Electromagnetic induction in accelerated conductors

H. E. Wilhelm
Michelson Laboratory, Naval Weapons Center, China Lake, California 93555
(Received 7 October 1981)

Boundary conditions are derived for the interfaces of a conductor moving across an external magnetic field in an ambient medium (vacuum or nonconductor), which consider the emission of electromagnetic waves from the conductor surface as a result of electromagnetic induction. These boundary conditions are applied to the initial-boundary-value problem for the electromagnetic induction in a conducting slab, which is accelerated across a homogeneous magnetic field to a nonrelativistic velocity. Fourier-series solutions are presented for the transient electromagnetic fields in the moving conductor and the discontinuous electromagnetic waves in the ambient space. It is shown that the transient electromagnetic fields inside and outside the conductor are due to two mechanisms, i.e., "velocity induction" (ordinary induction) and "acceleration induction" $[d\vec{v}(t)/dt\neq\vec{0}]$. The latter result cannot be explained by means of the Lorentz transformation, which is valid only for constant conductor velocities (inertial frames).

INTRODUCTION

Although the theoretical foundations for electromagnetic induction in conductors moving across magnetic fields were formulated in 1908 by Minkowski,[1] only simple problems such as stationary unipolar induction in rotating disks have been discussed.[2] The electrodynamics of moving media has been a subject of basic research and also controversy to date.[3] In the treatment of electromagnetic induction, e.g., in liners of magnetic field compressors[4,5] and transient plasma shock waves interacting with external magnetic fields $\vec{B}_0$,[6,7] it has become customary to use the boundary condition for the tangential magnetic field in the form $\vec{n}\times(\vec{B}-\vec{B}_0)=\vec{0}$ at the conductor-gas interface, where $\vec{B}$ is the transient magnetic field in the conductor and $\vec{B}_0$ is the unperturbed external magnetic field. This boundary condition leads to electromagnetic solutions $\vec{B},\vec{E}$ in the conductor which do not satisfy the corresponding boundary condition $\vec{n}\times(\vec{E}-\vec{E}_0)=\vec{0}$ for the tangential electric fields at the moving interface. These "conventional" boundary conditions produce approximate to incorrect results, depending on the physical situation.

For an illustration of the problematics of the conventional boundary conditions,[4,7] which are also being used in the analysis of magnetic field diffusion into conductors at rest,[8] consider a conducting slab $\Delta x=2a$ with its surfaces initially at $x=\pm a$ in a transverse homogeneous magnetic field $\vec{B}_0=(0,B_0,0)$ for $|x|\leq\infty$ (Fig. 1.). At time $t=0$, this conductor is set into motion with a velocity $\vec{v}(t)=[d\hat{x}(t)/dt,0,0]$ so that its front and rear surfaces are at $x=\hat{x}(t)\pm a$ at time $t\geq 0$ where $\hat{x}(t=0)=0$. No matter whether the induction of the transient magnetic field $\vec{B}(x,t)$ in the moving conductor is described by the relativistic wave equation[9] or the nonrelativistic diffusion equation,[9] the initial condition $\vec{B}(x,t=0)=\vec{B}_0$, $-a<x<+a$, and the conventional boundary conditions $\vec{B}[x=\hat{x}(t)\pm a,t]=\vec{B}_0, t>0$, permit only one and the same solution, $\vec{B}(x,t)=\vec{B}_0$, which implies $\vec{E}(x,t)=-\vec{v}(t)\times\vec{B}_0$ and $\vec{j}(x,t)=\vec{0}$ by Ohm's law for the moving conductor. These simple solutions are due to the conventional boundary conditions without external perturbations and are obviously not correct since the boundary condition for the tangential electric field is not satisfied,

$$\vec{n}\times(\vec{E}-\vec{E}_0)=v(t)B_0\vec{e}_y\neq\vec{0}$$

where $\vec{E}_0=\vec{0}$. Since Faraday, it is an experimental fact that transient electromagnetic fields and currents are induced in the conducting slab as soon as it is moved relative to the external magnetic field $\vec{B}_0$.

In the following, an analytical solution is presented for the initial-boundary-value problem of the electromagnetic induction in a conducting slab $\Delta x=2a$, which is at rest for $t<0$ in a transverse homogeneous magnetic field $\vec{B}_0$, and which is set in motion at $t=0$ with a nonrelativistic velocity $\vec{v}(t)$ of arbitrary (finite) acceleration $d\vec{v}(t)/dt$

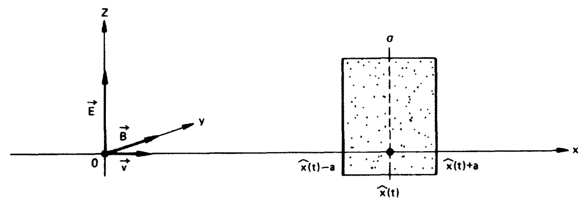

FIG. 1. Geometry of fields $\vec{B}$, $\vec{v}$, $\vec{E}$, and location $\hat{x}(t)$ of conductor for $t \geq 0$.

(Figs. 1 and 2). The electromagnetic induction in the conductor is shown to produce transient electromagnetic fields within the conductor and electromagnetic waves at the moving conductor surfaces $x=\hat{x}(t)\pm a$, which propagate with the speed of light in the surrounding space to infinity. Only if the electromagnetic waves outside of the moving conductor are taken into consideration, consistent solutions of Maxwell's equations exist which satisfy the boundary conditions for the tangential and normal electromagnetic fields at the conductor-nonconductor interfaces.

The analytical solutions for the moving conductor and surrounding spaces permit significant conclusions. The transient electric field induced in the conductor is the sum of a field which is proportional to the velocity $\vec{v}(t)$ and a field which is an integral functional of the acceleration $d\vec{v}(t)/dt$ of the conductor. A fundamental dimensionless group ($\sigma=$ conductivity, $\mu_1=$ permeability of conductor)

$$\mathscr{R}=\mu_1\sigma ac, \quad c=(\mu_2\epsilon_2)^{-1/2},$$

is found which represents a "magnetic Reynolds number" of (i) "free space" (if the conductor moves in vacuum or gas with permeabilities ϵ_0,μ_0), or (ii) "nonconducting space" (if the conductor moves in a nonconducting medium with permeabilities ϵ_2,μ_2). $\mathscr{R}$ determines the coupling of the transient electromagnetic fields inside and outside the conductor. The external electromagnetic waves would be negligible only in the limit $\mathscr{R}\rightarrow\infty$ which can, however, not be realized in actual experiments.

