PHYSICAL REVIEW A VOLUME 25, NUMBER 5 MAY 1982

Renormalization group for intermittency in area-preserving mappings

Albert B. Zisook and Scott J. Shenker
The James Franck Institute, The University of Chicago, 5640 South Ellis Avenue, Chicago, Illinois 60637
(Received 15 December 1981)

An exact solution is found for the functional renormalization-group equations describing intermittency in area-preserving mappings of the plane. Analysis of the solution yields two scaling laws for the duration of laminar flow in between chaotic bursts.

Many authors have discussed a behavior resembling experimentally observed intermittency which arises in one-dimensional maps or recursion relations.[1-3] In this scheme a recursion relation produces long strings of nearly periodic iterations, corresponding to laminar flow, interrupted by bursts of chaotic iterations. The maximum length of the nearly periodic sequences is found to diverge as a power law when a parameter approaches a critical value, at which a pair of cycles form via a tangent bifurcation.[1-3] Recently, Hirsch, Nauenberg, and Scalapino have explained the scaling behavior with an exact, analytic renormalization group.[4] The functional equation used is the same one used to describe period doubling in one- and two-dimensional maps.[5-8]

In another recent paper, a similar intermittent behavior in area-preserving maps is described.[9] The tangent bifurcations in these maps lie in another universality class and have different scaling properties than the one-dimensional ones. In this paper, a renormalization group is used to describe these scaling behaviors.[10]

The treatment begins with the same functional equation used to describe intermittency in one-dimensional maps[4]:

$$\phi(x,y)=L^{-1}\phi\circ\phi L(x,y)\ . \tag{1}$$

Here $\phi(x,y)$ is an area-preserving map of the plane and L is a constant 2×2 matrix. The strategy is to expand around the trivial solution $\phi\equiv$identity. In order to find the matrix L, we demand that the operators $\delta\phi(x,y)=(y,0)$ and $\delta\phi(x,y)=(0,x^2)$ be marginal (are perturbations of the fixed point with eigenvalue one).[11] These operators are being singled out as the physically important ones because they are the leading terms in the generic area-preserving map at a tangent bifurcation.[12] This marginality requirement uniquely specified L (which may be assumed to be diagonal), which must have eigenvalues $\frac{1}{4}$ and $\frac{1}{8}$. Table I contains a listing of all the relevant and marginal eigenvalues and eigenvectors around the trivial solution $\phi\equiv$identity.

To obtain the interesting nontrivial solution, one perturbs the trivial solution in the marginal direction. Beginning with the trial solution

$$\phi(\lambda,x,y)=(x+\lambda y,y+\lambda x^2)$$

one can compute $\phi(\lambda,x,y)$ self-consistently, order by order. The result, up to third order in λ, is

$$\phi(\lambda,x,y)=(f(\lambda,x,y),g(\lambda,x,y))\ ,$$

where

$$f(\lambda,x,y)=x+\lambda y+\frac{\lambda^2x^2}{2}+\frac{\lambda^3xy}{3}\ ,$$

TABLE I. Relevant and marginal perturbations around the trivial solution.

Eigenvalue Λ	Eigenvector $\delta\phi(x,y)$	Significance
16	(0,1)	Produces $(R-R_c)^{-1/4}$ singularity
8	(1,0)	Produces $\lvert A-A_c\rvert^{-1/6}$ singularity
4	$(0,x)$	Equivalent to (0,1) (Ref. 13)
2	$(x,0)$	Violates area preservation
2	$(0,y)$	Violates area preservation
1	$(y,0)$	Marginal
1	$(0,x^2)$	Marginal

$$g(\lambda,x,y)=y+\lambda x^2+\lambda^2 xy+\frac{\lambda^3 x^2}{3}+\frac{\lambda^3 y^2}{3} . \tag{2}$$

One notices a perhaps unexpected symmetry built into the solution. That is,

$$\begin{aligned} f(\lambda,x,y)&=l^2 f(l\lambda,xl^{-2},yl^{-3}) ,\\ g(\lambda,x,y)&=l^3 g(l\lambda,xl^{-2},yl^{-3}) . \end{aligned} \tag{3}$$

This scaling behavior has strong consequences. When applied with ($l=2$) to the original functional equation (1), one obtains the remarkable result that:

$$\phi(\lambda,x,y)\circ\phi(\lambda,x,y)=\phi(2\lambda,x,y) . \tag{4}$$

In addition, one sees that any function $\phi(\lambda,x,y)$ which satisfies both Eqs. (3) and (4) must in fact be a solution of Eq. (1).

