^2(m_2+1)]^{1/2}\tau\}\ .\end{aligned} \tag{13}$$

Here the field is considered to be in a mixed state with P_ψ representing the corresponding probability. Likewise, we have

$$\begin{aligned}\rho^{(a)}_{b,n_1,n_2;b,m_1,m_2}(t+\tau)&=\rho_{n_1-1,n_2;m_1-1,m_2}(t)\frac{|g_1|^2\sqrt{n_1m_1}}{\{[|g_1|^2n_1+|g_2|^2(n_2+1)][|g_1|^2m_1+|g_2|^2(m_2+1)]\}^{1/2}}\\&\quad\times\sin\{[|g_1|^2n_1+|g_2|^2(n_2+1)]^{1/2}\tau\}\sin\{[|g_1|^2m_1+|g_2|^2(m_2+1)]^{1/2}\tau\}\ ,\end{aligned} \tag{14}$$

$$\begin{aligned}\rho^{(a)}_{c,n_1,n_2;c,m_1,m_2}(t+\tau)&=\rho_{n_1,n_2-1;m_1,m_2-1}(t)\frac{|g_2|^2\sqrt{n_2m_2}}{\{[|g_1|^2(n_1+1)+|g_2|^2n_2][|g_1|^2(m_1+1)+|g_2|^2m_2]\}^{1/2}}\\&\quad\times\sin\{[|g_1|^2(n_1+1)+|g_2|^2n_2]^{1/2}\tau\}\sin\{[|g_1|^2(m_1+1)+|g_2|^2m_2]^{1/2}\tau\}\ .\end{aligned} \tag{15}$$

Substituting Eqs. (13)–(15) into Eq. (6) and carrying out the integration we obtain

$$\dot{\rho}^{(a)}_{n_1,n_2;m_1,m_2}(t) = -R_a\rho_{n_1,n_2;m_1,m_2}(t) \times \frac{\frac{1}{\gamma^2}[\,|g_1|^2(n_1+m_1+2)+|g_2|^2(n_2+m_2+2)]+\frac{1}{\gamma^4}[\,|g_1|^2(n_1-m_1)+|g_2|^2(n_2-m_2)]^2}{1+\frac{2}{\gamma^2}[\,|g_1|^2(n_1+m_1+2)+|g_2|^2(n_2+m_2+2)]+\frac{1}{\gamma^4}[\,|g_1|^2(n_1-m_1)+|g_2|^2(n_2-m_2)]^2}$$

$$+R_a\rho_{n_1-1,n_2;m_1-1m_2}(t) \times \frac{\frac{2}{\gamma^2}|g_1|^2\sqrt{n_1m_1}}{1+\frac{2}{\gamma^2}[\,|g_1|^2(n_1+m_1)+|g_2|^2(n_2+m_2+2)]+\frac{1}{\gamma^4}[\,|g_1|^2(n_1-m_1)+|g_2|^2(n_2-m_2)]^2}$$

$$+R_a\rho_{n_1,n_2-1;m_1,m_2-1}(t) \times \frac{\frac{2}{\gamma^2}|g_2|^2\sqrt{n_2m_2}}{1+\frac{2}{\gamma^2}[\,|g_1|^2(n_1+m_2+2)+|g_2|^2(n_2+m_2)]+\frac{1}{\gamma^4}[\,|g_1|^2(n_1-m_1)+|g_2|^2(n_2-m_2)]^2}\,. \qquad (16)$$

Put

$$A_1=2R_a\left|\frac{g_1}{\gamma}\right|^2,\quad A_2=2R_a\left|\frac{g_2}{\gamma}\right|^2,\quad \frac{B_1}{A_1}=4\left|\frac{g_1}{\gamma}\right|^2,\quad \frac{B_2}{A_2}=4\left|\frac{g_2}{\gamma}\right|^2. \qquad (17)$$

We now consider the diagonal elements and abbreviate $\rho_{n_1,n_2;m_1,m_2}(t)$ as $p(n_1,n_2)$. $p(n_1,n_2)$ has the meaning of the probability of finding n_1 photons in mode 1 and n_2 protons in mode 2. Thus the following equation can be deduced:

$$\dot{p}^{(a)}(n_1,n_2) = -\frac{A_1(n_1+1)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)}p(n_1,n_2) - \frac{A_2(n_2+1)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)}p(n_1,n_2)$$
$$+\frac{A_1n_1}{1+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}(n_2+1)}p(n_1-1,n_2)+\frac{A_2n_2}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}n_2}p(n_1,n_2-1)\,. \qquad (18)$$

B. Calculation of $\dot{\rho}_f^{(b)}$

The atom is initially in state $|b\rangle$:

$$|\psi_{Af}(t)\rangle = \left[\sum_{n_1,n_2} F_{n_1,n_2}(t)\,|n_1,n_2\rangle\right]|b\rangle\,. \qquad (19)$$

Set

$$|\psi_{Af}(t+\tau)\rangle = \sum_{n_1,n_2}[\,a_{n_1-1,n_2}(t+\tau)|a\rangle|n_1-1,n_2\rangle + b_{n_1,n_2}(t+\tau)|b\rangle|n_1,n_2\rangle$$
$$+c_{n_1-1,n_2+1}(t+\tau)|c\rangle|n_1-1,n_2+1\rangle]\,. \qquad (20)$$

From Eq. (19), we know

$$a_{n_1-1,n_2}(t)=c_{n_1-1,n_2+1}(t)=0,\quad b_{n_1,n_2}(t)=F_{n_1,n_2}(t)\ . \tag{21}$$

By virtue of Eqs. (A8)–(A10) of Appendix A, the following equations can be obtained:

$$a_{n_1-1,n_2}(t+\tau)=F_{n_1,n_2}(t)\frac{-ig_1\sqrt{n_1}}{[\,|g_1|^2n_1+|g_2|^2(n_2+1)]^{1/2}}\sin\{[\,|g_1|^2n_1+|g_2|^2(n_2+1)]^{1/2}\tau\}\ , \tag{22}$$

$$b_{n_1,n_2}(t+\tau)=F_{n_1,n_2}(t)\frac{1}{|g_1|^2n_1+|g_2|^2(n_2+1)}\times(|g_1|^2n_1\cos\{[\,|g_1|^2n_1+|g_2|^2(n_2+1)]^{1/2}\tau\}+|g_2|^2(n_2+1))\ , \tag{23}$$

$$c_{n_1-1,n_2}(t+\tau)=F_{n_1,n_2}(t)\frac{g_1g_2^*\sqrt{n_1(n_2+1)}}{|g_1|^2n_1+|g_2|^2(n_2+1)}(\cos\{[\,|g_1|^2n_1+|g_2|^2(n_2+1)]^{1/2}\tau\}-1)\ . \tag{24}$$

