Lagrange functions of a class of dynamical systems with limit-cycle and chaotic behavior

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It is shown that a class of dissipative dynamical systems with limit-cycle and chaotic behavior can be derived from a Lagrange function. A particular system is investigated numerically.

Recently several authors have studied chaotic states of nonlinear oscillators in an external periodic field.[1–4] Holmes[1] investigated the forced Duffing equation with damping

$$\ddot{x}+\alpha\dot{x}-\beta x+\gamma x^3=f\cos(\Omega t)\ , \tag{1}$$

where α, β, $\gamma>0$. Ω is the frequency of the external periodic field and α is the damping coefficient. For fixed α, β, γ, Ω, he showed that in the phase plane (x,y) (where $\dot{x}=y$), an extremely complicated nonperiodic motion arises for a wide range of moderate f. Equation (1) can also be written as

$$\ddot{x}+\alpha\dot{x}+\frac{dV}{dx}=f\cos(\Omega t)\ , \tag{2}$$

where the potential V is given by $V(x)=\gamma x^4/r-\beta x^2/2$. Thus we find global stability, since $V(x)\rightarrow\infty$ as $|x|\rightarrow\infty$. Huberman and Crutchfield[2] considered the system

$$\ddot{x}+\alpha\dot{x}+\beta x-\gamma x^3=f\cos(\Omega t)\ , \tag{3}$$

where α, β, γ, $f>0$. For fixed α, γ, and f the authors found limit-cycle behavior or chaotic behavior according to the ratio Ω/β^2. But in the present equation of motion there is no global stability, i.e., when we use sufficiently large initial values the solution has an exploding amplitude. This is due to the fact that the potential V is given by

$$V(x)=-\gamma x^4/4+\beta x^2/2\ ,$$

and therefore $V(x)\rightarrow-\infty$ as $|x|\rightarrow\infty$.

Using the Poincaré mapping, Hayashi[3] discussed a special case of the equations of motion given above, namely, $\ddot{x}+\alpha\dot{x}+x^3=f\cos t$. Holmes and Marsden[4] extended the work of Holmes[1] and studied a class of time-periodically perturbed evolution equations whose associated Poincaré map contains a Smale horseshoe.

In the present paper we show that the present evolution equations and extensions, which also show chaotic behavior, can be derived from a Lagrangian. Since we consider dissipative systems, the Lagrangian depends explicitly on time. From the Lagrangian we can calculate a Hamiltonian. Moreover, we study briefly an equation of motion which can be derived from a Lagrangian and which is an extension of the models given above.

Before considering dynamical systems with limit-cycle and chaotic behavior which can be derived from a Lagrangian, we recall a dissipative system without limit-cycle and chaotic behavior which can be derived from a Lagrangian. Later on we extend this dynamical system and include limit-cycle behavior and chaotic behavior.

The equation of motion

$$\ddot{x}+\alpha\dot{x}+\beta x=0 \tag{4}$$

can be derived from the Lagrangian

$$L=\tfrac{1}{2}e^{\alpha t}(\dot{x}^2-\beta x^2)\ . \tag{5}$$

When $\beta>0$ we describe a damped harmonic oscillator. A constant of motion[5] is given by

$$f(x,\dot{x},t)=e^{\alpha t}(\dot{x}^2+\beta x^2+\alpha x\dot{x})\ . \tag{6}$$

The corresponding Hamiltonian is given by

$$H=\tfrac{1}{2}(p^2e^{-\alpha t}+\beta x^2e^{\alpha t})\ . \tag{7}$$

It is worthwhile mentioning that the equation of motion can be derived from a Lagrangian which is independent of time, but this Lagrangian is not globally defined. The more general equation of motion

$$\ddot{x}+\alpha\dot{x}+f(x)=0 \quad (8)$$

can be derived from Lagrangian

$$L=\tfrac{1}{2}e^{\alpha t}[\dot{x}^2-V(x)]\ , \quad (9)$$

where

$$V(x)=\int_0^x f(\xi)d\xi\ . \quad (10)$$

Thus we are able to derive a class of nonlinear damped harmonic oscillators from a Lagrangian. The Lagrangian given by Eq. (9) can be found in the literature.[6] The corresponding Hamiltonian is given by

$$H=\tfrac{1}{2}p^2e^{-\alpha t}+e^{\alpha t}V(x)\ . \quad (11)$$

Consider now the case where the Lagrangian leads to an equation of motion which shows chaotic behavior and limit-cycle behavior. For this purpose we consider the following Lagrangian:

$$L=\tfrac{1}{2}e^{\alpha t}[\dot{x}^2-V(x,t)]\ . \quad (12)$$

The equation of motion is given by

$$\ddot{x}+\alpha\dot{x}+\frac{\partial V(x,t)}{\partial x}=0\ . \quad (13)$$

If we choose

$$V(x,t)=-\frac{\beta x^2}{2}+\frac{\lambda x^4}{4}-fx\cos(\Omega t)\ ,$$

then we obtain the equation of motion studied by Holmes.[2] On the other hand, if we choose