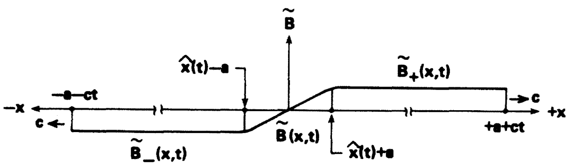

FIG. 2. Qualitative representation of induced field $\tilde{B}(x,t)$ in moving conductor and external electromagnetic waves $\tilde{B}_{\pm}(x,t)$ with fronts at $x=\pm(a+ct)$.

Thus, the conventional boundary conditions which assume that the external magnetic field $\vec{B}_0$ remains unperturbed outside of the conductor while transient electromagnetic processes occur in the latter, are not correct within rigorous electromagnetic theory.

BOUNDARY CONDITIONS

For the analysis of electromagnetic induction in moving conductors (conductivity $\sigma<\infty$, permittivities ϵ_1,μ_1) which move with a velocity field $\vec{v}(\vec{r},t)$ relative to the "laboratory system" in a nonconducting medium (vacuum, gas, or fluid; $\sigma=0,\epsilon_2,\mu_2$), the boundary conditions for the electromagnetic fields $\vec{B}_1,\vec{E}_1$ (conductor) and $\vec{B}_2,\vec{E}_2$ (nonconductor) are required in the L frame. Integration of Maxwell's equations with displacement current[10] over the interface 1-2 with velocity $\vec{v}(\vec{r},t)$ and normal vector $\vec{n}(\vec{r},t)$ (direction $1\rightarrow 2$) yields for the tangential and normal electromagnetic field components the boundary conditions in the L systems:

$$\vec{n}\times(\vec{E}_2-\vec{E}_1)=+(\vec{n}\cdot\vec{v})(\vec{B}_2-\vec{B}_1)\,, \quad (1)$$

$$\vec{n}\times(\vec{B}_2/\mu_2-\vec{B}_1/\mu_1)=\vec{j}^{\,*}-(\vec{n}\cdot\vec{v})(\epsilon_2\vec{E}_2-\epsilon_1\vec{E}_1)\,, \quad (2)$$

$$\vec{n}\cdot(\epsilon_2\vec{E}_2-\epsilon_1\vec{E}_1)=\rho^*\,, \quad (3)$$

$$\vec{n}\cdot(\vec{B}_2-\vec{B}_1)=0\,, \quad (4)$$

where

$$\vec{j}^{\,*}=\rho^*\vec{v}, \quad \rho^*=\lim_{\Delta s\rightarrow 0;\,|\rho|\rightarrow\infty}\rho\Delta s\,, \quad (5)$$

are the surface current and charge densities of the interface $\Delta s\rightarrow 0$. A conductive surface current density does not exist at a conductor of finite conductivity

$$\vec{j}^{\,*}_{\sigma}=\lim_{\Delta s\rightarrow 0}\sigma\vec{E}\Delta s=\vec{0}$$

for $\sigma<\infty$ and $\vec{E}$ bounded.

In many cases, the permittivities of good conductors and their ambient atmospheres equal the free-space values $\epsilon_{1,2}=\epsilon$ and $\mu_{1,2}=\mu$. With this simplification in notation, the boundary conditions (1)–(4) are applied to the front (+) and rear (−) surfaces $x=\hat{x}(t)\pm a$ of a conducting slab ($\sigma<\infty$) with the fields (Fig. 1)

$$\begin{aligned}&\vec{B}=[0,B(x,t),0],\ \vec{E}=[0,0,E(x,t)],\ \hat{x}(t)-a<x<\hat{x}(t)+a\\&\vec{B}(x,t=0)=\vec{B}_0,\ \vec{E}(x,t=0)=\vec{0},\ -a<x<+a\end{aligned}\tag{6}$$

which moves with the velocity $\vec{v}(t)=[d\hat{x}(t)/dt,0,0]$ in an ambient medium ($\sigma=0$) with the fields (Fig. 1)

$$\begin{aligned}&\vec{B}_\pm=[0,B_\pm(x,t),0],\ \vec{E}_\pm=[0,0,E_\pm(x,t)],\ x\gtrless x(t)\pm a\\&\vec{B}_\pm(x,t=0)=\vec{B}_0,\ \vec{E}_\pm(x,t=0)=\vec{0},\ x\gtrless\pm a\ .\end{aligned}\tag{7}$$

$\vec{B}_0$ is an external homogeneous magnetic field which fills uniformly the conductor (1) and the medium (2). The boundary conditions (3) and (4) are satisfied identically since $\vec{B},\vec{E}$ and $\vec{B}_\pm,\vec{E}_\pm$ are $\perp\vec{n}$ so that $\vec{j}^*=\vec{0}$ and $\rho^*=0$ by Eq. (5). The tangential boundary conditions (1) and (2) yield for the fields (6) and (7)

$$E_\pm(x=\hat{x}(t)\pm a,t)-E(x=\hat{x}(t)\pm a,t)=-v(t)[B_\pm(x=\hat{x}(t)\pm a,t)-B(x=\hat{x}(t)\pm a,t)]\ ,\tag{8}$$

$$B_\pm(x=\hat{x}(t)\pm a,t)-B(x=\hat{x}(t)\pm a,t)=-v(t)c^{-2}[E_\pm(x=\hat{x}(t)\pm a,t)-E(x=\hat{x}(t)\pm a,t)]\ ,\tag{9}$$

where

$$c=(\mu\epsilon)^{-1/2}\tag{10}$$

is the speed of light. For nonrelativistic conductor motions, Eqs. (8) and (9) reduce to

$$E_\pm(x=\hat{x}(t)\pm a,t)-E(x=\hat{x}(t)\pm a,t)=0,\ \ v(t)^2<<c^2\tag{11}$$

$$B_\pm(x=\hat{x}(t)\pm a,t)-B(x=\hat{x}(t)\pm a,t)=0,\ \ v(t)^2<<c^2\ .\tag{12}$$