From Eqs. (22)–(24), the diagonal elements are found to be

$$\begin{aligned}\rho^{(b)}_{a,n_1,n_2;a,n_1,n_2}(t+\tau)&=\sum_\psi P_\psi a_{n_1,n_2}(t+\tau)a^*_{n_1,n_2}(t+\tau)\\&=\rho_{n_1+1,n_2;n_1+1,n_2}(t)\frac{|g_1|^2(n_1+1)}{|g_1|^2(n_1+1)+|g_2|^2(n_2+1)}\sin^2\{[\,|g_1|^2(n_1+1)+|g_2|^2(n_2+1)]^{1/2}\tau\},\end{aligned} \tag{25}$$

$$\rho^{(b)}_{b,n_1,n_2;b,n_1,n_2}(t+\tau)=\rho_{n_1,n_2;n_1,n_2}(t)\frac{1}{[\,|g_1|^2n_1+|g_2|^2(n_2+1)]^2}\times(|g_1|^2n_1\cos\{[\,|g_1|^2n_1+|g_2|^2(n_2+1)]^{1/2}\tau\}+|g_2|^2(n_2+1))^2\ , \tag{26}$$

$$\rho^{(b)}_{c,n_1,n_2;c,n_1,n_2}(t+\tau)=\rho_{n_1+1,n_2;n_1+1,n_2}(t)\frac{|g_1|^2|g_2|^2n_2(n_1+1)}{[\,|g_1|^2(n_1+1)+|g_2|^2n_2]^2}(\cos\{[\,|g_1|^2(n_1+1)+|g_2|^2n_2]^{1/2}\tau\}-1)^2\ . \tag{27}$$

Substituting Eqs. (25)–(27) into Eq. (6), we arrive at

$$\begin{aligned}\dot p^{(b)}(n_1,n_2)={}&\frac{A_1'(n_1+1)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)}p(n_1+1,n_2)\\&-\frac{A_1'n_1}{\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}(n_2+1)}\left[\frac{\frac{B_1}{A_1}n_1}{1+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}(n_2+1)}+\frac{\frac{B_2}{A_2}(n_2+1)}{1+\frac{1}{4}\left[\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}(n_2+1)\right]}\right]p(n_1,n_2)\\&+A_1'(n_1+1)\frac{B_2}{A_2}n_2\left[\frac{1}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}n_2}-\frac{1}{4+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}n_2}\right]p(n_1+1,n_2-1)\ ,\end{aligned} \tag{28}$$

where

$$A_1'=2R_b\left|\frac{g_1}{\gamma}\right|^2 . \tag{29}$$

C. Calculation of $\dot\rho_f^{(c)}$

Using the same method, and Eqs. (A11)–(A13) of Appendix A we obtain

$$\dot{p}^{(c)}(n_1,n_2)=\frac{A_2''(n_2+1)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)}p(n_1,n_2+1)$$
$$-\frac{A_2''n_2}{\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}n_2}\left[\frac{\frac{B_2}{A_2}n_2}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}n_2}+\frac{\frac{B_1}{A_1}(n_1+1)}{1+\frac{1}{4}\left[\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}n_2\right]}\right]p(n_1,n_2)$$
$$+A_2''(n_2+1)\frac{B_1}{A_1}n_1\left[\frac{1}{1+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}(n_2+1)}-\frac{1}{4+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}(n_2+1)}\right]p(n_1-1,n_2+1)\,, \quad (30)$$

where

$$A_2''=2R_c\left|\frac{g_2}{\gamma}\right|^2 . \quad (31)$$

The change of the density-matrix elements caused by cavity losses is[3]

$$\dot{p}^{l}(n_1,n_2)=C_1(n_1+1)p(n_1+1,n_2)+C_2(n_2+1)p(n_2,n_2+1)-C_1n_1p(n_1,n_2)-C_2n_2p(n_1,n_2)\,. \quad (32)$$

Combining Eqs. (18), (28), (30), and (32), the master equation of two-mode lasers in a three-level atomic system with a common upper level is as follows:

$$\dot{p}(n_1,n_2)=-\frac{A_1(n_1+1)p(n_1,n_2)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)}-\frac{A_2(n_2+1)p(n_1,n_2)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)}+\frac{A_1n_1p(n_1-1,n_2)}{1+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}(n_2+1)}$$
$$+\frac{A_2n_2p(n_1,n_2-1)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}n_2}+C_1(n_1+1)p(n_1+1,n_2)+C_2(n_2+1)p(n_1,n_2+1)-C_1n_1p(n_1,n_2)$$
$$-C_2n_2p(n_1,n_2)+\frac{A_1'(n_1+1)p(n_1+1,n_2)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)}-\frac{A_1'n_1p(n_1,n_2)}{1+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}(n_2+1)}$$
$$+\frac{\frac{3}{4}A_1'(n_1+1)\frac{B_2}{A_2}n_2p(n_1+1,n_2-1)}{\left[1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}n_2\right]\left[1+\frac{1}{4}\frac{B_1}{A_1}(n_1+1)+\frac{1}{4}\frac{B_2}{A_2}n_2\right]}$$
$$-\frac{\frac{3}{4}A_1'n_1\frac{B_2}{A_2}(n_2+1)p(n_1,n_2)}{\left[1+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}(n_2+1)\right]\left[1+\frac{1}{4}\frac{B_1}{A_1}n_1+\frac{1}{4}\frac{B_2}{A_2}(n_2+1)\right]}+\frac{A_2''(n_2+1)p(n_1,n_2+1)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)}$$
$$-\frac{A_2''n_2p(n_1,n_2)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}n_2}+\frac{\frac{3}{4}A_2''(n_2+1)\frac{B_1}{A_1}n_1p(n_1-1,n_2+1)}{\left[1+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}(n_2+1)\right]\left[1+\frac{1}{4}\frac{B_1}{A_1}n_1+\frac{1}{4}\frac{B_2}{A_2}(n_2+1)\right]}$$
$$-\frac{\frac{3}{4}A_2''n_2\frac{B_1}{A_1}(n_1+1)p(n_1,n_2)}{\left[1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}n_2\right]\left[1+\frac{1}{4}\frac{B_1}{A_1}(n_1+1)+\frac{1}{4}\frac{B_2}{A_2}n_2\right]} \quad (33)$$

On the right-hand side of the master equation there are 16 terms, and they can be interpreted as probability flows which can be expressed by arrows in a two-dimensional probability flow diagram that, in its turn, can be extended to infinity as shown in Fig. 2.

Each arrow in Fig. 2(a) represents a term on the right-hand side of Eq. (33). The numbers attached to each arrow indicate to which term in Eq. (33) it corresponds. Though the fifth term has the same physical meaning as the ninth term, they are of different origin and are represented, therefore, by two different arrows in Fig. 2(a). These two terms are combined into one in Fig. 2(b). The same is true with the following pairs: the sixth and thirteenth, the seventh and the tenth, and the eighth and the fourteenth. Figure 2(b) is the total probability flow diagram extended to infinity. Figure 2(a) is the probability flow diagram of state (n_1,n_2).

The 11th, 12th, 15th, and 16th terms in (33) represent two-photon processes between the two modes, which have not been considered in Ref. 3. The atom pumped into one lower state can make a transition to an upper state by absorbing a photon of one mode, then make another transition to the other lower state by emitting a photon of the other mode. Each process plays a role of a loss to one mode, while a gain to the other mode. These processes are in fact saturation effects of absorption.

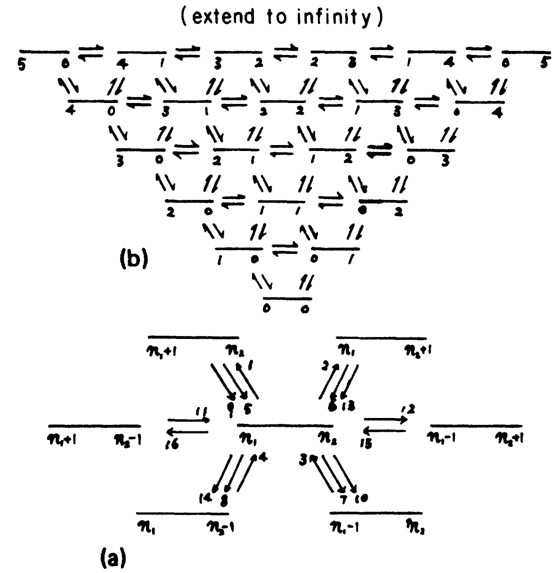


FIG. 2. Diagram of probability flow in two dimensions. The numbers under each bar (state) indicate the photon number (n_1,n_2). (a) The probability flow of state (n_1,n_2). (b) The total probability flow scheme.

