# Wave functions for weakly coupled bound states

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An inhomogeneous integral equation has been derived for the bound-state wave functions. Iterative solutions to this equation provide wave functions for bound states of weakly coupled Hamiltonians in one and two dimensions.

## I. INTRODUCTION

It has been observed[1] that a short-range attractive potential always produces a bound state in one or two dimensions. This property has been investigated in the weak-coupling limit, with the intention of obtaining perturbative expressions for the bound-state wave functions and energies. In particular, Hausmann[2] has analyzed the bound states in one-dimensional, short-range potentials by using the homogeneous Lippmann-Schwinger equation for the bound-state wave functions. He showed that for the one-dimensional Hamiltonian

$$H=-\frac{1}{2}\frac{d^2}{dx^2}+\lambda V(x)\ , \tag{1}$$

the ground-state energy $E(\lambda)$ is of the form

$$-\epsilon=\lambda\int dx\, V(x) +\lambda^2\int dx\, dy\, V(x)\,|x-y|\,V(y)+O(\lambda^3)\ , \tag{2}$$

where $\epsilon=[-2E(\lambda)]^{1/2}$, and the wave function is of the form

$$\psi(x)=1+\lambda\int_{-\infty}^{\infty}dx'\,V(x')\,|x-x'|+O(\lambda^2)\ , \tag{3}$$

apart from normalization. Here, $\lambda$ is introduced so as to keep track of the order of the perturbation, and may be taken to be positive without any loss of generality. These results are modified for long-range potentials. For example, it was shown by Blankenbecler, Goldberger, and Simon[3] that if

$$V(x)\to -ax^{-2} \text{ as } x\to\pm\infty,$$

$$-\epsilon=(\lambda+4a\lambda^2\ln\lambda)\int dx\, V(x)+O(\lambda^2)\ . \tag{4}$$

Several other significant results have been obtained by Simon[4] and by Klaus,[5] in particular, about the uniqueness of the bound states, from a rigorous mathematical basis. In contrast, much less is known about the bound state in two dimensions. Apart from the original result of Landau and Lifshitz,[1] the only other results are those obtained by Simon.[4] For the two-dimensional Hamiltonian

$$H=-\frac{1}{2}\left[\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right]+\lambda V(r)\ , \tag{5}$$

Simon showed that if

$$\int |V(r)|^{1+\delta}d^2r<\infty\ ,\qquad \int(1+r^2)^{\delta}|V(r)|\,d^2r<\infty\ ,\quad \delta>0 \tag{6}$$

then a bound state exists for all positive $\lambda$, if and only if

$$\lambda\int V(r)d^2r\le 0\ . \tag{7}$$

No significant results are known about the corresponding wave functions.

In two recent papers,[6,7] it was pointed out that the Noyes form[8] of the $T$ matrix was very convenient for the analysis of the bound states for weakly coupled potentials in one and two dimensions. It was shown that for Hamiltonian (1), the ground-state energy is given by

$$-\epsilon=\lambda\int dx\,V(x)+\lambda^2\int dx\,dy\,V(x)V(y)\,|x-y|$$
$$+\tfrac{1}{6}\lambda^3\int dx\,dy\,dz\,V(x)V(y)V(z)(\,|x-y|+|y-z|+|z-x|\,)^2$$
$$+\tfrac{1}{6}\lambda^4\int dx\,dy\,dz\,dt\,V(x)V(y)V(z)V(t)(\,|x-y|^3+6|x-y|^2|x-z|+3|x-y|^2|z-t|$$
$$+6|x-y|\,|y-z|\,|z-t|\,)+O(\lambda^5)\,, \quad (8)$$

and for the Hamiltonian (5) one has an asymptotic series

$$\ln[-E(\lambda)]\simeq\frac{2\pi}{\lambda V_o}\left[1-\lambda\frac{C}{\pi}V_0+\sum_{n=1}^{\infty}\lambda^n D_n\right],\quad \lambda\to 0 \quad (9)$$

where

$$V_0=\int V(r)d^2r\,, \quad (10)$$

$$D_1=-\int\frac{d^2r_1d^2r_2}{V_0}V(r_1)P(r_{12})V(r_2)\,,$$

$$D_n=-\int\frac{d^2r_1\cdots d^2r_{n+1}}{V_0^n}V(r_1)P(r_{12})W(r_2,r_3)W(r_3,r_4)\cdots W(r_n,r_{n+1})V(r_{n+1})\,, \quad (11)$$

$$P(r_{ij})=\frac{1}{2\pi}\ln\left[\frac{|\vec{r}_i-\vec{r}_j|^2}{2}\right], \quad (12)$$

$$W(r_n,r_{n+1})=\frac{1}{\pi}V(r_n)\left[V_o\ln|\vec{r}_n-\vec{r}_{n+1}|-\int d^2r'V(r')\ln|\vec{r}\,'-\vec{r}_{n+1}|\right],$$

and $C\approx 0.5772$ is the Euler constant.