According to the nonrelativistic Ohm's law for conductors which are moving with a velocity $\vec{v}$ relative to the L system, $\vec{j}=\sigma(\vec{E}+\vec{v}\times\vec{B})$, the electric field $\vec{E}(x,t)$ in the conductor is expressed in terms of $\vec{B}(x,t)$:

$$E=-vB+(\mu\sigma)^{-1}\partial B/\partial x,\ \hat{x}(t)-a<x<\hat{x}(t)+a\ ,\ \ v(t)^2\ll c^2\ .\tag{13}$$

The electromagnetic field $\vec{B}_\pm(x,t)$, $\vec{E}_\pm(x,t)$ in the adjacent semi-infinite half-spaces is determined by the hyperbolic initial-boundary-value problems:

$$\partial^2B_\pm/\partial t^2=c^2\partial^2B_\pm/\partial x^2,\ x\gtrless\hat{x}(t)\pm a,\ \ t>0\tag{14}$$

$$B_\pm(x,t=0)=B_0,\ \ x\gtrless\pm a\tag{15}$$

$$B_\pm(x=\hat{x}(t)\pm a,t)=B(x=\hat{x}(t)\pm a,t)\ ,\ \ t>0\tag{16}$$

since

$$\partial E_\pm/\partial x=\partial B_\pm/\partial t,\ \ \partial B_\pm/\partial x=c^{-2}\partial E_\pm/\partial t\tag{17}$$

by Maxwell's equations for homogeneous nonconductors. Equation (16) couples the solutions $B_\pm(x,t)$ in the semi-infinite spaces $x\gtrless\hat{x}(t)\pm a$ to the solution $B(x,t)$ in the conductor $\hat{x}(t)-a<x<\hat{x}(t)+a$. By Eqs. (14)–(17), the ambient electromagnetic field transients are of the form

$$B_\pm(x,t)=\begin{cases}B_0+\Psi_\pm\left[t\mp\dfrac{x\mp a}{c}\right], & \hat{x}(t)\pm a\lesseqgtr x\lesseqgtr\pm(a+ct)\\ B_0, & a+ct<\pm x<\infty\end{cases}\tag{18}$$

and

$$E_\pm(x,t)=\begin{cases}\mp c\Psi_\pm\left[t\mp\dfrac{x\mp a}{c}\right], & \hat{x}(t)\pm a\lesseqgtr x\lesseqgtr\pm(a+ct)\\ 0, & a+ct<\pm x<\infty\end{cases}\tag{19}$$

where

$$\Psi_{\pm}\left[t\mp\frac{\hat{x}(t)}{c}\right]=B(x=\hat{x}(t)\pm a,t)-B_0 \qquad (20)$$

by Eq. (16) determines the form of the wave functions $\Psi_{\pm}(\eta_{\pm})$ of the "self-similar" arguments $\eta_{\pm}=t\mp x/c+a/c$ from the conductor solution $B(x,t)$.

The solutions (18) and (19) are typical for hyperbolic initial-boundary-value problems, i.e., the boundary values $\Psi_{\pm}(t\mp\hat{x}(t)/c)$ are transported with the speed of light c into the half-spaces $x\gtrless\hat{x}(t)\pm a$, so that discontinuous wave fronts result at $x=\pm(a+ct)$.

By means of Eqs. (18)–(20), the boundary values $B_{\pm}(x=\hat{x}(t)\pm a,t)$, $E_{\pm}(x=\hat{x}(t)\pm a,t)$, and $\Psi_{\pm}(t\mp\hat{x}(t)/c)$ are eliminated from the boundary conditions (11) and (12):

$$\frac{\partial B(x=\hat{x}(t)\pm a,t)}{\mu\sigma\partial x}\pm[c\mp v(t)]B(x=\hat{x}(t)\pm a,t)=\pm cB_0,\quad v(t)^2<<c^2 . \qquad (21)$$

These are the fundamental new boundary conditions for moving conductors which (i) involve only boundary values of the magnetic field $B(x,t)$ of the conductor, and (ii) consider the emission of electromagnetic waves

$$\tilde{B}_{\pm}(x,t)=\Psi_{\pm}\left[t\mp\frac{x\mp a}{c}\right],$$
$$\tilde{E}_{\pm}(x,t)=\mp c\Psi\left[t\mp\frac{x\mp a}{c}\right], \qquad (22)$$

from the conductor surfaces $x=\hat{x}(t)\pm a$ into the ambient spaces $x\gtrless\hat{x}(t)\pm a$. Since the magnetic field in the conductor is the sum of the external B_0 and a transient $\tilde{B}(x,t)$,

$$B(x,t)=B_0+\tilde{B}(x,t),\quad \hat{x}(t)-a<x<\hat{x}(t)+a \qquad (23)$$

Eq. (21) gives for the transient conductor field the boundary conditions

$$\frac{\partial\tilde{B}(x=\hat{x}(t)\pm a,t)}{\mu\sigma\partial x}\pm[c\mp v(t)]\tilde{B}(x=\hat{x}(t)\pm a,t)=v(t)B_0,\quad v(t)^2<<c^2 . \qquad (24)$$

Equation (24) indicates that $c\tilde{B}(x=\hat{x}(t)\pm a,t)\sim v(t)B_0$, i.e., $v(t)/c$ in the coefficient $[1\mp v(t)/c]$ contributes a negligible term of order $v(t)^2/c^2$. Accordingly, Eqs. (21) and (24) reduce to the boundary conditions

$$(\mu\sigma)^{-1}\frac{\partial B(x=\hat{x}(t)\pm a,t)}{\partial x}\pm c[B(x=\hat{x}(t)\pm a,t)-B_o]=v(t)B_0,\quad v(t)^2<<c^2 \qquad (25)$$

and

$$(\mu\sigma)^{-1}\frac{\partial\tilde{B}(x=\hat{x}(t)\pm a,t)}{\partial x}\pm c\tilde{B}(x=\hat{x}(t)\pm a,t)=v(t)B_0,\quad v(t)^2<<c^2 . \qquad (26)$$

If the $\nabla\times\vec{B}$ and $v(t)B_0$ terms are omitted, Eqs. (25) and (26) reduce to the conventional boundary conditions[4–7] $B(x=\hat{x}(t)\pm a,t)=B_0$ and $\tilde{B}(x=\hat{x}(t)\pm a,t)=0$, respectively. Comparison shows that the conductor currents $\nabla\times\vec{B}/\mu$ and the induced currents $\sigma v(t)B_0$ at the conductor surface are the sources of the emitted electromagnetic waves.