From Eq. (33) the following equations can be obtained:

$$\dot{p}(n_1)=\sum_{n_2}\dot{p}(n_1,n_2)$$

$$=-A_1(n_1+1)\sum_{n_2}\frac{p(n_1,n_2)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)}+A_1n_1\sum_{n_2}\frac{p(n_1-1,n_2)}{1+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}(n_2+1)}+C_1(n_1+1)p(n_1+1)$$

$$-C_1n_1p(n_1)+A_1'(n_1+1)\sum_{n_2}\frac{p(n_1+1,n_2)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)}-A_1'n_1\sum_{n_2}\frac{p(n_1,n_2)}{1+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}(n_2+1)}$$

$$+\tfrac{3}{4}A_1'(n_1+1)\sum_{n_2}\frac{\frac{B_2}{A_2}(n_2+1)p(n_1+1,n_2)}{\left[1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)\right]\left[1+\frac{1}{4}\frac{B_1}{A_1}(n_1+1)+\frac{1}{4}\frac{B_2}{A_2}(n_2+1)\right]}$$

$$-\tfrac{3}{4}A_1'n_1\sum_{n_2}\frac{\frac{B_2}{A_2}(n_2+1)p(n_1,n_2)}{\left[1+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}(n_2+1)\right]\left[1+\frac{1}{4}\frac{B_1}{A_1}n_1+\frac{1}{4}\frac{B_2}{A_2}(n_2+1)\right]}$$

$$+\tfrac{3}{4}A_2''\frac{B_1}{A_1}n_1\sum_{n_2}\frac{n_2p(n_1-1,n_2)}{\left[1+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}n_2\right]\left[1+\frac{1}{4}\frac{B_1}{A_1}n_1+\frac{1}{4}\frac{B_2}{A_2}n_2\right]}$$

$$-\tfrac{3}{4}A_2'' \frac{B_1}{A_1}(n_1+1)\sum_{n_2} \frac{n_2 p(n_1,n_2)}{\left[1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}n_2\right]\left[1+\frac{1}{4}\frac{B_1}{A_1}(n_1+1)+\frac{1}{4}\frac{B_2}{A_2}n_2\right]}, \tag{34}$$

$$\dot p(n_2)=\sum_{n_1}\dot p(n_1,n_2)$$

$$=-A_2(n_2+1)\sum_{n_1}\frac{p(n_1,n_2)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)}+A_2 n_2\sum_{n_1}\frac{p(n_1,n_2-1)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}n_2}$$

$$+C_2(n_2+1)p(n_2+1)-C_2 n_2 p(n_2)+A_2''(n_2+1)\sum_{n_1}\frac{p(n_1,n_2+1)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)}$$

$$-A_2'' n_2\sum_{n_1}\frac{p(n_1,n_2)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}n_2}$$

$$+\tfrac{3}{4}A_2''(n_2+1)\sum_{n_1}\frac{\frac{B_1}{A_1}(n_1+1)p(n_1,n_2+1)}{\left[1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)\right]\left[1+\frac{1}{4}\frac{B_1}{A_1}(n_1+1)+\frac{1}{4}\frac{B_2}{A_2}(n_2+1)\right]}$$

$$-\tfrac{3}{4}A_2'' n_2\sum_{n_1}\frac{\frac{B_1}{A_1}(n_1+1)p(n_1,n_2)}{\left[1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}n_2\right]\left[1+\frac{1}{4}\frac{B_1}{A_1}(n_1+1)+\frac{1}{4}\frac{B_2}{A_2}n_2\right]}$$

$$+\tfrac{3}{4}A_1'\frac{B_2}{A_2}n_2\sum_{n_1}\frac{n_1 p(n_1,n_2-1)}{\left[1+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}n_2\right]\left[1+\frac{1}{4}\frac{B_1}{A_1}n_1+\frac{1}{4}\frac{B_2}{A_2}n_2\right]}$$

$$-\tfrac{3}{4}A_1'\frac{B_2}{A_2}(n_2+1)\sum_{n_1}\frac{n_1 p(n_1,n_2)}{\left[1+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}(n_2+1)\right]\left[1+\frac{1}{4}\frac{B_1}{A_1}n_1+\frac{1}{4}\frac{B_2}{A_2}(n_2+1)\right]}. \tag{35}$$

Equation (35) can be represented by a probability flow diagram in one dimension, as shown in Fig. 3. Each arrow in Fig. 3 represents a term in Eq. (35). The numbers attached to it indicate to which term in (35) the arrow corresponds. Equation (34) is represented and interpreted in the same way.

III. STEADY-STATE EQUATION OF MOTION

When the equilibrium state is reached, macroscopically the intensities of each mode will not change; microscopically, it means that $\dot p(n_1)=0$ and $\dot p(n_2)=0$. From the physical meaning of every term in (34) and (35) and the principle of detailed balance, which is the microscopical reflection of the macroscopical equilibrium, two equations of detailed balance, which are equivalent to Eqs. (34) and (35), can be obtained. They are

$$A_1 \sum_{n_2} \frac{p(n_1,n_2)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)} + \tfrac{3}{4}A_2'' \frac{B_1}{A_1} \sum_{n_2} \frac{n_2 p(n_1,n_2)}{\left[1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}n_2\right]\left[1+\frac{1}{4}\frac{B_1}{A_1}(n_1+1)+\frac{1}{4}\frac{B_2}{A_2}n_2\right]}$$
$$= C_1 p(n_1+1) + A_1' \sum_{n_2} \frac{p(n_1+1,n_2)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)}$$
$$+\tfrac{3}{4}A_1' \sum_{n_2} \frac{\frac{B_2}{A_2}(n_2+1)p(n_1+1,n_2)}{\left[1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)\right]\left[1+\frac{1}{4}\frac{B_1}{A_1}(n_1+1)+\frac{1}{4}\frac{B_2}{A_2}(n_2+1)\right]}\,, \tag{36}$$

$$A_2 \sum_{n_1} \frac{p(n_1,n_2)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)} + \tfrac{3}{4}A_1' \frac{B_2}{A_2} \sum_{n_1} \frac{n_1 p(n_1 n_2)}{\left[1+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}(n_2+1)\right]\left[1+\frac{1}{4}\frac{B_1}{A_1}n_1+\frac{1}{4}\frac{B_2}{A_2}(n_2+1)\right]}$$
$$= C_2 p(n_2+1) + A_2'' \sum_{n_1} \frac{p(n_1,n_2+1)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)}$$
$$+\tfrac{3}{4}A_2'' \sum_{n_1} \frac{\frac{B_1}{A_1}(n_1+1)p(n_1,n_2+1)}{\left[1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)\right]\left[1+\frac{1}{4}\frac{B_1}{A_1}(n_1+1)+\frac{1}{4}\frac{B_2}{A_2}(n_2+1)\right]}\,. \tag{37}$$

The solutions of these two equations are quite difficult, but some general properties of laser operation can be drawn from them.

