PHYSICAL REVIEW A VOLUME 25, NUMBER 6 JUNE 1982

Effect of fluctuating space-charge fields on sideband instabilities in free-electron lasers

H. P. Freund,* P. Sprangle, and C. M. Tang
Naval Research Laboratory, Washington, D.C. 20375
(Received 14 August 1981)

We consider the sideband instabilities that can occur in free-electron lasers which operate in the regime where the bulk of the energetic electron beam is trapped in the ponderomotive potential generated by the beating of the electromagnetic pulse with the static wiggler field. The general dynamical equations which govern the evolution of the system in the presence of sideband modes are derived for modes propagating along the axis of symmetry. All space-charge effects are included in a self-consistent manner, and the gains for both the primary signal and the upper and lower sidebands are computed. We find the lower sideband to be excited, and discuss the consequences for operation of both free-electron-laser amplifiers and oscillators. The effect of the space-charge waves on the gain depends upon the number of bounces undergone by the trapped electrons in the length of the system, and can act either to enhance or reduce the gain of the sideband.

I. INTRODUCTION

Several recently proposed free-electron-laser concepts[1–4] involve sufficiently large amplitude signal strengths that electron trapping in the ponderomotive wave, which results from the beating of the electromagnetic pulse and the static helical magnetic pump field, can have a significant effect on the operation of the device. Trapped electrons will execute periodic oscillations in the pondermotive potential which can give rise to excitation of sidebands of the primary electromagnetic spectrum at harmonics of the electron bounce frequency. The trapped electron bounce frequency is anharmonic, and depends upon the amplitude of both the static magnetic field and the radiation spectrum. Therefore, system response may be quite sensitive to the radiation amplitude.

Our purpose in this work is to examine the gain of the sideband modes which result from the trapped electron motion under a variety of operating conditions, and subject to the inclusion of a fluctuating space-charge field. In this sense our work represents an extension over a previous analysis of sideband instabilities by Kroll and Rosenbluth,[5] in which the space-charge effect was ignored. We find, typically, that the space-charge effect can be important when the beam density is sufficiently high that the invariant plasma frequency is comparable to the bounce frequency of the energetic electrons in the ponderomotive potential.

The organization of this paper is as follows. In Sec. II, we derive the general dynamical equations which govern the evolution of the primary and sideband signals in a system in which an energetic electron beam is propagating along the axis of a static helical magnetic field. The electron beam is assumed to be cold, and electrostatic (i.e., space-charge) effects are included in a self-consistent manner. The equation governing the electron trajectories is derived in Sec. III, and solved for the case in which both space-charge contributions and a tapered wiggler (i.e., helical pump) field are included. Anharmonic effects on the bounce motion of electrons in the ponderomotive potential are included to lowest order. The gain for both the primary and sideband signals are computed in Sec. IV. A summary and discussion is provided in Sec. V, and the consequences for the operation of both free-electron-laser amplifiers and oscillators are discussed.

II. GENERAL EQUATIONS

The physical configuration we study consists of a relativistic electron beam interacting with a static, helical magnetic field which can be represented by the vector potential

$$\vec{A}_w(z) = -A_w(z)\left[\hat{e}_x \cos\left[\int_0^z dz' k_w\right] + \hat{e}_y \sin\left[\int_0^z dz' k_w\right]\right], \tag{1}$$

as well as scattered electromagnetic and electrostatic fields of the form[6]

$$\vec{A}(z,t)=A(z)\left[\hat{e}_x\cos\left[\int_0^z dz'k-\omega t\right]-\hat{e}_y\sin\left[\int_0^z dz'k-\omega t\right]\right],$$
$$\vec{A}_s(z,t)=A_s(z)\left[\hat{e}_x\cos\left[\int_0^z dz'k_s-\omega_s t+\theta\right]-\hat{e}_y\sin\left[\int_0^z dz'k_s-\omega_s t+\theta\right]\right], \tag{2}$$

and

$$\Phi(z,t)=\Phi(z)\cos\left[\int_0^z dz'\kappa-\omega t+\alpha\right],$$
$$\Phi_s(z,t)=\Phi_s(z)\cos\left[\int_0^z dz'\kappa_s-\omega_s t+\alpha_s\right], \tag{3}$$

where the amplitudes of the scalar and vector potentials $\Phi(z)$, $\Phi_s(z)$, $A_w(z)$, $A(z)$, and $A_s(z)$ and the wave vectors of these potentials $k_w(z)$, $k(z)$, $k_s(z)$, $\kappa(z)$, and $\kappa_s(z)$ are assumed to be slowly varying functions of z. The frequencies ω and ω_s as well as the relative phases θ, α, and α_s of the wave fields are taken to be independent of z. It should be noted that our assumption that the spatial variation of the fields depends solely on z is valid, strictly speaking, only near the axis of a realizable free-electron laser. In the interests of computational simplicity, we choose to write the electrostatic fields in the alternate form