$$V(x,t)=\frac{\beta x^2}{2}-\frac{\lambda x^4}{4}-fx\cos(\Omega t)\ ,$$

then we obtain the equation of motion studied by Huberman and Crutchfield.[2] The corresponding Hamiltonian is given by

$$H=\tfrac{1}{2}p^2e^{-\alpha t}+\tfrac{1}{2}V(x,t)e^{\alpha t}\ . \quad (14)$$

Since H depends explicitly on t it is obvious that H is not a constant of the motion. In general, the dynamical system described by Eqs. (12) or (14) does not possess global constants of motion. This due to the fact that chaotic behavior occurs. However, for the well-known special case with

$$V(x,t)=\frac{\beta x^2}{2}-fx\cos(\Omega t)$$

we find a global constant of motion. This system, however, does not show chaotic behavior, but only limit-cycle behavior.

Let us give a further interesting equation of motion which can be derived from the Lagrangian given above. Consider

$$V(x,t)=-\beta\cos x-fx\cos(\Omega t)\ . \quad (15)$$

Then we find the equation of motion

$$\ddot{x}+\alpha\dot{x}+\beta\sin x=f\cos(\Omega t)\ . \quad (16)$$

Similar to Huberman and Crutchfield[2] we can interpret Eq. (16) as follows: Consider a particle of mass m and charge Q moving in a one-dimensional potential V given by $V(x)=-\beta\cos x$, where x denotes the particle displacement (with respect to a typical length of the system). The particle is subjected to a periodic electric field of frequency Ω. Since the model should describe conduction in a solid, we introduce a phenomenological damping coefficient α. Utilizing these assumptions we find the equation of motion given by Eq. (16), where the mass m is included in the parameters α, β, and f. The parameter f is given by $f=EQ/ma$, where Q is the charge, E the electric field, and a the lattice constant. We have studied Eq. (16) numerically in the phase plane (x,y) where $y=\dot{x}$. This means we have considered the system of differential equations

$$\dot{x}=y\ ,$$
$$\dot{y}=-\alpha y-\beta\sin x-f\cos(\Omega t)\ . \quad (17)$$

The points $(n\pi,0)$ $(n\in Z)$ are critical points of the system (17) when we put $f=0$.

We mention that Huberman *et al.*[7] have solved Eq. (16) using a hybrid digital-analog computer system. They have characterized the chaotic behavior with the help of the power spectrum which is the Fourier transform of the autocorrelation function. We have characterized the chaotic behavior with the help of the Lyapunov exponent.[8] We can see that for the present model both concepts coincide (positive Lyapunov exponent, autocorrelation function decays).

For fixed values of α, β, and Ω we have calculated the trajectories in the phase plane. We have put $\alpha=0.2$, $\beta=1$, and $\Omega=0.8$. For $f=0.5$, 1, and 1.5 we have plotted the phase portraits in Figs. 1–3. For $f=0.5$ (Fig. 1) we find that the trajectories surround the origin (0,0). The system shows limit-cycle behavior. Within the numerical accuracy the Lyapunov exponents χ is given by $\chi=0$. When we increase the amplitude f of the external

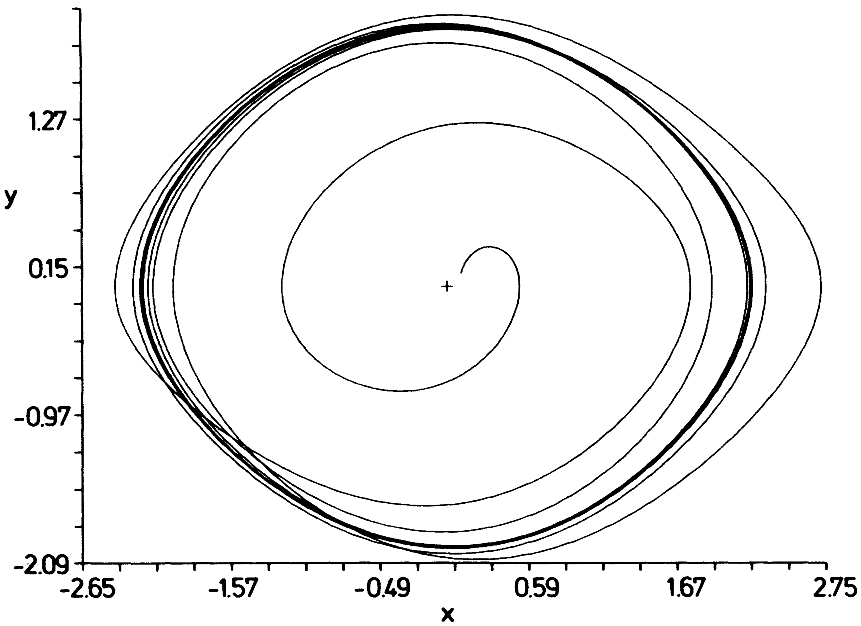

FIG. 1. Trajectories in the phase plane for $f=0.5$ (limit-cycle behavior).

perturbation, we find for $f=1$ that the trajectories surround the points (0,0), $(\pi,0)$, and $(-\pi,0)$. Again we find limit-cycle behavior (Fig. 2). Therefore, the Lyapunov exponent χ is given by $\chi=0$.