INITIAL-BOUNDARY-VALUE PROBLEM

Consider a rigid conducting slab of width $\Delta x=2a$ with surfaces in the planes $x=\hat{x}(t)\pm a$ at time $t\geq 0$ (Fig. 1). This conductor is exposed to an external magnetic field $\vec{B}_0=\{0,B_0,0\}$ which is homogeneous throughout the space $-\infty<x<+\infty$, and is accelerated to a (nonrelativistic) velocity $\vec{v}(t)=[v(t),0,0]$ from an initial position at rest $\hat{x}(t=0)=0$ so that

$$v(t)=\frac{d\hat{x}(t)}{dt},$$
$$\hat{x}(t)=\int_0^t v(t')dt' , \qquad (27)$$

with

$$\hat{x}(t=0)=0 ,$$
$$v(t=0)=0,\quad \frac{dv(t=0)}{dt}>0 . \qquad (28)$$

The electromagnetic induction of the transient electromagnetic fields $\vec{B}=[0,B(x,t),0]$ and $\vec{E}=[0,0,E(x,t)]$ in the conductor of finite width, as a result of its accelerated motion $\vec{v}(t)$ across the external magnetic field $\vec{B}_0$, is determined by the parabolic initial-boundary-value problem for $B(x,t)$:

$$\frac{\partial B}{\partial t}+v(t)\frac{\partial B}{\partial x}=\kappa\frac{\partial^2 B}{\partial x^2} ,$$
$$\hat{x}(t)-a<x<\hat{x}(t)+a,\quad t>0 \qquad (29)$$

$$B(x,t=0)=B_0\ ,\quad -a<x<+a \tag{30}$$

$$\frac{\partial B(x=\hat{x}(t)\pm a,t)}{\partial x}\pm h[B(x=\hat{x}(t)\pm a,t)-B_0]=\kappa^{-1}v(t)B_0,\quad t>0 \tag{31}$$

where

$$\kappa=1/\mu\sigma,\quad h=c/\kappa\ . \tag{32}$$

Equation (29) follows from Maxwell's equations and Ohm's law $\vec{j}=\sigma(\vec{E}+\vec{v}\times\vec{B})$ for moving conductors in the diffusion approximation,[9] $\tau_R=\epsilon/\sigma<<1$. The boundary conditions (31) couple the electromagnetic induction process in the conductor to the external transients in the ambient medium [Eq. (25)]. The transformation,

$$B(x,t)=B_0+\tilde{B}(\mathscr{x},t),\quad -a\le \mathscr{x}\le +a,\quad t\ge 0 \tag{33}$$

$$\mathscr{x}=x-\hat{x}(t),\quad t\ge 0 \tag{34}$$

reduces Eqs. (29)–(31) to the initial-boundary-value problem (IBVP):

$$\partial\tilde{B}/\partial t=\kappa\partial^2\tilde{B}/\partial\mathscr{x}^2,\quad -a<\mathscr{x}<+a,\quad t>0 \tag{35}$$

$$\tilde{B}(\mathscr{x},t=0)=0,\quad -a<\mathscr{x}<+a \tag{36}$$

$$\partial\tilde{B}(\mathscr{x}=\pm a,t)/\partial\mathscr{x}\pm h\tilde{B}(\mathscr{x}=\pm a,t)=\kappa^{-1}v(t)B_0,\quad t>0\ . \tag{37}$$

The linear IBVP (35)–(37) is decomposed into a BVP and an IBVP by means of the ansatz

$$\tilde{B}(\mathscr{x},t)=F(\mathscr{x},t)+G(\mathscr{x},t),\quad -a\le\mathscr{x}\le +a,\quad t\ge 0 \tag{38}$$

where

$$\partial^2F/\partial\mathscr{x}^2=0,\quad -a<\mathscr{x}<+a,\quad t>0 \tag{39}$$

$$\partial F(\mathscr{x}=\pm a,t)/\partial\mathscr{x}\pm hF(\mathscr{x}=\pm a,t)=\kappa^{-1}v(t)B_0,\quad t>0 \tag{40}$$

and

$$\partial G/\partial t=\kappa\partial^2G/\partial\mathscr{x}^2-\partial F/\partial t,\quad -a<\mathscr{x}<+a,\quad t>0 \tag{41}$$

$$G(\mathscr{x},t=0)=-F(\mathscr{x},t=0),\quad -a<\mathscr{x}<+a \tag{42}$$

$$\partial G(\mathscr{x}=\pm a,t)/\partial\mathscr{x}\pm hG(\mathscr{x}=\pm a,t)=0,\quad t>0\ . \tag{43}$$

The solution $F(\mathscr{x},t)$ of Eqs. (29) and (30), the source $\partial F(\mathscr{x},t)/\partial t$ in Eq. (41), and the initial value $F(\mathscr{x},t=0)$ in Eq. (42) are

$$F(\mathscr{x},t)=[v(t)B_0/(\kappa+ac)]\mathscr{x},\quad -a\le\mathscr{x}\le +a,\quad t\ge 0 \tag{44}$$

$$\partial F(\mathscr{x},t)/\partial t=\left[\frac{dv(t)}{dt}B_0/(\kappa+ac)\right]\mathscr{x},\quad -a\le\mathscr{x}\le +a,\quad t\ge 0 \tag{45}$$

and

$$F(\mathscr{x},t=0)=0\ ,\quad -a\le\mathscr{x}\le +a \tag{46}$$

since $v(t=0)=0$. With $\partial F(\mathscr{x},t)/\partial t$ odd in $\mathscr{x}$ and $F(\mathscr{x},t=0)=0$, the initial-boundary-value problem (41)–(43) is solved by means of the Fourier expansions

$$G(\mathscr{x},t)=\sum_{n=1}^{\infty}G_n(t)\sin k_n\mathscr{x},\quad -a\le\mathscr{x}\le +a,\quad t\ge 0 \tag{47}$$