$$\Phi(z,t)=\Phi_1(z)\cos\psi(z,t)+\Phi_2(z)\sin\psi(z,t),$$
$$\Phi_s(z,t)=\Phi_{s1}(z)\cos\psi_s(z,t)+\Phi_{s2}(z)\sin\psi_s(z,t), \tag{4}$$

where

$$\psi(z,t)\equiv\int_0^z dz'(k_w+k)-\omega t$$

and

$$\psi_s(z,t)\equiv\int_0^z dz'(k_w+k_s)-\omega_s t+\theta .$$

In this form, $\vec{A}$ and Φ represent the primary radiation spectrum which may result either from coherent Raman scattering of low-frequency noise on the electron beam or from a coherent external source of radiation. For a sufficiently large amplitude primary spectrum, electron trapping in the ponderomotive wave produced by the beating of the static pump and primary radiation fields can give rise to the excitation of sidebands of the primary spectrum at harmonics of the electron bounce frequency. As a result, in the subsequent analysis we shall assume that $|\omega_s-\omega|\ll\omega$ and adopt the ordering $|A_s|\ll|A|\ll|A_w|$.

The fields satisfy Maxwell's equations

$$(\partial_t^2-c^2\partial_z^2)\vec{A}_T=4\pi c\vec{J}_\perp$$

and

$$\partial_z\partial_t\Phi_T=4\pi J_z ,$$

where $\vec{A}_T$ and Φ_T represent the total vector and scalar potentials of the wave fields, and $\vec{J}(z,t)$ denotes the driving current. After substituting the potentials from Eqs. (2) and (4) and averaging over the long-time scale $T=2\pi/\delta\omega$, where $\delta\omega\equiv|\omega-\omega_s|$, we obtain

$$(\omega^2-c^2k^2)A=-2c\delta\omega\int_{-T/2}^{T/2}dt\left[J_x\cos\left[\int_0^z dz'k-\omega t\right]-J_y\sin\left[\int_0^z dz'k-\omega t\right]\right],$$
$$(\omega_s^2-c^2k_s^2)A_s=-2c\delta\omega\int_{-T/2}^{T/2}dt\left[J_x\cos\left[\int_0^z dz'k_s-\omega_s t+\theta\right]-J_y\sin\left[\int_0^z dz'k_s-\omega_s t+\theta\right]\right], \tag{5}$$

$$2k^{1/2}\partial_z(k^{1/2}A)=\frac{2\delta\omega}{c}\int_{-T/2}^{T/2}dt\left[J_x\sin\left[\int_0^z dz'k-\omega t\right]+J_y\cos\left[\int_0^z dz'k-\omega t\right]\right],$$
$$2k_s^{1/2}\partial_z(k_s^{1/2}A_s)=\frac{2\delta\omega}{c}\int_{-T/2}^{T/2}dt\left[J_x\sin\left[\int_0^z dz'k_s-\omega_s t+\theta\right]+J_y\cos\left[\int_0^z dz'k_s-\omega_s t+\theta\right]\right], \tag{6}$$

and

$$(k_w+k)\begin{bmatrix}\Phi_1\\ \Phi_2\end{bmatrix}=4\frac{\delta\omega}{\omega}\int_{-T/2}^{T/2}dt\,J_z\begin{bmatrix}\cos\psi\\ \sin\psi\end{bmatrix}, \tag{7}$$

$$(k_w+k_s)\begin{bmatrix}\Phi_{s1}\\ \Phi_{s2}\end{bmatrix}=\frac{\delta\omega}{\omega_s}\int_{-T/2}^{T/2}dt\,J_z\begin{bmatrix}\cos\psi_s\\ \sin\psi_s\end{bmatrix}. \tag{8}$$

In the above, it has been assumed that the frequency shift between the primary and sideband fields satisfies the conditions $2\pi/\delta\omega=2\pi N/\omega=2\pi(N+m)/\omega_s$, where N and m are integers ($N>>1$ and $N>>m$), and that the gradients of the scalar potentials may be ignored.

The nonlinear driving current is derived under the assumption that the initial velocities of all the electrons are identical, and that only the initial phases vary. Therefore, the electron distribution can be written in the form[7]

$$f(z,\vec{p},t)=n_b v_{z0}\int_{-\infty}^{\infty}dt_0\,W(t_0)\delta[z-\xi(t_0,t)]\delta[\vec{p}-\vec{\eta}(t_0,t)], \tag{9}$$

where $W(t_0)$ is a weighting function which describes the distribution of initial phases, n_b and v_{z0} are the initial beam density and axial velocity, $\xi(t_0,t)$ and $\vec{\eta}(t_0,t)$ are the axial position and momentum of a particle at time t which entered the interaction region (i.e., $z\geq 0$) at time t_0. The driving current associated with such a distribution is

$$\vec{J}(z,t)=-en_b v_{z0}\int_{-\infty}^{\infty}dt_0\,W(t_0)\vec{\eta}(t_0,z)\frac{\delta[t-\tau(t_0,z)]}{\eta_z(t_0,z)}, \tag{10}$$

where $\tau(t_0,z)\equiv t_0+\int_0^z dz'/v_z(t_0,z')$ is the time required for a particle which entered the interaction region at time t_0 to travel an axial distance z, and $v_z(t_0,z)$ is the axial velocity of a particle at position z which entered the interaction region at time t_0.