A. The case of no pumping to the lower levels

The condition of no pumping to the lower levels means that $A_1' = A_2'' = 0$. Thus Eqs. (36) and (37) are now reduced to

$$A_1 \sum_{n_2} \frac{p(n_1,n_2)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)} = C_1 p(n_1+1)\,, \tag{38}$$

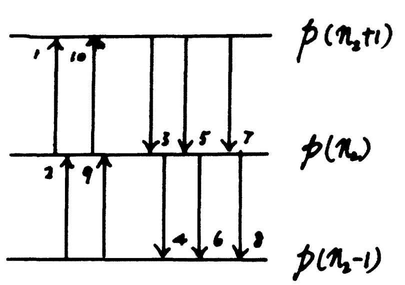


FIG. 3. Diagram of probability flow in one dimension.

$$A_2 \sum_{n_1} \frac{p(n_1,n_2)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)} = C_2 p(n_2+1)\,. \tag{39}$$

When both modes are above the threshold, Eqs. (38) and (39) can be simplified by assuming the existence of the equalities

$$\frac{A_1 p(n_1)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}H_2(n_1)\langle n_2\rangle} = C_1 p(n_1+1)\,, \tag{40}$$

$$\frac{A_2 p(n_2)}{1+\frac{B_1}{A_1}H_1(n_2)\langle n_1\rangle+\frac{B_2}{A_2}(n_2+1)} = C_2 p(n_2+1)\,, \tag{41}$$

where the symbol $\langle\ \rangle$ represents the peak position in the curve of the photon-statistical distribution and $H(n)$'s are newly introduced parameters. The validity of equalities (40) and (41) lies in the fact that we can always choose parameters H to ensure that the equalities are correct. The value of H_1 is dependent on n_2 as well as on the photon-statistical distribution of mode 1, so it designated by a symbol $H_1(n_2)$. $H_2(n_1)$ has the same meaning as $H_1(n_2)$ except the roles of n_1 and n_2 are interchanged. The assumption of the existence of equalities (40) and (41) is equivalent to the existence of the equalities

$$\sum_{n_2} \frac{p(n_1,n_2)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)} = \frac{p(n_1)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}H_2(n_1)\langle n_2\rangle}, \tag{42a}$$

$$\sum_{n_1} \frac{p(n_1,n_2)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)} = \frac{p(n_2)}{1+\frac{B_1}{A_1}H_1(n_2)\langle n_1\rangle+\frac{B_2}{A_2}(n_2+1)}. \tag{42b}$$

From the condition $p(n)=p(n-1)$, the peak position of the photon-statistical distribution can be found in

$$\frac{A_1}{1+\frac{B_1}{A_1}\langle n_1\rangle+\frac{B_2}{A_2}H_2\langle n_2\rangle}=C_1\,, \tag{43}$$

$$\frac{A_2}{1+\frac{B_1}{A_1}H_1\langle n_1\rangle+\frac{B_2}{A_2}\langle n_2\rangle}=C_2\,, \tag{44}$$

where $H_1=H_1(\langle n_2\rangle)$ and $H_2=H_2(\langle n_1\rangle)$ are constants. In Appendix B we proved that $H>1$.

Equations (40), (41), (43), and (44) can be suited to the cases above the threshold. Below threshold there exists no peak. At threshold the peak position is just at the original point $\langle n\rangle=0$. However, by inspection of Eqs. (42a) and (42b) one can see that $H\langle n\rangle$ must be positive and finite. So for the case of threshold operation we have $H=\infty$. Below threshold, from (40) and (41), it may be assumed that the peak position is in the region where n is negative. Extending $\langle n\rangle$ into negative value range means that we have included the cases for which the operation is below threshold. Thus Eqs. (40)–(44) can be applied to all cases. But it has already been pointed out that $H\langle n\rangle$ should always be positive. If $\langle n\rangle$ is positive, H must be positive also, if $\langle n\rangle$ is negative H must be negative too, i.e., H has the same sign as $\langle n\rangle$. When the pumping increases from a value below the threshold up to threshold and then above it, H changes from negative to $-\infty$ $(+\infty)$, and then rapidly decreases from $+\infty$ to a value slightly larger than 1.

From Eqs. (38) and (39), a formal solution can be found:

$$p(n_1)=N_1^{-1}\prod_{j=1}^{n_1}\frac{A_1}{C_1}\frac{1}{1+\frac{B_1}{A_1}j+\frac{B_2}{A_2}H_2(j)\langle n_2\rangle}\,, \tag{45}$$

$$p(n_2)=N_2^{-1}\prod_{j=1}^{n_2}\frac{A_2}{C_2}\frac{1}{1+\frac{B_1}{A_1}H_1(j)\langle n_1\rangle+\frac{B_2}{A_2}j}\,. \tag{46}$$

1. The threshold condition

Every factor of the multiplier in Eq. (40) [or in Eq. (41)], $(A_1/C_1)/[1+(B_1/A_1)j+(B_2/A_2)H_2(j)\langle n_2\rangle]$ decreases as j increases. The threshold conditions are

$$\frac{A_1}{C_1}\frac{1}{1+\frac{B_2}{A_2}H_2(0)\langle n_2\rangle}=1 \tag{47}$$

for mode 1,

$$\frac{A_2}{C_2}\frac{1}{1+\frac{B_1}{A_1}H_1(0)\langle n_1\rangle}=1 \tag{48}$$

for mode 2. The threshold condition of one mode is related not only to corresponding A and C, but also to the intensity of the other mode. The stronger the other mode is, the higher the threshold will be. Let $H\langle n\rangle=0$ in Eq. (43) [or (44)], that is to say, no influence of the other mode exists, the maximum peak value of photon-statistical distribution can be obtained as follows:

$$\langle n_1\rangle_{\max}=\frac{A_1}{B_1}\left[\frac{A_1}{C_1}-1\right],\qquad \langle n_2\rangle_{\max}=\frac{A_2}{B_2}\left[\frac{A_2}{C_2}-1\right]. \tag{49}$$

2. The condition of one-mode operation

The term of one-mode operation means that only one curve of the photon-statistical distribution of the two modes has a peak. From Eq. (43) we have

$$\frac{B_1}{A_1}\langle n_1\rangle=\frac{A_1}{C_1}-1-\frac{B_2}{A_2}H_2\langle n_2\rangle \geq\frac{A_1}{C_1}-1-\frac{B_2}{A_2}\langle n_2\rangle_{\max} =\frac{A_1}{C_1}-\frac{A_2}{C_2}\ . \tag{50}$$

According to Eq. (48) one can see that mode 2 could not oscillate if

$$\frac{A_2}{C_2}\frac{1}{1+\frac{B_1}{A_1}H_1(0)\langle n_1\rangle}<1\ ,$$

i.e.,

$$\frac{A_2}{C_2}<1+\frac{B_1}{A_1}H_1(0)\langle n_1\rangle\approx 1+\frac{B_1}{A_1}\langle n_1\rangle\ . \tag{51}$$

With (51) and (50), we obtain

$$\frac{A_1}{C_1}\geq 2\frac{A_2}{C_2}-1\ . \tag{52}$$

Thus, when (52) is valid, we have a one-mode operation. If $(A_1/C_1)/(A_2/C_2)\geq 2$, no matter how strong the pumping is, only mode 1 can oscillate. Equations (43) and (44) can be solved formally to obtain

$$\frac{B_1}{A_1}\langle n_1\rangle=\frac{H_2\frac{A_2}{C_2}-\frac{A_1}{C_1}+1-H_2}{H_1H_2-1}\ , \tag{53}$$

$$\frac{B_2}{A_2}\langle n_2\rangle=\frac{H_1\frac{A_1}{C_1}-\frac{A_2}{C_2}+1-H_1}{H_1H_2-1}\ . \tag{54}$$

Some characteristics for the two-mode operation can be drawn from these two formal solutions.