$$\partial F(\mathscr{x},t)/\partial t=\sum_{n=1}^{\infty}S_n(t)\sin k_n\mathscr{x},\quad -a\le\mathscr{x}\le +a,\quad t\ge 0 \tag{48}$$

where

$$k_na\ \cot k_na=-ha,\quad n=1,2,3,\ldots \tag{49}$$

determines the eigenvalues k_n of the eigenfunctions $\sin k_n x$ associated with the boundary conditions (43). Substitution of Eqs. (47) and (48) into Eqs. (41) and (42) yields by means of the inverse Fourier theorem

$$dG_n(t)/dt+\kappa k_n^2 G_n(t)=-S_n(t),\quad t>0 \tag{50}$$

$$G_n(t=0)=0\ , \tag{51}$$

where

$$S_n(t)=(a/\kappa)K_n B_0 dv(t)/dt,\quad t\geq 0 \tag{52}$$

$$K_n=2(h^2+k_n^2)\sin k_n a/(k_n a)^2[(h^2+k_n^2)+(h/a)]\ , \tag{53}$$

by Eqs. (45) and (48). The solution of Eqs. (50) and (51) is

$$G_n(t)=-\int_0^t e^{-\kappa k_n^2(t-t')}S_n(t')dt',\quad t\geq 0\ . \tag{54}$$

Combining of Eqs. (47), (52), and (54) yields as solution of the initial-boundary-value problem (41)–(43):

$$G(x,t)=-\frac{a}{\kappa}B_0\sum_{n=1}^{\infty}K_n\left[\int_0^t \frac{dv(t')}{dt'}e^{-\kappa k_n^2(t-t')}dt'\right]\sin k_n x,\quad -a\leq x\leq +a,\quad t\geq 0\ . \tag{55}$$

By Eqs. (33), (34), (38), (44), and (55), the magnetic field solution satisfying Eqs. (29)–(31) is

$$B(x,t)=B_0+[v(t)B_0/(\kappa+ac)][x-\hat{x}(t)]-\frac{a}{\kappa}B_0\sum_{n=1}^{\infty}K_n\left[\int_0^t \frac{dv(t')}{dt'}e^{-\kappa k_n^2(t-t')}dt'\right]\sin k_n[x-\hat{x}(t)]\ ,$$
$$\hat{x}(t)-a\leq x\leq \hat{x}(t)+a,\quad t\geq 0\ . \tag{56}$$

This is a fundamental result which shows that the transient magnetic field $\tilde{B}(x,t)$ is the sum of a field $F(x,t)$ induced by the motion $v(t)$ and a field $G(x,t)$ induced by the acceleration $dv(t)/dt$ of the conductor in the external magnetic field B_0. Similar decompositions exist for the electric field $E(x,t)$ and current density $j(x,t)$ in the accelerated conductor.

The still unknown forms ($\pm$) of the two external wave functions $\Psi_\pm(t\mp x/c+a/c)$ in the spaces $x\gtrless\hat{x}(t)\pm a$ are determined from the solution (56) by means of the boundary condition (20), which gives

$$\Psi_\pm\left[t\mp\frac{\hat{x}(t)}{c}\right]=\tilde{B}(x=\pm a,t),\quad t\geq 0\ . \tag{57}$$

The transformations $t_\pm=t_\pm(t)$ and its inverses $t=f_\pm\{t_\pm\}$ for the two ($\pm$) waves defined by

$$t_\pm=t\mp\frac{\hat{x}(t)}{c},\quad t=f_\pm\{t_\pm\},\quad t\geq 0 \tag{58}$$

where $t_+=t_-=0$ for $t=0$ [$\hat{x}(t=0)=0$] but $t_+\neq t_-$ for $t>0$, show that the wave functions are of the form

$$\Psi_\pm(t_\pm)=\tilde{B}(x=\pm a,t=f_\pm\{t_\pm\})\ .$$

Accordingly,

$$\Psi_\pm\left[t\mp\frac{x\mp a}{c}\right]=\tilde{B}\left[x=\pm a,t=f_\pm\left[t\mp\frac{x\mp a}{c}\right]\right]\ ,$$
$$\hat{x}(t)\pm a\lesseqgtr x\lesseqgtr \pm(a+ct),\quad t\geq 0\ . \tag{59}$$

Since $\tilde{B}(x,t)=B(x,t)-B_0$, substitution of Eq. (56) into Eq. (59) gives the wave functions $\Psi_\pm$ as functionals of $f_\pm(t\mp x/c+a/c)$:

$$\Psi_\pm\left[t\mp\frac{x\mp a}{c}\right]=\pm[aB_0/(\kappa+ac)]v\left[f_\pm\left[t\mp\frac{x\mp a}{c}\right]\right]$$
$$\mp\frac{a}{\kappa}B_0\sum_{n=1}^{\infty}K_n\sin k_n a e^{-\kappa k_n^2 f_\pm[t\mp(x\mp a)/c]}\int_0^{f_\pm[t\mp(x\mp a)/c]}\frac{dv(t')}{dt'}e^{\kappa k_n^2 t'}dt',$$
$$\hat{x}(t)\pm a\gtrless x\gtrless \pm(a+ct),\quad t\geq 0\ . \tag{60}$$

The solutions (60) determine the propagation of the emitted electromagnetic waves outside of the moving conductor $x \gtrless \hat{x}(t) \pm a$. Again, these waves each consist of a "velocity" wave $[v(t)]$ and an "acceleration" wave $[dv(t)/dt]$. They satisfy all boundary and initial conditions,

$$\Psi_{\pm|x=\hat{x}(t)\pm a} = \tilde{B}(x=\pm a, t), \quad t \geq 0 \tag{61}$$

$$\Psi_{\pm|t=0} = 0, \quad x \gtrless \pm a \tag{62}$$

since $t_{\pm}=0$ and $x=\pm a$ [Eq. (60)] for $t=0$, and, hence, $f_{\pm}=f_{\pm}(0)=0$ [Eq. (58)] and $v(f_{\pm}) = v(0) = 0$.