If the beam electrons initially have negligible transverse momentum, then we may approximate

$$\eta_x\simeq-(eA_w/c)\cos\left[\int_0^z dz'k_w(z')\right],$$

$$\eta_y\simeq-(eA_w/c)\sin\left[\int_0^z dz'k_w(z')\right],$$

and the nonlinear current

$$\vec{J}(z,t)\simeq en_b v_{z0}\left[\frac{e}{c}\vec{A}_w(z)\int_{-\infty}^{\infty}dt_0\,W(t_0)\frac{\delta[t-\tau(t_0,z)]}{\eta_z(t_0,z)}-\hat{e}_z\int_{-\infty}^{\infty}dt_0\,W(t_0)\delta[t-\tau(t_0,z)]\right]. \tag{11}$$

Substitution of (11) into Maxwell's equations (5)–(8) yields the following set of equations:

$$(\omega^2-c^2k^2)A=-\frac{\omega_b^2 v_{z0}A_w}{2\pi N}\int_{-\pi N}^{\pi N}d\psi_0\frac{\cos\psi(\psi_0,z)W(\psi_0)}{\gamma(\psi_0,z)v_z(\psi_0,z)},$$

$$(\omega_s^2-c^2k_s^2)A_s=-\frac{\omega_b^2 v_{z0}A_w}{2\pi N}\int_{-\pi N}^{\pi N}d\psi_0\frac{\cos\psi_s(\psi_0,z)W(\psi_0)}{\gamma(\psi_0,z)v_z(\psi_0,z)}, \tag{12}$$

$$2k^{1/2}\partial_z(k^{1/2}A)=\frac{\omega_b^2 v_{z0}A_w}{2\pi Nc^2}\int_{-\pi N}^{\pi N}d\psi_0\frac{\sin\psi(\psi_0,z)W(\psi_0)}{\gamma(\psi_0,z)v_z(\psi_0,z)},$$

$$2k_s^{1/2}\partial_z(k_s^{1/2}A_s)=\frac{\omega_b^2 v_{z0}A_w}{2\pi Nc^2}\int_{-\pi N}^{\pi N}d\psi_0\frac{\sin\psi_s(\psi_0,z)W(\psi_0)}{\gamma(\psi_0,z)v_z(\psi_0,z)}, \tag{13}$$

and

$$(k_w+k)\begin{bmatrix}\Phi_1\\ \Phi_2\end{bmatrix}=-\frac{4en_b v_{z0}}{N\omega}\int_{-\pi N}^{\pi N}d\psi_0\,W(\psi_0)\begin{bmatrix}\cos\psi(\psi_0,z)\\ \sin\psi(\psi_0,z)\end{bmatrix}, \tag{14}$$

$$(k_w+k_s)\begin{bmatrix}\Phi_{s1}\\ \Phi_{s2}\end{bmatrix}=-\frac{4en_b v_{z0}}{N\omega_s}\int_{-\pi N}^{\pi N}d\psi\, W(\psi_0)\begin{bmatrix}\cos\psi_s(\psi_0,z)\\ \sin\psi_s(\psi_0,z)\end{bmatrix}, \tag{15}$$

where $\psi_0\equiv-\omega t_0$ is the initial phase, $\omega_b^2\equiv 4\pi e^2 n_b/m$,

$$\psi(\psi_0,z)=\psi_0+\int_0^z dz'\left[k_w+k-\frac{\omega}{v_z(\psi_0,z')}\right] \tag{16}$$

and

$$\psi_s(\psi_0,z)=\frac{\omega_s}{\omega}\psi(\psi_0,z)+\left[\frac{\omega-\omega_s}{\omega}\right]\int_0^z dz' k_w+\omega_s\int_0^z dz'\left[\frac{k_s}{\omega_s}-\frac{k}{\omega}\right]+\theta\ . \tag{17}$$

The problem, therefore, reduces to solution of the axial orbit equation for $v_z(\psi_0,z)$ or, alternately, for $\psi(\psi_0,z)$.