3. The case $A_1/C_1=A_2/C_2=A/C>1$

In this case Eqs. (53) and (54) now read

$$\frac{B_1}{A_1}\langle n_1\rangle=\frac{\left[\frac{A}{C}-1\right](H_2-1)}{H_1H_2-1}\ , \tag{55}$$

$$\frac{B_2}{A_2}\langle n_2\rangle=\frac{\left[\frac{A}{C}-1\right](H_1-1)}{H_1H_2-1}\ . \tag{56}$$

Both H_1 and H_2 could not be negative, otherwise H_1 (or H_2) would have the opposite sign with $\langle n_1\rangle$ (or $\langle n_2\rangle$). Therefore, both modes exist simultaneously,

$$\frac{B_1}{A_1}\langle n_1\rangle+\frac{B_2}{A_2}\langle n_2\rangle=\frac{H_1+H_2-2}{H_1H_2-1}\left[\frac{A}{C}-1\right] \approx\frac{A}{C}-1\ . \tag{57}$$

When $A_1=C_1$ and $A_2=C_2$ the above equation can be deduced strictly.

4. The cases $A_1/C_1>A_2/C_2>1$, $A_1/C_1=\chi A_2/C_2$, and $2>\chi>1$

It can be seen from Eqs. (53) and (54) that H_1 must be positive, or $\langle n_1\rangle$ and H_1 would have the opposite sign. However, H_2 is not prevented from being negative. Therefore, mode 1 definitely exists. If the two modes are asked to appear simultaneously, the following inequalities are required:

$$H_2>\chi\ ,\qquad H_1>1/\chi\ . \tag{58}$$

Doubtlessly the inequality $H_1>1/\chi$ is valid. However, the inequality $H_2>\chi$, according to Eq. (42), indicates that the "center of gravity" of the curve of the photon-statistical distribution [or first moment $\sum_n np(n)$] is shifted from the peak position by a larger value, in this case, than in that of a single mode. This, in turn, according to Eq. (B12) demands that $\langle n_1\rangle$ should be considerably larger. Thus Eqs. (53) and (54) can be written as

$$\frac{B_1}{A_1}\langle n_1\rangle=\frac{\frac{A_2}{C_2}(H_2-\chi)-(H_2-1)}{H_1H_2-1}\ , \tag{59}$$

$$\frac{B_2}{A_2}\langle n_2\rangle=\frac{\frac{A_2}{C_2}(H_1\chi-1)-(H_1-1)}{H_1H_2-1}\ . \tag{60}$$

5. The change of the photon-statistical distribution

When both modes exist the photon-statistical distribution of mode 1 is determined by Eq. (40).

Though the strict results have not been found, some characteristics of it can be understood by comparing it with the photon statistics of a single-mode operation with its peak at the same position.

In single-mode operation, the photon statistics are determined by

$$p_s(n+1)=\frac{A_s}{C_s}\frac{p_s(n)}{1+\frac{B_s}{A_s}(n+1)} . \tag{61}$$

Letting $B_1/A_1=B_s/A_s$, from the condition $\langle n_1\rangle=\langle n_2\rangle$ one obtains

$$\frac{A_s}{C_s}=\frac{A_1}{C_1}-\frac{B_2}{A_2}H_2\langle n_2\rangle . \tag{62}$$

Consider the ratio k, which is proportional to the ratio of the slopes of the two curves (It is the ratio of the two increments of the two curves. It will not influence the correctness of our results):

$$k=\frac{\frac{A_s}{C_s}\frac{1}{1+\frac{B_s}{A_s}n}-1}{\frac{A_1}{C_1}\frac{1}{1+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}H_2(n_1)\langle n_2\rangle}-1}$$

$$=\frac{1+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}H_2(n_1)\langle n_2\rangle}{1+\frac{B_1}{A_1}n_1}\;\frac{\frac{A_1}{C_1}-1-\frac{B_1}{A_1}n_1-\frac{B_2}{A_2}H_2\langle n_2\rangle}{\frac{A_1}{C_1}-1-\frac{B_1}{A_1}n_1-\frac{B_2}{A_2}H_2(n_1)\langle n_2\rangle}$$

$$=\frac{1+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}H_2\langle n_2\rangle}{1+\frac{B_1}{A_1}n_1}>1 , \tag{63}$$

in which the approximation, $H_2(n_1)=H_2$ has been used. This approximation introduces little error, since the change of $H_2(n_1)$ with n_1 is very small, especially in the vicinity of the peak value.

Equation (63) tells us that the absolute value of the slope of the photon-statistical distribution under the existence of the other mode is smoother than that of single-mode operation having the same peak value. This means that the photon-statistics-distribution curve is stiff for single-mode operation, but smooth for the two-mode coexistence operation, as shown in Fig. 4. It can also be seen from Eq. (B12) that the shift of the center of gravity of the curve from its peak position is also larger for the two-mode operation. The stronger the other mode is, the more pronounced these two phenomena will be.

B. The case of same pumping to the two lower levels $R_b=R_c=R'$

Under this condition, the second term on the left-hand side of Eq. (37) [or Eq. (36)] is approximately equal to the third term on the right-hand side of it. The two two-photon processes between the two modes originating from pumping to the two lower levels canceled each other out. Therefore, Eqs. (37) and (36) become

$$\frac{A_1p(n_1)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_1}{A_2}H_2(n_1)\langle n_2\rangle}=C_1p(n_1+1)+\frac{A_1'p(n_1+1)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}H_2(n_1)\langle n_2\rangle} , \tag{64}$$

$$\frac{A_2p(n_2)}{1+\frac{B_1}{A_1}H_1(n_2)\langle n_1\rangle+\frac{B_2}{A_2}(n_2+1)}=C_2p(n_2+1)+\frac{A_2''p(n_2+1)}{1+\frac{B_1}{A_1}H_1(n_2)\langle n_1\rangle+\frac{B_2}{A_2}(n_2+1)} . \tag{65}$$

By letting $p(n)=p(n-1)$ the peak position can be found to be

$$\frac{A_1-A_1'}{1+\frac{B_1}{A_1}\langle n_1\rangle+\frac{B_2}{A_2}H_2\langle n_2\rangle}=C_1\ , \tag{66}$$

$$\frac{A_2-A_2''}{1+\frac{B_1}{A_1}H_1\langle n_1\rangle+\frac{B_2}{A_2}\langle n_2\rangle}=C_2\ . \tag{67}$$

This is equivalent to changing the pumping rates to the upper level from R_a to R_a-R', and to the lower levels from R' to 0. Therefore, their peak positions and basic characteristics are similar. However, the photon statistics of the pumping rates differ from each other somewhat.