For a brief illustration of the transformation (58) consider the simple conductor motion $v(t)=v_0$, $\hat{x}(t)=v_0 t$, $t \geq 0$. In this case, $t_{\pm}=(1 \mp v_0/c)t$ and $t = t_{\pm}/[1 \mp (v_0/c)]$, i.e., $f_{\pm}(t_{\pm})$ is proportional to t.

ANALYTICAL SOLUTIONS

For the most general representation of the electromagnetic fields in the moving conductor and the ambient medium, dimensional independent and dependent variables are introduced by

$$\xi = x/a, \quad \tau = \kappa t/a^2, \quad \hat{\xi}(\tau) = \hat{x}(t)/a, \quad \alpha_n = k_n a, \quad \nu(\tau) = v(t)/v_0 , \tag{63}$$

$$\mathscr{B}(\xi,\tau) = B(x,t)/B_0, \quad \mathscr{E}(\xi,\tau) = E(x,t)/(\kappa B_0/a), \quad \jmath(\xi,\tau) = j(x,t)/(B_0/\mu a), \quad \psi(\xi,\tau) = \Psi(x,t)/B_0 . \tag{64}$$

Conductor solutions

According to Eq. (56), the dimensionless electromagnetic fields $\mathscr{B}(\xi,\tau)$, $\jmath(\xi,\tau) = \partial \mathscr{B}(\xi,\tau)/\partial \xi$, and $\mathscr{E}(\xi,\tau)$ in the accelerated conductor are

$$\mathscr{B}(\xi,\tau) = 1 + \frac{\mathscr{M}}{(1+\mathscr{R})} \nu(\tau)[\xi - \hat{\xi}(\tau)] - \mathscr{M} \sum_{n=1}^{\infty} K_n \left[\int_0^{\tau} \frac{d\nu(\tau')}{d\tau'} e^{-\alpha_n^2(\tau-\tau')} d\tau' \right] \sin\alpha_n [\xi - \hat{\xi}(\tau)], \quad \hat{\xi}(\tau) - 1 \leq \xi \leq \hat{\xi}(\tau) + 1, \quad \tau \geq 0 \tag{65}$$

$$\jmath(\xi,\tau) = \frac{\mathscr{M}}{(1+\mathscr{R})} \nu(\tau) - \mathscr{M} \sum_{n=1}^{\infty} \alpha_n K_n \left[\int_0^{\tau} \frac{d\nu(\tau')}{d\tau'} e^{-\alpha_n^2(\tau-\tau')} d\tau' \right] \cos\alpha_n [\xi - \hat{\xi}(\tau)], \quad \hat{\xi}(\tau) - 1 \leq \xi \leq \hat{\xi}(\tau) + 1, \quad \tau \geq 0 \tag{66}$$

$$\mathscr{E}(\xi,\tau) = -\mathscr{M} \nu(\tau) \mathscr{B}(\xi,\tau) + \jmath(\xi,\tau), \quad \hat{\xi}(\tau) - 1 \leq \xi \leq \hat{\xi}(\tau) + 1, \quad \tau \geq 0 \tag{67}$$

where

$$\alpha_n \cot\alpha_n = -\mathscr{R}, \quad n = 1,2,3,\ldots \tag{68}$$

$$K_n = 2(\mathscr{R}^2 + \alpha_n^2)\sin\alpha_n / \alpha_n^2[(\mathscr{R}^2 + \alpha_n^2) + \mathscr{R}] , \tag{69}$$

and

$$\mathscr{M} = \mu\sigma a v_0 , \tag{70}$$

$$\mathscr{R} = \mu\sigma a c . \tag{71}$$

$\mathscr{M}$ is known as the magnetic Reynolds number of the conductor with characteristic speed $\sim v_0$. Equations (65)–(67) indicate that $\mathscr{M}/(1+\mathscr{R})$ determines the order of the ratio $\tilde{B}/B_0$ of induced and external magnetic fields. The steady-state induction in moving conductors[11,12] is determined only by $\mathscr{M}$.

$\mathscr{R}$ is a new dimensionless group which involves the velocity of light $c = (\epsilon\mu)^{-1/2}$ so that $\mathscr{R} >> \mathscr{M}$ for $v_0 << c$. $\mathscr{R}$ has the physical meaning of a "magnetic Reynolds number" of the ambient nonconducting space; i.e., $\mathscr{R}$ is a coupling parameter between the conductor $(0 < \sigma < \infty)$ and its external medium $(\sigma = 0)$, which determines the magnitudes of the external electromagnetic transients [Eqs. (75)–(76)].

The net electric current flowing through the conductor is per unit width $\Delta\eta = 1$

$$\mathscr{I}(\tau)=\int_{\hat{\xi}(\tau)-1}^{\hat{\xi}(\tau)+1} j(\xi,\tau)d\xi=2\mathscr{M}v(\tau)/(1+\mathscr{R})-2\mathscr{M}\sum_{n=1}^{\infty}\sin\alpha_n K_n \int_0^{\tau}\frac{dv(\tau')}{d\tau'}e^{-\alpha_n^2(\tau-\tau')}d\tau' , \quad \tau\geq 0 . \tag{72}$$

External solutions

For the space surrounding the moving conductor, the wave functions $\Psi_{\pm}(x,t)$ of the electromagnetic transients are by Eqs. (60), (63), and (64) in dimensionless representation

$$\psi_{\pm}\left[\tau\mp\frac{\xi\mp 1}{\mathscr{R}}\right]=\pm\frac{\mathscr{M}}{(1+\mathscr{R})}v\left[\phi_{\pm}\left[\tau\mp\frac{\xi\mp 1}{\mathscr{R}}\right]\right]$$
$$\mp\mathscr{M}\sum_{n=1}^{\infty}\sin\alpha_n K_n e^{-\alpha_n^2\phi_{\pm}[\tau\mp(\xi\mp 1)/\mathscr{R}]}\int_0^{\phi_{\pm}[\tau\mp(\xi\mp 1)/\mathscr{R}]}\frac{dv(\tau')}{d\tau'}e^{\alpha_n^2\tau'}d\tau' ,$$
$$\hat{\xi}(\tau)\pm 1\lesseqgtr\xi\lesseqgtr\pm(1+\mathscr{R}\tau),\quad \tau\geq 0 \tag{73}$$

where

$$\phi_{\pm}\left[\tau\mp\frac{\xi\mp 1}{\mathscr{R}}\right]=(\kappa/a^2)f_{\pm}\left[t\mp\frac{x\mp a}{c}\right] . \tag{74}$$