It will be evident in Sec. III that the equation describing the evolution of ψ is invariant under the transformation $\psi\to\psi+2\pi,\theta\to\theta+2\pi(\omega-\omega_s)/\omega$. As a consequence, we have the symmetry

$$\psi(\psi_0+2\pi,\theta+2\pi(\omega-\omega_s)/\omega,z)=\psi(\psi_0,\theta,z)+2\pi\ , \tag{18}$$

which is also possessed by ψ_s. If it is also required that $W(\psi_0+2\pi)=W(\psi_0)$, then the integrals in Eqs. (12)–(15) are of the general form $\int_{-\pi N}^{\pi N}d\psi_0 F(\psi_0,\theta,z)$, where $F(\psi_0,\theta,z)$ satisfies (18). The decomposition of these source integrals into a sum of N integrals over intervals of 2π is possible, and we find that for large N

$$\frac{1}{2\pi N}\int_{-\pi N}^{\pi N}d\psi_0 F(\psi_0,\theta,z)=\frac{1}{(2\pi)^2}\int_0^{2\pi}d\theta\int_{-\pi}^{\pi}d\psi_0 F(\psi_0,\theta,z)\ .$$

Thus, for sufficiently large N (i.e., for sufficiently small $\delta\omega/\omega$) Eqs. (12)–(15) can be written as

$$(\omega^2-c^2k^2)a=-\omega_b^2 v_{z0}a_w\langle\cos\psi/\gamma v_z\rangle\ ,$$
$$(\omega_s^2-c^2k_s^2)a_s=-\omega_b^2 v_{z0}a_w\langle\cos\psi_s/\gamma v_z\rangle\ , \tag{19}$$

$$2k^{1/2}\partial_z(k^{1/2}a)=\omega_b^2\frac{v_{z0}}{c}a_w\langle\sin\psi/\gamma v_z\rangle\ ,$$
$$2k_s^{1/2}\partial_z(k_s^{1/2}a_s)=\omega_b^2\frac{v_{z0}}{c^2}a_w\langle\sin\psi_s/\gamma v_z\rangle\ , \tag{20}$$

and

$$\varphi_1=-\frac{2\omega_b^2 v_{z0}}{\omega c^2(k_w+k)}\langle\cos\psi\rangle\ ,$$
$$\varphi_2=-\frac{2\omega_b^2 v_{z0}}{\omega c^2(k_w+k)}\langle\sin\psi\rangle\ , \tag{21}$$

$$\varphi_{s1}=-\frac{2\omega_b^2 v_{z0}}{\omega_s c^2(k_w+k_s)}\langle\cos\psi_s\rangle\ ,$$
$$\varphi_{s2}=-\frac{2\omega_b^2 v_{z0}}{\omega_s c^2(k_w+k_s)}\langle\sin\psi_s\rangle\ , \tag{22}$$

where the averaging operation is defined by

$$\langle\cdots\rangle\equiv(2\pi)^{-2}\int_0^{2\pi}d\theta\int_{-\pi}^{\pi}d\psi_0 W(\psi_0)\ ,$$

and we have defined the a's and φ's as the normalized amplitudes $a\equiv eA/mc^2$ and $\varphi\equiv e\Phi/mc^2$.

III. AXIAL ELECTRON TRAJECTORIES

Since the vector and scalar potentials are independent of the transverse coordinates, these coordinates are cyclic and the corresponding canonical momenta are constants of the motion. In view of this, the axial momentum equation takes the form

$$\frac{d}{dt}p_z=-mc^2\left[\frac{\partial}{\partial z}\psi(\varphi_1\sin\psi-\varphi_2\cos\psi)+\frac{\partial}{\partial z}\psi_s(\varphi_{s1}\sin\psi_s-\varphi_{s2}\cos\psi_s)\right]$$
$$-\frac{mc^2}{2\gamma}\frac{d}{dz}\mu^2-\frac{mc^2}{\gamma}[(k_w+k)a_w a\sin\psi+(k_w+k_s)a_w a_s\sin\psi_s]\ , \tag{23}$$

where $\mu^2 \equiv 1+a_w^2$, and terms of second order in the wave fields and in the gradient of the amplitudes of the wave fields have been neglected. The variation in total electron energy is given by

$$\frac{d}{dt}\gamma = -v_z\left[\frac{\partial}{\partial z}\psi(\varphi_1 \sin\psi - \varphi_2\cos\psi) + \frac{\partial}{\partial z}\psi_s(\varphi_{s1}\sin\psi_s - \varphi_{s2}\cos\psi_s)\right] - \frac{1}{\gamma}(\omega a_w a \sin\psi + \omega_s a_w a_s \sin\psi_s)\ . \tag{24}$$

Combination of (23) and (24) yields

$$\begin{aligned}\frac{d}{dt}v_z = &-\frac{c^2\mu^2}{\gamma^3}[(k+k_w)(\varphi_1\sin\psi - \varphi_2\cos\psi) + (k_s+k_w)(\varphi_{s1}\sin\psi_s - \varphi_{s2}\cos\psi_s)]\\ &-\frac{c^2}{2\gamma^2}\frac{d}{dz}\mu^2 - \frac{c^2}{\gamma^2}\left[\left[k_w + k - \frac{\omega v_z}{c^2}\right]a_w a \sin\psi + \left[k_w + k_s - \frac{\omega_s v_z}{c^2}\right]a_w a_s \sin\psi_s\right]\ ,\end{aligned} \tag{25}$$

where terms in the product of $(\omega-\omega_s)/\omega$ and either ψ_{s1} or ψ_{s2} have been ignored.