We put $p(n_1+1)=K'p(n_1)$, where K' is a parameter very near to 1. In fact, it is larger than 1 before the peak value, and smaller than 1 behind the peak value. Equation (64) now reads as

$$(A_1-K'A_1')\frac{p(n_1)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}H_2(n_1)\langle n_2\rangle}=C_1p(n_1+1)\ . \tag{68}$$

If $K'\equiv 1$, we would have the case of no pumping to lower level and a pumping rate, R_a-R', to upper level. Now it has been stated that $K'>1$ before the peak value, this shows that the rise of the curve becomes comparatively slow for two mode operation, while $K'<1$ behind the peak value, thus the decrease of the curve also slows down, as shown in Fig. 5. Therefore the curve becomes a little flattened. This is consistent with single-mode situation.

C. Different pumping rates to the two lower levels $R_b>R_c$

Equations (36) and (37) now become

$$\begin{aligned}A_1p(n_1)=\ &C_1p(n_1+1)\left[1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}H_2(n_1)\langle n_2\rangle\right]+A_1'p(n_1+1)\\&-\tfrac{3}{4}A_2''p(n_1)\frac{\frac{B_1}{A_1}H_2'(n_1)\langle n_2\rangle}{1+\frac{1}{4}\frac{B_1}{A_1}(n_1+1)+\frac{1}{4}\frac{B_2}{A_2}H_2(n_1)\langle n_2\rangle}\\&+\tfrac{3}{4}A_1'p(n_1+1)\frac{\frac{B_2}{A_2}H_2'(n_1)\langle n_2\rangle}{1+\frac{1}{4}\frac{B_1}{A_1}(n_1+1)+\frac{1}{4}\frac{B_2}{A_2}H_2(n_1)\langle n_2\rangle}\ ,\end{aligned} \tag{69}$$

$$\begin{aligned}A_2p(n_2)=\ &C_2p(n_2+1)\left[1+\frac{B_1}{A_1}H_1(n_2)\langle n_1\rangle+\frac{B_2}{A_2}(n_2+1)\right]+A_2''p(n_2+1)\\&+\tfrac{3}{4}A_2''p(n_2+1)\frac{\frac{B_1}{A_1}H_1'(n_2)\langle n_1\rangle}{1+\frac{1}{4}\frac{B_1}{A_1}H_1(n_2)\langle n_1\rangle+\frac{1}{4}\frac{B_2}{A_2}(n_2+1)}\\&-\tfrac{3}{4}A_1'p(n_2)\frac{\frac{B_2}{A_2}H_1'(n_2)\langle n_1\rangle}{1+\frac{1}{4}\frac{B_1}{A_1}H_1(n_2)\langle n_1\rangle+\frac{1}{4}\frac{B_2}{A_2}(n_2+1)}\ .\end{aligned} \tag{70}$$

Here Eq. (42) has been used. $H_1'(n_2)$ and $H_2'(n_1)$ are parameters, too.

The peak position can be found to be

$$A_1=C_1\left[1+\frac{B_1}{A_1}\langle n_1\rangle+\frac{B_2}{A_2}H_2\langle n_2\rangle\right]+A_1'+\tfrac{3}{4}A_1'\frac{B_2}{A_2}\frac{H_2'\langle n_2\rangle(1-R_c/R_b)}{1+\frac{1}{4}\frac{B_1}{A_1}\langle n_1\rangle+\frac{1}{4}\frac{B_2}{A_2}H_2\langle n_2\rangle}\,, \tag{71}$$

$$A_2=C_2\left[1+\frac{B_1}{A_1}H_1\langle n_1\rangle+\frac{B_2}{A_2}\langle n_2\rangle\right]+A_2''-\tfrac{3}{4}A_1'\frac{B_2}{A_2}\frac{H_1'\langle n_1\rangle(1-R_c/R_b)}{1+\frac{1}{4}\frac{B_1}{A_1}H_1\langle n_1\rangle+\frac{1}{4}\frac{B_2}{A_2}\langle n_2\rangle}\,. \tag{72}$$

The last term on the right-hand side of Eq. (71) [and of Eq. (72)] originates from the two-photon processes between the two modes. One of them is negative, the other positive. (If $R_b=R_c$, they are all equal to zero.) The imbalance between the two two-photon processes gives rise to an additional gain to one mode while an additional loss to the other. If $R_b>R_c$, the curve of the output power of mode 1 versus pumping will bend down, owing to the decrease of the net gain, and the second will rise up. But the bending down of the first mode weakens the effect of the last term in (72). Therefore, the bending of the second curve is small. How far the bending will be, is related to the last terms in (71) and (72), i.e., related to A_1'/C. If this number is small, the contribution of the two-photon process can be neglected. When it reaches some value, their contribution can be remarkable. These conclusions are consistent with the experimental results of Ref. 4.

In Ref. 4 the lower level populations originating from thermal distribution can be regarded as a result of two processes: a pumping to it, plus a decay from it. When there is no lasing operation the number pumped to it is equal to the number decayed from it. If the decay constants are the same for all lower levels, and the populations of the lower levels obey Boltzmann's distribution, the origin of the populations of the lower levels can be attributed to different pumping rates with the lower level having a larger one. In Ref. 4, $R_1>R_2>R_3$. According to the earlier discussions, curve 1, having a larger pumping rate to its lower level, may appear as bending down.

Figure 6 shows the experimental results of Ref. 4 (Fig. 10 of Ref. 4). The order has been rearranged. The lengths of the crystals are same for each rank, which means they possess the same A_1', and the transmittances are same for each column, i.e., each set has the same C. The experiments are performed under the experimental conditions such that

$$A_1>A_2>A_3\ (g_1>g_2),\ A_1'>A_2'>A_3'\ .$$

(They originate from thermal distributions. The larger the crystal is, the larger A' will be. However, the ratio A_i'/A_j' has no relation to the length of the crystal.) Also, $C_1=C_2=C$. (They are determined by the transmittance.) The ratio A_1'/C increases from the top to the bottom and from right to left. Along the diagonal [Figs. 6(b), 6(c), and 6(d)] the ratio A_1'/C is the same for these three

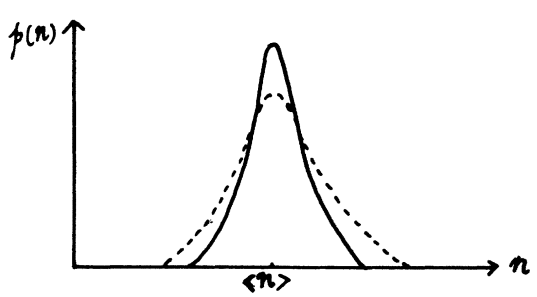


FIG. 4. The comparison of photon statistical distribution. Solid line: in the single-mode operation, dotted line: in the two-mode operation.