When we increase f further we see that the trajectories surround further points of the type $(n\pi,0)$, where $n\in Z$. For $f=1.5$ we find chaotic behavior (Fig. 3). The Lyapunov exponent χ is positive

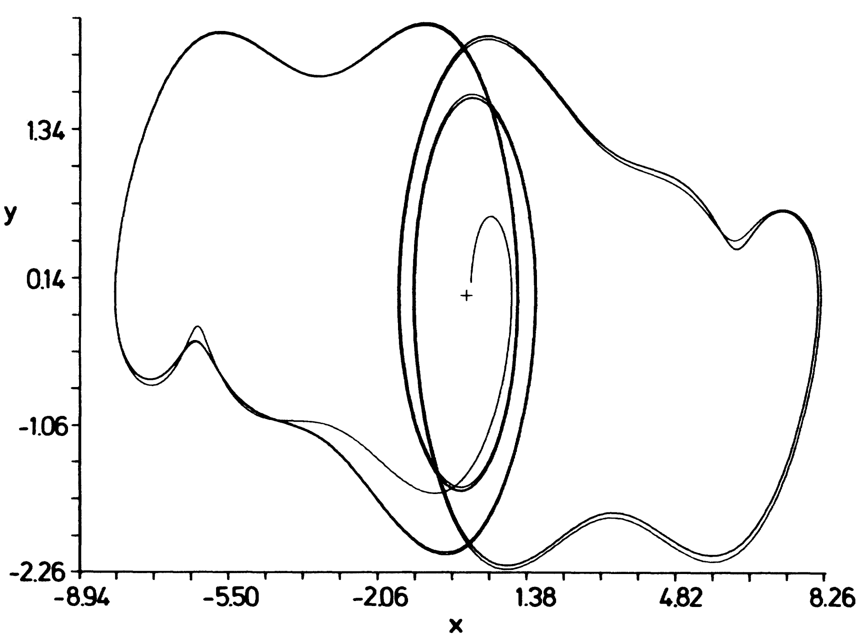

FIG. 2. Trajectories in the phase plane for $f=1$ (limit-cycle behavior).

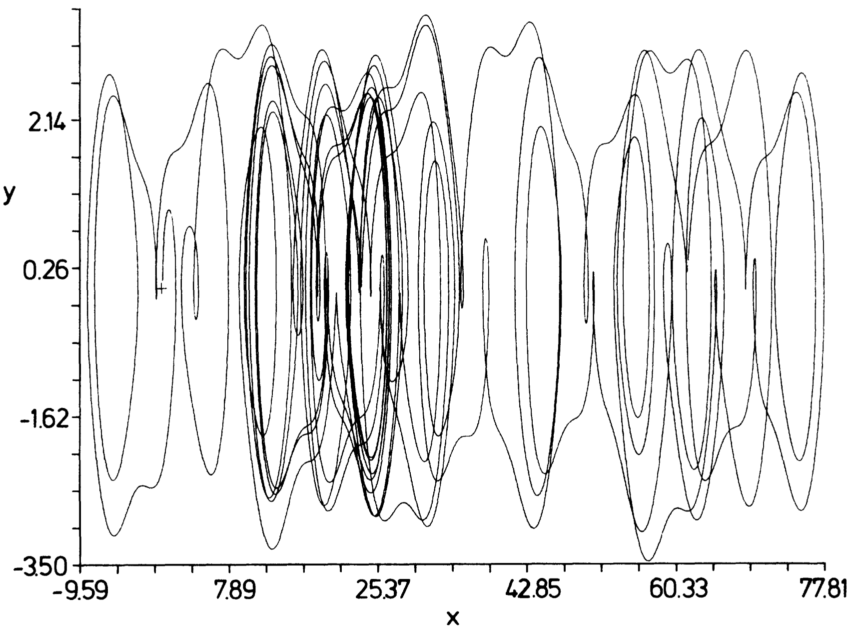

FIG. 3. Trajectories in the phase plane for $f=1.5$ (chaotic behavior).

($\chi=0.16$).

It is consistent with the fact that the system (17) shows chaotic behavior that no global constant of motion exists. Chaotic behavior excludes the existence of a global constant of motion. Even if a Lagrangian (or Hamiltonian) exists for deriving a dissipative system, in general, a global constant of motion does not exist.

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