Substitution of the axial position for the time as the independent variable and the phase for the axial velocity can be made by noting that $d/dt = v_z d/dz$ and $d\psi/dz = k_w + k - \omega/v_z$. Therefore, Eq. (25) can be cast into the form

$$\begin{aligned}\frac{d^2}{dz^2}\psi \simeq &\frac{d}{dz}(k_w+k) - \frac{(k_w+k)c^2}{2\gamma_r^2 v_r^2}\frac{d}{dz}\mu^2 - \frac{\mu^2 c^2 (k_w+k)^2}{\gamma_r^4 v_r^2}a_w a\left[\sin\psi + \epsilon\frac{\omega_s}{\omega}\sin\psi_s\right]\\ &+\frac{2\omega_b^2\mu^2 v_{z0}}{\gamma_r^3 v_r^3}\left[\sin\psi\langle\cos\psi\rangle - \cos\psi\langle\sin\psi\rangle + \frac{\omega}{\omega_s}(\sin\psi_s\langle\cos\psi_s\rangle - \cos\psi_s\langle\sin\psi_s\rangle)\right]\ ,\end{aligned} \tag{26}$$

where $\epsilon \equiv a_s/a$, $v_r \equiv \omega/(k_w+k)$ is the velocity of the ponderomotive wave so that $\gamma_r \equiv \mu(1-v_r^2/c^2)^{-1/2}$ is the relativistic factor corresponding to the ponderomotive frame. In the derivation of (26) terms of order $(k_w+k)^{-1}d\psi/dz$ have been ignored. Equation (26) reduces to that found by Kroll and Rosenbluth[5] in the limit in which space-charge effects can be ignored, i.e., $\omega_b^2 << 2\gamma_{z0}^4 a_w a k_w^2 c^2/\gamma_r$, where $\gamma_{z0} \equiv (1-v_{z0}^2/c^2)^{-1/2}$.

We include the space-charge effect for a diffuse beam limit in which we may approximate $\omega \simeq ck$ and $\omega_s \simeq ck_s$. Therefore, under the assumptions that $k_w << k, k_s$ and $dk/kz \simeq 0$ we find

$$\begin{aligned}\frac{d^2}{dz^2}\psi \simeq &-\frac{K^2}{\cos\psi_r}\left[\sin\psi - \sin\psi_r + \epsilon\frac{\omega_s}{\omega}\sin\psi_s\right]\\ &+\delta K^2\left[\sin\psi\langle\cos\psi\rangle - \cos\psi\langle\sin\psi\rangle + \frac{\omega_s}{\omega}\sin\psi_s\langle\cos\psi_s\rangle - \frac{\omega_s}{\omega}\cos\psi_s\langle\sin\psi_s\rangle\right]\ ,\end{aligned} \tag{27}$$

where $K^2 \equiv 4k_w^2 a_w a \cos\psi_r/\mu^2$ measures the strength of the ponderomotive potential, $\delta K^2 \equiv 2\mu^2\omega_b^2/\gamma_0^3 c^2$ measures the electrostatic potential, $\psi_s \simeq (\omega_s/\omega)\psi + \Delta kz + \theta$, $\Delta k \equiv k_w(\omega-\omega_s)/\omega$, and

$$\sin\psi_r \equiv \frac{-1}{2a_w a}\left[\frac{a_w}{k_w}\frac{d}{dz}a_w - \frac{\mu^2}{2k_w^2}\frac{d}{dz}k_w\right] \tag{28}$$

describes the effects of the gradients in the wiggler amplitude and period (hence, the acceleration or deceleration of the ponderomotive frame). Solution to (27) is found by expansion of ψ about the resonant phase, and we assume $\psi = \psi_r + \delta\psi$, where $|\delta\psi| << \pi$. As a consequence,

$$\frac{d^2}{dz^2}\delta\psi \simeq -K^2\delta\psi + \frac{1}{2}K^2\tan\psi_r\delta\psi^2 + \frac{1}{6}K^2\delta\psi^3 - \frac{\epsilon K^2}{\cos\psi_r}\sin\left[\frac{\omega_s}{\omega}\psi_r + \Delta kz + \theta\right] + \delta K^2(\delta\psi - \langle\delta\psi\rangle)\ . \tag{29}$$

Observe that upon linearization of (27) $2\pi/K$ is the bounce period in the ponderomotive potential, and

$2\pi/\delta K$ is the oscillation period (i.e., the invariant plasma period) in the space-charge wave.