Now the solution to the functional equation can be given explicitly. Define $\phi(\lambda,x,y)$ as the point obtained by starting at the initial condition (x,y) and integrating the flow

$$\begin{aligned} \frac{dx}{dt}&=y ,\\ \frac{dy}{dt}&=x^2 , \end{aligned} \tag{5}$$

for $\Delta t=\lambda$ units of time. Equation (4) now states the simple fact that to follow a trajectory for λ seconds twice, is the same as following a trajectory for 2λ seconds. Equation (3) also follows readily as the substitutions $t\to lt$, $x\to xl^{-2}$, and $y\to yl^{-3}$ leave the differential equation (5) invariant. Since $\phi(\lambda,x,y)$ so defined obeys Eqs. (3) and (4), it is the solution of interest to Eq. (1). In fact this solution $\phi(\lambda,x,y)$ obeys a stronger law than Eq. (4):

$$\phi^l(\lambda,x,y)=\phi(l\lambda,x,y) . \tag{6}$$

In combination with the scaling property (3), this implies $\phi(\lambda,x,y)$ obeys the functional equation

$$(L_l)^{-1}\phi^l L_l=\phi , \tag{7}$$

where

$$L_l=\begin{vmatrix} l^{-2} & 0\\ 0 & l^{-3}\end{vmatrix} . \tag{8}$$

One should have expected intermittency to be described by this more general equation (7), as the phenomenon is not associated with any fixed-length period, as in the case of period doubling.

One may use the linearization around the trivial fixed point to explain the scaling behaviors associated with relevant perturbations. The linearization around the nontrivial solution has the same eigenvalues, and hence the same scaling behavior. The eigenvector with the largest eigenvalue, $\Lambda=16$, will correspond to the difference $(R-R_c)$, where R is the control parameter and R_c is the value for which the tangent bifurcation takes place. If one defines $n(R-R_c)$ as the number of iterations required to bring the origin out to some large fixed distance, the renormalization group says

$$2n[16(R-R_c)]=n(R-R_c) . \tag{9}$$

In other words, it takes twice as many iterations with $(R-R_c)$ one-sixteenth as small. The scaling solution is

$$n(R-R_c)=(R-R_c)^{-1/4}\text{const} .$$

In the exceptional case that no $\Lambda=16$ eigenvector is present, the $\Lambda=8$ eigenvector will control the behavior. This $\Lambda=8$ eigenvector is related to the choice of initial conditions when $R=R_c$. The solution curves of Eq. (5) have the form $y=\pm(\frac{2}{3}x^3+A)^{1/2}$, where A corresponds to the choice of initial conditions. Changing A from zero is equivalent to adding the perturbation with eigenvalue $\Lambda=8$ and amplitude $A^{1/2}$. So one must have

$$2n(8|A|^{1/2})=n(|A|^{1/2}) . \tag{10}$$

And hence $n(|A|^{1/2})=|A|^{-1/3}$, or $n(A)=|A|^{-1/6}$. These scaling results are the same as reported earlier.[9]

ACKNOWLEDGMENTS

The authors are grateful to Leo P. Kadanoff for several helpful discussions. They also acknowledge their Robert R. McCormick Fellowships. S. J. S. acknowledges his National Science Foundation Fellowship. Computations were supported by the Materials Research Laboratory Program of the National Science Foundation at the University at the University of Chicago under Grant 79-24007.

[1]P. Manneville and Y. Pomeau, Phys. Lett. 75A, 1 (1979).
[2]P. Manneville and Y. Pomeau, Comm. Math. Phys. 74, 189 (1980)
[3]J. E. Hirsch, B. A. Huberman, and D. J. Scalapino, Phys. Rev. A 25, 519 (1982).
[4]J. E. Hirsch, M. Nauenberg, and D. J. Scalapino, Phys. Lett. 87A, 391 (1982).
[5]M. J. Feigenbaum, J. Stat. Phys. 19, 25 (1978).
[6]M. J. Feigenbaum, J. Stat. Phys. 21, 669 (1979).
[7]J. M. Greene, R. S. MacKay, F. Vivaldi, and M. J. Feigenbaum, Physica D 3, 468 (1981).
[8]P. Collet, J.-P. Eckmann, and H. Koch, Physica D 3, 457 (1981).
[9]A. B. Zisook, Phys. Rev. A 25, 2289 (1982).
[10]Some of the qualitative aspects of these bifurcations have been discussed in the mathematical literature. For a review of codimension two bifurcations see J. Guckenheimer, Santa Cruz report (unpublished).
[11]F. Wegner, *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1976), Vol. 6, p. 8.
[12]See Eq. (2) in Ref. 9.
[13]The perturbation $2\epsilon(0,x)$ is equivalent to the perturbation $-\epsilon^2(0,1)$ under the change of variables $\bar{x}=x+\epsilon$.