By Eqs. (18), (19), and (73) the dimensionless solutions for the electromagnetic field outside the moving conductor are

$$\mathscr{B}_{\pm}(\xi,\tau)=\begin{cases}1+\psi_{\pm}\left[\tau\mp\frac{\xi\mp 1}{\mathscr{R}}\right], & \hat{\xi}(\tau)\pm 1\lesseqgtr\xi\gtreqless\pm(1+\mathscr{R}\tau),\quad \tau\geq 0\\ 1, \quad 1+\mathscr{R}\tau\leq\pm\xi\leq\infty, & \tau\geq 0\end{cases} \tag{75}$$

and

$$\mathscr{E}_{\pm}(\xi,\tau)=\begin{cases}\mp\mathscr{R}\psi_{\pm}\left[\tau\mp\frac{\xi\mp 1}{\mathscr{R}}\right], & \hat{\xi}(\tau)\pm 1\lesseqgtr\xi\gtreqless\pm(1+\mathscr{R}\tau),\quad \tau\geq 0\\ 0, \quad 1+\mathscr{R}\tau\leq\pm\xi\leq\infty, & \tau\geq 0 .\end{cases} \tag{76}$$

DISCUSSION

The magnetic Reynolds number is $\mathscr{M}=\mu\sigma a v_0 \gtrless 1$ for conductors and the magnetic coupling number $\mathscr{R}=\mu\sigma a c >> 1$ is large, depending on the parameters σ, a, and v_0. In the most general case, electromagnetic induction in a moving conductor is determined both by $\mathscr{M}$ and $\mathscr{R}$, where $\mathscr{M} << \mathscr{R}$ since $v_0 << c$.

1. Case $\mathscr{R}>>1$. For most macroscopic conductors, it is $\sigma > 10^4\ \Omega^{-1}/\mathrm{m}$ and $a > 10^{-4}$ m ($\mu=4\pi 10^{-7}$ V sec/A m, $c=3\times 10^8$ m/sec) so that $\mathscr{R}>10^2$, and by Eqs. (68) and (69)

$$\alpha_n=n\pi(1-\mathscr{R}^{-1}),\quad \mathscr{R}>>1,\quad n=1,2,3,\ldots \tag{77}$$

$$K_n=2(-1)^{n+1}(n\pi)^{-1}\mathscr{R}^{-1},\quad \mathscr{R}>>1 ,\quad n=1,2,3,\ldots . \tag{78}$$

It is seen that the "velocity" and acceleration" fields in Eqs. (65)–(67) are of the same order since

$$\mathscr{M}/(1+\mathscr{R})\sim\mathscr{M}(+K_1)\sim\mathscr{M}/\mathscr{R} =v_0/c<<1,\quad R>>1 . \tag{79}$$

In the magnetic field solution (65), the dominant term is the external $\mathscr{B}_0=1>>\mathscr{M}/\mathscr{R}$ and in the electric field solution (67) the dominant term is the motion-induced field $|\mathscr{M}v\mathscr{B}_0| >> |j|$ if $\mathscr{R}>>1$. Although significant electric fields are induced in the conductor, the induced magnetic field is small,

$$\tilde{\mathscr{B}}\sim\mathscr{M}/(1+\mathscr{R})<<\mathscr{B}_0=1,\quad \mathscr{R}>>1$$

and therefore, the current density $j=\partial\mathscr{B}/\partial\xi$ is small, too.

2. Case $\mathscr{R}=\infty$. In an actual experiment, $\mathscr{R}=\infty$

can never be reached but only asymptotically approached. In this hypothetical situation, the conductor [Eqs. (65)–(67)] and external [Eqs. (75) and (76)] solutions reduce to

$$\mathscr{B}(\xi,\tau)=1,\ \mathscr{j}(\xi,\tau)=0\ ,$$
$$\mathscr{E}(\xi,\tau)=-\mathscr{M}v(\tau),\ \ \hat{\xi}(\tau)-1\le\xi\le\hat{\xi}(\tau)+1,\qquad \tau\ge 0 \tag{80}$$

and

$$\mathscr{B}_{\pm}(\xi,\tau)=1,\ \ \mathscr{E}_{\pm}(\xi,\tau)=-\mathscr{M}v(\tau),$$
$$\hat{\xi}(\tau)\pm 1 \lessgtr \xi \lessgtr \pm\infty,\ \ \tau\ge 0 \tag{81}$$

since

$$\lim_{\mathscr{R}\to\infty}\alpha_n=n\pi,\ \ \lim_{\mathscr{R}\to\infty}K_n=0\ ,$$
$$\lim_{\mathscr{R}\to\infty}\mathscr{R}K_n=2(-1)^{n+1}/n\pi,\ \ n=1,2,3,\ldots. \tag{82}$$

It should be noted that for $\mathscr{R}\to\infty$, only $\tilde{\mathscr{B}}_{\pm}=0$ but $\mathscr{E}_{\pm}\neq 0$, i.e., the external electric transients are (behind their wave fronts) of the same order of magnitude as the electric field $\mathscr{E}$ in the conductor. Since $\nabla\times\vec{B}_{\pm}=c^{-2}\partial\vec{E}_{\pm}/\partial t$, the electric transients $\mathscr{E}_{\pm}$ always coexist with (no matter how small) magnetic transients $\tilde{\mathscr{B}}_{\pm}$. For this reason, the limit $\mathscr{R}=\infty$ has no physical meaning, quite apart from the fact that always $\mathscr{R}<\infty$ for $\sigma,a,c<\infty$. The conventional boundary conditions[4,5,8] for electromagnetic diffusion processes in conductors imply $\mathscr{R}=\infty$ and $\mathscr{E}_{\pm}\equiv 0$, and are, therefore, physically incorrect.