If we write $\delta\psi=\delta\psi^{(0)}+\delta\psi^{(1)}$, where $\delta\psi^{(0)}$ and $\delta\psi^{(1)}$ denote the contribution to the phase to zeroth and first order in the effects due to the sideband modes and the space-charge fluctuations, then it follows that

$$\delta\psi^{(0)}\simeq\frac{\alpha^2}{4}\tan\psi_r+\alpha\cos K_a z-\frac{\alpha^2}{12}\tan\psi_r\cos 2K_a z \tag{30}$$

for small deviations from ψ_r. In Eq. (30)

$$K_a\equiv K-\frac{\alpha^2 K}{16}(1+\tfrac{5}{3}\tan^2\psi_r) \tag{31}$$

is the anharmonic bounce period in the ponderomotive wells, and α is a constant fixed by the initial condition $\psi^{(0)}(z=0)=\psi_0$ which implies that $\psi_0-\psi_r=\alpha+\alpha^2\tan\psi_r/6$ for $\alpha<<1$. The first-order correction is given by

$$\begin{aligned}\delta\psi^{(1)}\simeq{}&\frac{\epsilon}{2\cos\psi_r}\frac{K}{\Delta K_\pm}\left[1\pm\frac{\alpha^2K}{16\Delta K_\pm}(1+\tfrac{5}{3}\tan^2\psi_r)\right]\\&\times\left[\left[1\pm\frac{\alpha^2K}{8\Delta K_\pm}(1+\tfrac{5}{3}\tan^2\psi_r)\right][\cos(\Delta K_\pm z+\phi_r)-\cos\phi_r]\sin Kz\right.\\&\qquad\left.\mp[\sin(\Delta K_\pm z+\phi_r)-\sin\phi_r]\cos Kz\pm\frac{\alpha^2K}{8\Delta K_\pm}(1+\tfrac{5}{3}\tan^2\psi_r)Kz\sin\phi_r\sin Kz\right]\\&+\frac{\delta K^2}{2K^2}(\alpha-\langle\alpha\rangle)Kz\sin Kz\ ,\end{aligned} \tag{32}$$

where $\phi_r\equiv\omega_s\psi_r/\omega+\theta$, $\Delta K_\pm\equiv\Delta k\pm K$, and $\delta\psi^{(1)}(z=0)=\partial_z\delta\psi^{(1)}(z=0)=0$. It is important to recognize that this solution requires $\alpha^2<\Delta K_\pm/K$, which implies that all electrons are deeply trapped and have phases close to ψ_r.

IV. SMALL-SIGNAL GAIN

The gain per pass of the primary and sideband waves in a system of length L follows from (20):

$$g\simeq\frac{\omega_b^2}{2\omega c\gamma_0}\frac{a_w}{a}\int_0^L dz\langle\sin\psi\rangle \tag{33}$$

and

$$g_s\simeq\frac{\omega_b^2}{2\omega_s c\gamma_0}\frac{a_w}{a}\int_0^L dz\langle\sin\psi_s\rangle\ , \tag{34}$$

where the radiation wave vectors, electron energy, and wave amplitudes have been assumed to be slowly varying functions of z. These expressions may be readily evaluated using the solutions for the phase described in (30) and (32).

In the case of the gain per pass of the primary wave, we have to lowest nontrivial order

$$\langle\sin\psi\rangle\simeq\sin\psi_r(1-\tfrac{1}{2}\langle\delta\psi^{(0)2}\rangle-\langle\psi^{(1)}\delta\psi^{(0)}\rangle)\ .$$

Because of the average over the relative phase between the primary and sideband signals, this implies that only those components of $\psi^{(1)}$ due to the space-charge potential can contribute to g to this order. Specifically, the coupling between the primary and sideband waves cannot affect the gain of the primary wave to first order in ϵ. In addition, the space-charge effect is manifested by means of a beating between the bounce motion of electrons in the ponderomotive potential and the space-charge wave itself. In evaluating these averages, we shall assume a distribution of initial phases of the form

$$W(\psi_0)=\frac{\pi}{\Delta}H(\psi_0-\psi_r+\Delta)H(\psi_r+\Delta-\psi_0)\ , \tag{35}$$

in the range $-\pi\leq\psi_0\leq\pi$, where $\Delta<<1$ measures the spread in phases, and H is the Heaviside function. This distribution describes a tight, uniform bunching of electrons about the resonant phase and is depicted schematically in Fig. 1. As a consequence, g can be written in the form

$$g\simeq\frac{\omega_b^2L}{2\omega\gamma_0c}\frac{a_w}{a}\sin\psi_r\left\{1+\frac{\Delta^2}{12}\left[\frac{\sin KL}{KL}+\frac{2\sin 2KL}{3KL}+\frac{\delta K^2}{2K^2}\left[\cos 2KL-\frac{\sin 2KL}{2KL}\right]\right]\right\}\ , \tag{36}$$

which vanishes when $\sin\psi_r=0$ (i.e., for untapered wigglers). Evidently, the space-charge contribution to the gain vanishes in the limit in which $\Delta\rightarrow 0$. This conclusion is quite general and not dependent on the particular choice of $W(\psi_0)$. It can be shown from (32) that the contribution to the gain from the space-charge potential must vanish whenever the spread in ψ_0 vanishes. Finally, we observe that the space-charge tends to reduce the gain whenever $\sin 2KL>2KL\cos 2KL$. This inequality holds for $KL<\pi/4$ (when the bounce length of electrons in the ponderomotive well is longer than the system) but not in general and, for proper choices of the parameters, the space-charge effect could be made to enhance the gain.