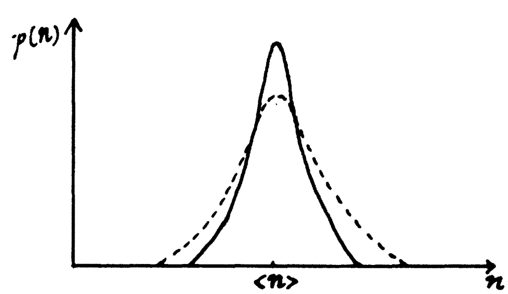
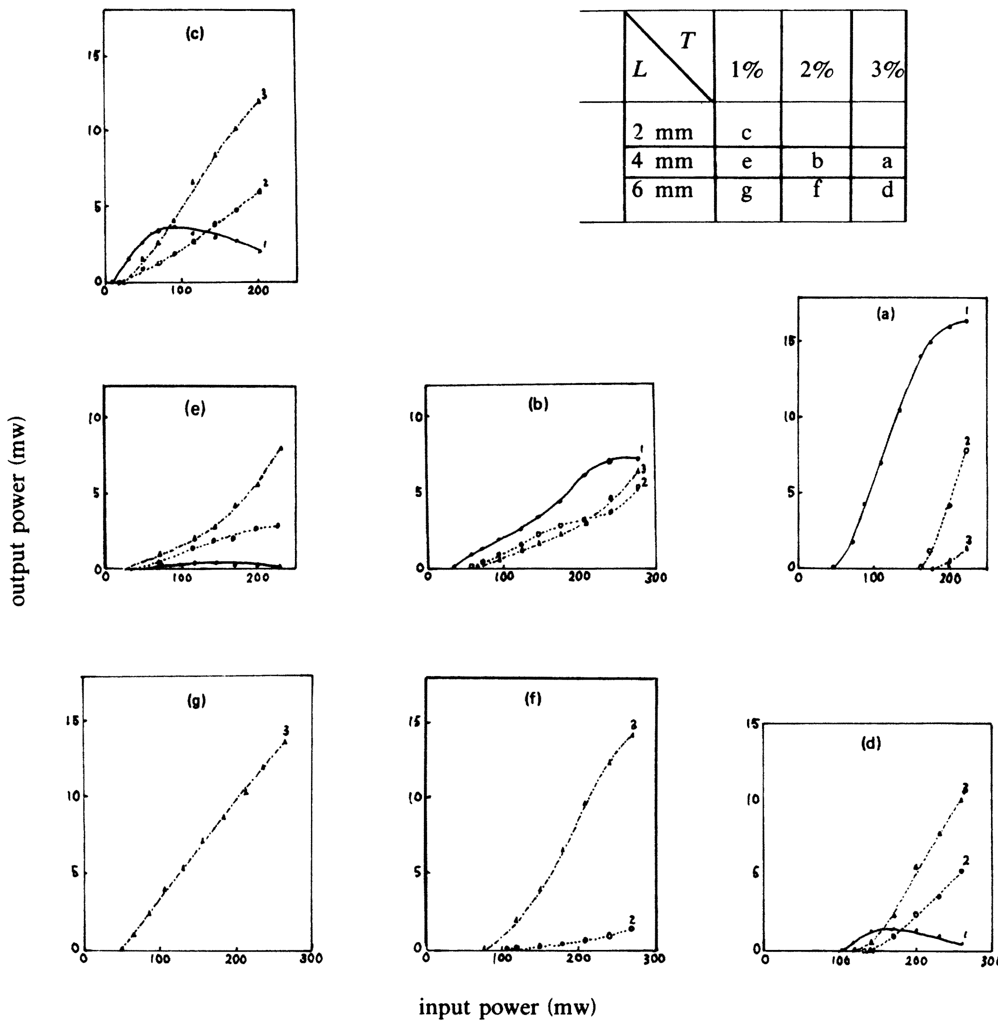


FIG. 5. The comparison of photon-statistical distribution. Solid line: with no pumping to the lower level, dotted line: with pumping to the lower level.



L \ T	1%	2%	3%
2 mm	c		
4 mm	e	b	a
6 mm	g	f	d

FIG. 6. LNP laser output powers as a function of pump power.

cases. They are of the same type and demonstrate similar properties: mode 1 oscillates first, then as the pumping increases, the curve of mode 1 shows a remarkable bending down. In Fig. 6(a), A_1'/C is comparatively small and the bending down does not appear. In Figs. 6(e), 6(f), and 6(g), A_1'/C is so large that modes 2 and 3 oscillate first and mode 1 is suppressed. [Compare Fig. 6(a) with Fig. 6(e).]

It would be beneficial to estimate the possibility that mode 1 oscillates first and then is suppressed as the pumping increases. In order to investigate the effect of the two-photon processes, we express the last term of Eq. (71) [or Eq. (72)] by an abbreviation $T_1(n_2)$ [or $T_2(n_1)$]. Evidently T is dependent on $\langle n \rangle$; $T(0)=0$ and $T(\infty)=T_0$, where T_0 is a definite number. Equations (71) and (72) may now be written as

$$\frac{\bar{A}_1}{C}=\frac{A_1-A_1'-T_1A_1'}{C}=1+\frac{B_1}{A_1}\langle n_1\rangle+\frac{B_2}{A_2}H_2\langle n_2\rangle \ , \tag{73}$$

$$\frac{\bar{A}_2}{C}=\frac{A_2-A_2''+T_2A_1'}{C}=1+\frac{B_1}{A_1}H_1\langle n_1\rangle+\frac{B_2}{A_2}\langle n_2\rangle . \tag{74}$$

They can be treated equivalently as the cases of no pumping to the lower levels and modified pumping rates $\bar{A}_1$ and $\bar{A}_2$ to the upper levels. Equations (52) can be used to estimate the operation properties of the laser. Mode 1 will not oscillate when the condition of Eq. (52) is satisfied, i.e., when

$$\frac{A_2-A_2''}{C}\geq\frac{2(A_1-A_1'-T_1A_1')}{C}-1\ , \tag{75}$$

where the assumption $T_2=0$ has been used. It is legal to set $T_2=0$, since we are now considering the case where mode 1 is suppressed. Considering the experimental situation in Ref. 4, it is adequate to put

$$A_1'=3C,\ A_2''=\tfrac{6}{5}C,\ A_1=2A_2\ . \tag{76}$$

As the pumping increases from zero, mode 1 oscillates first, then when the pumping rates reach $A_1=9C$ and $A_2=4.5C$, Eq. (75) is satisfied once $T_1\geq\frac{77}{66}=1.28$. In fact, T_1 can reach 2 with the numerical values of the parameters given by Eq. (76). Thus we see that there indeed exists the case in which mode 1 oscillate first and then is suppressed.

IV. CONCLUSION

Having obtained the master equation, an essential step is to construct a correct probability flow diagram to assist the analysis of the roles of different terms in the master equation. This is done in this paper. Each photon state (n_1,n_2) is related to its six neighboring states. The links between it and its neighboring states are the probability flows, of which each is connected with one kind of a real physical process. It is worthwhile to mention that there are two current flows that correspond to the two-photon process between two neighboring state having the same total number of photons. These two-photon processes cause one mode to decrease by one photon while the other increases by one. It is mentioned that these two-photon processes are responsible for the phenomena where in some cases a mode originally excited is eventually suppressed as pumping increases by the other mode that is excited sometime later.

All the above statements are discussed in a qualitative manner, since it seems momentarily difficult to solve the equation directly. However, it is believed that, having made some improvements to the present paper, a more detailed account should be possible.

APPENDIX A: MOTION EQUATION OF A THREE-LEVEL ATOM WITHOUT DECAY UNDER THE ACTION OF TWO RESONANT FIELDS

According to Eq. (2) in the text, the perturbation energy in the interaction picture becomes

$$V^I=g_1e^{-i(\Omega_1-\omega_1)t}a_1A_a^\dagger A_b+g_2e^{-i(\Omega_2-\omega_2)t}a_2A_a^\dagger A_c+\text{H.c.}\ , \tag{A1}$$

where $\omega_1=\omega_a-\omega_b$ and $\omega_2=\omega_a-\omega_c$. On resonance, $\omega_1=\Omega_1$ and $\omega_2=\Omega_2$. The state vector obeys the equation of motion

$$\frac{d}{dt}|\psi^I\rangle=-iV^I|\psi^I\rangle\ . \tag{A2}$$

Now we discuss its solution under different initial conditions.