As an explanation it is noted that, in conductors, magnetic field diffusion is a nonrelativistic process, as is electric conduction, $\vec{j}=\sigma(\vec{E}+\vec{v}\times\vec{B})$. The electric transients $\mathscr{E}_{\pm}$ in vacuum must be of the same order as the electric field $\mathscr{E}$ in the conductors, $\mathscr{E}_{\pm}\sim\mathscr{E}\sim\mathscr{M}\mathscr{R}^0$, since otherwise the tangential electric field would not be continuous across the conductor surface. On the other hand, the external magnetic transients $\tilde{\mathscr{B}}_{\pm}$ are small, of order $\mathscr{M}\mathscr{R}^{-1}=\mathscr{M}(\mu\sigma a)^{-1}c^{-1}$, since the electromagnetic energy flows with the speed of light in the ambient space. The deeper physical reason for the external electromagnetic transients is to be seen in the conservation laws for electromagnetic energy and momentum, which follow from Maxwell's equations.[9]

3. Diffusion approximation. It is known that Maxwell's equations with displacement current and the nonrelativistic Ohm's law, $\vec{j}=\sigma(\vec{E}+\vec{v}\times\vec{B})$, combine to a hyperbolic diffusion equation for the magnetic field $\vec{B}$ in conductors,[9] which reads in the considered one-dimensional case with $\vec{r}$-independent conductor velocity $v(t)$

$$\frac{\partial^2 B}{\partial t^2}+\frac{1}{\tau_R}\frac{\partial B}{\partial t}+\frac{v}{\tau_R}\frac{\partial B}{\partial x}=c^2\frac{\partial^2 B}{\partial x^2}\ , \tag{83}$$

where

$$\tau_R=\epsilon/\sigma<<1 \tag{84}$$

is the field relaxation time, which is extremely small for conductors. Equation (83) reduces to the parabolic diffusion equation in the limit $\tau_R<<1$:

$$\frac{\partial B}{\partial t}+v\frac{\partial B}{\partial x}=\tau_R c^2\frac{\partial^2 B}{\partial x^2},\ \ \tau_R<<1\ . \tag{85}$$

The parabolic diffusion equation is an excellent approximation, since the relaxation time of conductors is very small, $\tau_R<<1$. By Eqs. (84) and (85), the field relaxation time τ_R and the diffusion time τ_D are interrelated by

$$\tau_D^{-1}=\tau_R c^2/a^2,\ \ \tau_D=\mu\sigma a^2\ , \tag{86}$$

where a is the extension of the conductor. For conductors, the diffusion time is relatively large if a is not too small , i.e., $\tau_D>>\tau_R$.

Comparison of the neglected term $\partial^2 B/\partial t^2$ with the leading $(0\le|v|<<c)$ second and fourth terms of Eq. (83) reveals the relation of the parabolic diffusion approximation to the new coupling number $\mathscr{R}=\mu\sigma a c$:

$$\left|\frac{\partial^2 B}{\partial t^2}\right|\Big/\left|\tau_R^{-1}\frac{\partial B}{\partial t}\right|\sim\left|\frac{\partial^2 B}{\partial t^2}\right|\Big/\left|c^2\frac{\partial^2 B}{\partial x^2}\right|\sim\frac{\tau_R}{\tau_D}=\frac{\epsilon\mu}{(\mu\sigma a)^2}=\mathscr{R}^{-2}\ . \tag{87}$$

This result again confirms the validity of the parabolic diffusion equation for conductors, for which $\mathscr{R}=\mu\sigma a c>>1$. More important, Eq. (87) demonstrates that the neglected relativistic term $\partial^2 B/\partial t^2$ is small of order $\mathscr{R}^{-2}<<<<1$, whereas the calculated electromagnetic fields in the conductor are of order $\mathscr{R}\tilde{\mathscr{B}}\sim\mathscr{E}\sim\mathscr{M}\mathscr{R}^0$ [Eqs. (65)–(67)], and the external electromagnetic transients are of order $\tilde{B}_{\pm}\sim\mathscr{M}\mathscr{R}^{-1}$ and $\mathscr{E}_{\pm}\sim\mathscr{M}\mathscr{R}^0$ [Eqs. (73)– (76)].

Thus, consistent electromagnetic field solutions for the regions in and outside of the accelerated conductor have been obtained within the parabolic diffusion approximation, which satisfy the boundary conditions for the continuity of the tangential electric and magnetic fields. If an accuracy of higher order $\mathscr{R}^{-2}$ is to be achieved, then the hyperbolic diffusion equation (83) has to be used.

For nonrelativistic conductor motions, $v^2 << c^2$, however, an accuracy of order $\mathscr{R}^{-1}$ is completely sufficient. The mathematical advantages of the parabolic diffusion equation become obvious if it is used in connection with (time-dependent) moving boundary conditions (magnetic flux compressors, electromagnetic induction generators, etc.), which are extremely difficult to treat for the hyperbolic diffusion equation.

ACKNOWLEDGMENT

This work was supported by the U. S. Office of Naval Research.

---

[1]H. Minkowski, Goettinger Nachr. 1908, p. 53.

[2]W. F. Hughes and F. T. Young, *Electromagnetodynamics of Fluids* (Wiley, New York, 1966).

[3]P. Penfield and H. A. Haus, *Electrodynamics of Moving Media* (MIT, Cambridge, 1967).

[4]G. Lehner, J. G. Linhart, and T. P. Somon, Nucl. Fusion 4, 362 (1964).

[5]G. Lehner, Ergeb. Exakten Naturwiss. 40, 67 (1968).

[6]H. Matsuo, Phys. Fluids 22, 1618 (1979).

[7]A. Sakurai and T. Takao, J. Phys. Soc. Jpn. 28, 1329 (1970).

[8]W. N. Batygin and I. N. Toptygin, *Problems on Electrodynamics* (Academic, New York, 1964).

[9]L. D. Landau and E. M. Lifshitz, *Electrodynamics of Conducting Media* (Pergamon, Oxford, 1960).

[10]Equations (1)–(4) are derived by integrating Maxwell's equations over the *moving* interface. Transformation of the known boundary conditions for the electromagnetic fields in the frame of reference moving with the velocity of the interface to the laboratory system (Ref. 9) leads to the same formulas. However, the application of the Lorentz transformation to accelerated interfaces is questionable.

[11]H. E. Wilhelm, J. Math. Phys. 14, 1430 (1973).

[12]H. E. Wilhelm and S. H. Choi, Phys. Fluids 21, 1717 (1978).