In computing the gain of the sideband signal, we note that $\langle\sin\psi_s\rangle\simeq(\omega_s/\omega)\langle\psi^{(1)}\cos\psi_s^{(0)}\rangle$ and the space-charge effect does not contribute. Gain, therefore, arises because the presence of the sideband induces an oscillation in the electron bounce motion. We find that

$$g_s\simeq\mp\frac{\omega_b^2KL^2}{16\omega_s\gamma_0c\cos\psi_r}\frac{a_w}{a}\left[1\pm\frac{K\Delta^2}{48\Delta K_\pm}(1+\tfrac{5}{3}\tan^2\psi_r)\mp\frac{KL\Delta^2}{96}(1+\tfrac{5}{3}\tan^2\psi_r)\frac{d}{dx_\pm}\right]\left[\frac{\sin x_\pm}{x_\pm}\right]^2\ , \tag{37}$$

where $x_\pm\equiv\Delta K_\pm L/2$, and it is the lower sideband which is excited (i.e., $\Delta K\simeq K$).

The preceding applies to operation of a free-electron-laser amplifier. However, when operation is in the oscillator mode it is the relative gain $G(\equiv g_s/g)$ which is the significant quantity in determining whether the sideband signal will grow from noise. The close frequencies of these modes ensure nearly equal loss rates; hence, the relative gain must be near unity in the steady state and greater than unity if the sideband is expected to grow from noise. Comparison of Eqs. (36) and (37) shows that since $K>|\Delta K_\pm|$ the space-charge contribution to the relative gain will be important only when $\delta K>K$, and in this case

$$G\simeq\mp\frac{\omega KL}{4\omega_s\sin 2\psi_r}\left\{1\pm\frac{\Delta^2}{24}\left[\frac{K}{2\Delta K_\pm}(1+\tfrac{5}{3}\tan^2\psi_r)\mp\frac{\delta K^2}{K^2}\left[\cos 2KL-\frac{\sin 2KL}{2KL}\right]\right]\right\}\left[\frac{\sin x_\pm}{x_\pm}\right]^2\ , \tag{38}$$

and the effect of the space-charge field is to enhance the relative gain of the lower sideband when $KL<\pi/4$. Note that the only constraint on the magnitude of δK imposed by the analysis is that $\Delta^2\delta K^2<K^2$.

V. SUMMARY AND DISCUSSION

In this paper, we have investigated the question of free-electron-laser operation in the trapped particle regime from the standpoint of the excitation of sidebands shifted from the primary spectrum by the bounce frequency of the electrons trapped in the ponderomotive potential. The analysis described the interaction of a bunched cold electron beam with the static wiggler, the radiation fields of the primary and sideband modes, and with fluctuating space-charge fields. Gradients in the wiggler field have been included as well. Further, in order to make the problem analytically tractable, we have included the anharmonic contributions to the trapped electron trajectories only to lowest order. This necessitates the assumption of tightly bunched electrons about the resonant phase.

It was found in the computation of the gain per pass that the coupling of the electron orbits to the fields of the sideband modes had no effect on the gain of the primary spectrum to lowest order in a_s. It should be remarked here that the space-charge components of the sideband waves were found to be unimportant to the gain of either radiation mode for the parameter regimes under consideration. However, the space-charge fluctuations which couple to the primary radiation spectrum

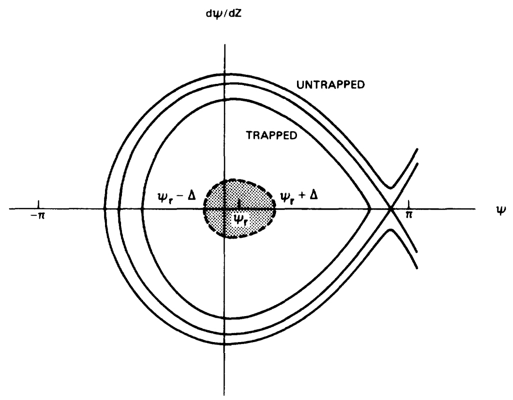

FIG. 1. Schematic representation of phase-space distribution about the resonant phase, showing the separatrix for a tapered wiggler.

are found to provide a contribution of order $\Delta^2\delta K^2/K^2$, where Δ^2 measures the trapped electron spread about the resonant phase. Since $\Delta^2 < 1$ in the present analysis, we conclude that the space-charge contribution to the gain of the primary wave can be important only if $\delta K^2 \gtrsim K^2$. It is instructive, therefore, to consider specific examples of experiments (either in current operation or proposed) to determine whether space-charge effects will be important in the trapped particle regime.