(1) The atom initially in state $|a\rangle$. For an atom initially in the state $|a\rangle$, we have the initial condition

$$|\psi^I(0)\rangle=|a\rangle|n_1,n_2\rangle\ . \tag{A3}$$

Let

$$|\psi^I(t)\rangle=a_{n_1,n_2}(t)|a\rangle|n_1,n_2\rangle+b_{n_1+1,n_2}(t)|b\rangle|n_1+1,n_2\rangle+c_{n_1,n_2+1}(t)|c\rangle|n_1,n_2+1\rangle\ . \tag{A4}$$

Substituting Eqs. (A1) and (A4) into Eq. (A2) and by virtue of (A3), we obtain

$$a_{n_1,n_2}(t)=\cos\{[|g_1|^2(n_1+1)+|g_2|^2(n_2+1)]^{1/2}t\}\ , \tag{A5}$$

$$b_{n_1+1,n_2}(t)=\frac{-ig_1^*\sqrt{n_1+1}}{[|g_1|^2(n_1+1)+|g_2|^2(n_2+1)]^{1/2}}\sin\{[|g_1|^2(n_1+1)+|g_2|^2(n_2+1)]^{1/2}t\}\ , \tag{A6}$$

$$c_{n_1,n_2+1}(t)=\frac{-ig_2^*\sqrt{n_2+1}}{[|g_1|^2(n_1+1)+|g_2|^2(n_2+1)]^{1/2}}\sin\{[|g_1|^2(n_1+1)+|g_2|^2(n_2+1)]^{1/2}t\}\ . \tag{A7}$$

(2) The atom initially in state $|b\rangle$. Similarly we get

$$a_{n_1-1,n_2}(t)=\frac{-ig_1\sqrt{n_1}}{[\,|g_1|^2n_1+|g_2|^2(n_2+1)]^{1/2}}\sin\{[\,|g_1|^2n_1+|g_2|^2(n_2+1)]^{1/2}t\}\ , \tag{A8}$$

$$b_{n_1,n_2}(t)=\frac{1}{|g_1|^2n_1+|g_2|^2(n_2+1)}(\,|g_1|^2n_1\cos\{[\,|g_1|^2n_1+|g_2|^2(n_2+1)]^{1/2}t\}+|g_2|^2(n_2+1))\ , \tag{A9}$$

$$c_{n_1-1,n_2}(t)=\frac{g_1g_2^*\sqrt{n_1(n_2+1)}}{|g_1|^2n_1+|g_2|^2(n_2+1)}(\cos\{[\,|g_1|^2n_1+|g_2|^2(n_2+1)]^{1/2}t\}-1)\ . \tag{A10}$$

(3) The atom initially in $|c\rangle$. Likewise, we find

$$a_{n_1,n_2-1}(t)=\frac{-ig_2\sqrt{n_2}}{[\,|g_1|^2(n_1+1)+|g_2|^2n_2]^{1/2}}\sin\{[\,|g_1|^2(n_1+1)+|g_2|^2n_2]^{1/2}t\}\ , \tag{A11}$$

$$b_{n_1+1,n_2-1}(t)=\frac{g_1^*g_2\sqrt{n_2(n_1+1)}}{|g_1|^2(n_1+1)+|g_2|^2n_2}(\cos\{[\,|g_1|^2(n_1+1)+|g_2|^2n_2]^{1/2}t\}-1)\ , \tag{A12}$$

$$c_{n_1,n_2}(t)=\frac{1}{|g_1|^2(n_1+1)+|g_2|^2n_2}(\,|g_2|^2n_2\cos\{[\,|g_1|^2(n_1+1)+|g_2|^2n_2]^{1/2}t\}+|g_1|^2(n_1+1))\ . \tag{A13}$$

APPENDIX B: PROOF OF THE INEQUALITY $H>1$

For the single-mode laser in the two-level atomic system, we have[2,7]

$$\frac{p(n)}{1+\frac{B}{A}(n+1)}=\frac{C}{A}p(n+1)\ . \tag{B1}$$

Let

$$\sum_{n=0}^{\infty}\frac{p(n)}{1+\frac{B}{A}(n+1)}=\frac{1}{1+\frac{B}{A}H\langle n\rangle}\ . \tag{B2}$$

Applying Eqs. (B1) in (B2), we find

$$\frac{1}{1+\frac{B}{A}H\langle n\rangle}=\frac{C}{A}[1-p(0)]\ . \tag{B3}$$

The peak position of the photon-statistical distribution is determined by the condition $p(n)=p(n-1)$ so that

$$\frac{A}{C}=1+\frac{B}{A}\langle n\rangle\ . \tag{B4}$$

From Eqs. (B3) and (B4), we obtain

$$\frac{1}{1+\frac{B}{A}H\langle n\rangle}=\frac{1-p(0)}{1+\frac{B}{A}\langle n\rangle}\ . \tag{B5}$$

Since $p(0)>0$, it can be concluded the $H>1$ in the case of the single mode. In addition, the average photon number is[2]

$$\bar{n}=\sum_{n=0}^{\infty}np(n)=\langle n\rangle+\frac{A}{B}p(0)\ , \tag{B6}$$

in which a bar is used to denote average quantities,

$$\frac{B}{A}(\bar{n}-\langle n\rangle)=p(0)\ . \tag{B7}$$

Thus it can be seen that the center of gravity of the curve of the photon-statistical distribution (the first moment) shifts from the peak position towards the positive direction of the coordinate axis.

For two-mode lasers in the three-level atomic system, we have

$$\frac{A_1}{C_1}p(n_1)=\left[1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}H_2(n_1)\langle n_2\rangle\right]p(n_1+1)\ , \tag{B8}$$

$$\sum_{n_1=0}^{\infty} \frac{A_1}{C_1} p(n_1) = \sum_{n_1=0}^{\infty} \frac{B_1}{A_1}(n_1+1)p(n_1+1) + \sum_{n_1=0}^{\infty} \left[1+\frac{B_2}{A_2}H_2(n_1)\langle n_2\rangle\right] p(n_1+1) , \tag{B9}$$

$$\frac{A_1}{C_1} = \frac{B_1}{A_1}\bar{n}_1 + \left[1+\frac{B_2}{A_2}H_2\langle n_2\rangle\right][1-p(n_1=0)] , \tag{B10}$$

where the approximation $H_2(n_1)=H_2$ is made. As already stated, the change of $H_2(n_1)$ with n_1 is very small. Equation (43) in the text can be rewritten as

$$\frac{B_1}{A_1}\langle n_1\rangle = \frac{A_1}{C_1} - 1 - \frac{B_2}{A_2}H_2\langle n_2\rangle . \tag{B11}$$

From Eqs. (B10) and (B11) we obtain

$$\frac{B_1}{A_1}(\bar{n}_1 - \langle n_1\rangle) = \left[1+\frac{B_2}{A_2}H_2\langle n_2\rangle\right] p(n_1=0) . \tag{B12}$$

Comparing Eq. (B12) with Eq. (B7), it can be shown that the center of gravity of the curve of the photon-statistical distribution shifts even more from the peak position towards the positive direction of the coordinate axis. By inspection of Eq. (42) we can conclude that $H>1$ is also valid in two-mode lasers when both modes are above threshold. Furthermore, it can be shown from Eq. (B12) that even if the peak position of mode 1 is small, the average photon number of mode 1 can still take a considerable value as long as mode 2 is strong enough.

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