The first example is the experiment being conducted at Los Alamos National Laboratory[8] which is intended to operate in the infrared (at 10.6 μm). In this case a constant amplitude linear wiggler of about 2.4 kG is allowed to vary in period from 2.7 to 2.4 cm over a length of 1 m. As a consequence, the resonant phase is such that $\cos\psi_r \sim 0.94$. The experiment employs an electron beam of 20-MeV energy ($\Delta\gamma/\gamma \sim 0.01$) with a radius of 0.5 mm, and the maximum anticipated current is 25 A. Finally, the input signal intended to bunch the electron beam is to provide between 0.5 and 1 GW at 10.6 μm into an optical system with a 56-cm Rayleigh length. It is readily apparent, therefore, that for an equivalent system using a helical wiggler $\delta K^2/K^2 \lesssim 0.009$ and the space-charge effect is likely to be negligible. Indeed, this appears to be the case for all present or future free-electron-laser experiments which are to operate in the infrared due to the high-energy, low-current electron beams employed in such experiments.

In contrast, however, the space-charge effects may be important in free-electron lasers intended to operate in the millimeter or submillimeter range of wavelengths. Such devices operate at much lower energies and are thus able to achieve higher beam currents. One example of such an experiment is to be conducted using a Van de Graaff accelerator[4] capable of producing a 2-A beam of 3-MeV electrons (with effective $\Delta\gamma/\gamma \sim 0.005$) with a radius of approximately 2 mm. The wiggler field to be employed is uniform and of approximately 4 m in length with a 2-cm period and an amplitude of between 400 and 500 G. The radiation field amplitude required to trap the beam is about 1.6×10^{-5} (at approximately a 0.2-mm wavelength). Using these parameters it follows that $\delta K^2/K^2 \lesssim 1.2$ and, therefore, this experiment constitutes an intermediate case in which the space-charge fields may have an effect on the emission.

For still higher currents, the space-charge effect becomes still more important. A free-electron-laser experiment is being planned[9] which makes use of the Experimental Test Accelerator at Lawrence Livermore National Laboratory to test efficiency enhancement schemes with a tapered wiggler. Typical beam properties are a 4-MeV energy at a current density of greater than 500 A cm^{-2}. Beam densities, therefore, are expected to reach about 10^{11} electrons/cm^3. The projected wiggler field is linear and of 3-kG maximum amplitude with a period of 10 cm, and an input radiation pulse of about 930 kW/cm^2 (at a wavelength of 3 mm) is to be used to trap the beam with an effective $\Delta\gamma/\gamma \sim 9\%$. As a consequence, $\delta K^2/K^2 \sim 4.1/\cos\psi_r$ (where ψ_r is variable, $0.5 \le \cos\psi_r \le 1$), and it is clear that space-charge effects are important for such an experiment.

In conclusion, space-charge effects are important only when $\delta K^2 \gtrsim K^2$ (i.e., the electron bounce period in the ponderomotive potential is longer than the invariant plasma period). Such a condition is not generally satisfied for free-electron-laser experiments which operate at infrared wavelengths where beam currents and energies are of the order of several tens of amperes and MeV's. However, experiments operating at submillimeter wavelengths employ beams of much lower energy (several MeV) and higher current (1 kA) and collective space-charge effects are important.

ACKNOWLEDGMENT

The present work was supported by the Defense Advanced Research Projects Agency under Contract No. 3817.

*Permanent address: Science Applications, Inc., McLean, Virginia 22102.

[1]H. Boehmer, M. Z. Caponi, J. Edighoffer, S. Fornaca, J. Munch, G. Neil, B. Saur, and C. Shih, IEEE Trans. Nucl. Sci. NS-28, 3156 (1981).

[2]C. A. Brau, T. J. Boyd, R. K. Cooper, and D. A. Swenson, *Proceedings of the International Conference on Lasers*, edited by V. J. Corcoran (STS Press, McLean, Virginia, 1980).

[3]J. Slater, J. Quantum Electron. (in press).

[4]L. R. Elias, Proceedings of the International School of Quantum Electronics, Erice, Sicily, Italy, 17–29 August, 1980 (in press).

[5]N. M. Kroll and M. N. Rosenbluth, in *Free-Electron Generators of Coherent Radiation, Physics of Quantum Electronics, Vol. 7,* edited by S. J. Jacobs, H. S. Pilloff, M. Sargent, M. O. Scully, and R. Spitzer (Addison-Wesley, Reading, Mass., 1980), p. 147.

[6]P. Sprangle, C. M. Tang, and W. M. Manheimer, in *Free-Electron Generators of Coherent Radiation, Physics of Quantum Electronics, Vol. 7,* edited by S. F. Jacobs, H. S. Pilloff, M. Sargent, M. O. Scully, and R. Spitzer (Addison-Wesley, Reading, Mass., 1980), p. 207.

[7]P. Sprangle, C. M. Tang, and W. M. Manheimer, Phys. Rev. A 21, 302 (1980).

[8]C. J. Elliot (private communication).

[9]A. Sessler (private communication).