In the present paper, we complement the above results by obtaining the corresponding wave functions. In obtaining the bound-state wave functions, one encounters two difficulties: Firstly, the wave function satisfies a homogeneous integral equation which does not directly yield iterative, perturbative solutions. Secondly, solutions of the type given in Eq. (3) are not truly perturbative since the perturbation terms are unbounded. We overcome the first difficulty by converting the homogeneous equation into an inhomogeneous equation which allows us to obtain perturbative solutions by iteration. The second difficulty is overcome by essentially separating out the asymptotic behavior and then carrying out perturbative expansions in terms of bounded functions. The results discussed are valid for short-range potentials, though modifications required by the special case of $V(x)\sim -x^{-2}$ are also considered.

## II. GENERAL RESULTS

We begin with the Lippmann-Schwinger equation for the state $|\psi\rangle$,

$$|\psi\rangle=|q\rangle+\int\frac{dk^n}{(2\pi)^n}\frac{|k\rangle\langle k|\lambda V|\psi\rangle}{E-\frac{1}{2}k^2+i\eta}\,, \quad (13)$$

where $E=\frac{1}{2}q^2$, and the corresponding equation for the $T$ matrix,

$$\langle p|T|q\rangle\equiv\langle p|\lambda V|\psi\rangle$$
$$=\langle p|\lambda V|q\rangle+\int\frac{dk^n}{(2\pi)^n}\frac{\langle p|\lambda V|k\rangle\langle k|T|q\rangle}{E-\frac{1}{2}k^2+i\eta}\,. \quad (14)$$

For analyzing the bound states, it is convenient to use the Noyes form[8] of the equation which is obtained by writing

$$\langle k \mid T \mid q \rangle = f(k,q)\langle q \mid T \mid q \rangle ,$$

where $f(q,q)=1$. This leads to

$$\langle q \mid T \mid q \rangle = \frac{\langle q \mid \lambda V \mid q \rangle}{1 - \int \frac{dk^n}{(2\pi)^n} \frac{\langle q \mid \lambda V \mid k \rangle f(k,q)}{E - \frac{1}{2}k^2 + i\eta}} , \tag{15}$$

with $f(k,q)$ satisfying the integral equation

$$f(k,q) = \frac{\langle k \mid \lambda V \mid q \rangle}{\lambda V_0} + \int \frac{dk'^n}{(2\pi)^n} \left[ \langle k \mid \lambda V \mid k' \rangle - \frac{\langle k \mid \lambda V \mid q \rangle \langle q \mid \lambda V \mid k' \rangle}{\lambda V_0} \right] \frac{f(k',q)}{E - \frac{1}{2}k'^2 + i\eta} , \tag{16}$$

where $V_0 = \langle q \mid \lambda V \mid q \rangle$. The state $\mid \psi \rangle$ can then be written as

$$\mid \psi \rangle = \mid q \rangle + \langle q \mid T \mid q \rangle \int \frac{dk^n}{(2\pi)^n} \frac{\mid k \rangle f(k,q)}{E - \frac{1}{2}k^2 + i\eta} . \tag{17}$$

The term which is of main interest, is the second term which apart from an overall constant is given by

$$\mid \phi \rangle = \int \frac{dk^n}{(2\pi)^n} \frac{\mid k \rangle f(k,q)}{E - \frac{1}{2}k^2 + i\eta} . \tag{18}$$

Substituting expression (16) for $f(k,q)$, we get

$$\mid \phi \rangle = \frac{1}{\lambda V_0} \int \frac{dk^n}{(2\pi)^n} \frac{\mid k \rangle \langle k \mid \lambda V \mid q \rangle}{E - \frac{1}{2}k^2 + i\eta} + \int \frac{dk^n}{(2\pi)^n} \frac{\mid k \rangle \langle k \mid \lambda V \mid \phi \rangle}{E - \frac{1}{2}k^2 + i\eta}$$

$$- \int \frac{dk^n}{(2\pi)^n} \frac{\mid k \rangle \langle k \mid \lambda V \mid q \rangle \langle q \mid \lambda V \mid \phi \rangle}{\lambda V_0 (E - \frac{1}{2}k^2 + i\eta)} . \tag{19}$$

One can now go over to the coordinate representation to get

$$\phi(r) = \int d^n r' g(\mid \vec{r} - \vec{r}' \mid) \frac{V(r')}{V_0} e^{i\vec{q}\cdot\vec{r}'}$$

$$+ \lambda \int d^n r' \left[ g(\mid \vec{r} - \vec{r}' \mid) - \int d^n r'' g(\mid \vec{r} - \vec{r}'' \mid) \frac{V(r'')}{V_0} e^{i\vec{q}\cdot(\vec{r}'' - \vec{r}')} \right] V(r')\phi(r') , \tag{20}$$

where $n$ is the dimensionality of the space ($n=1$, 2, or 3), and the Green's function $g(R)$ is given by

$$g(R) = \int \frac{d^n k}{(2\pi)^n} \frac{e^{i\vec{k}\cdot\vec{R}}}{(E - \frac{1}{2}k^2 + i\eta)} . \tag{21}$$

One can also write $\langle q \mid T \mid q \rangle$ in terms of $\phi(r)$ as

$$\langle q \mid T \mid q \rangle = \frac{\langle q \mid \lambda V \mid q \rangle}{1 - \lambda \int d^n r' e^{-i\vec{q}\cdot\vec{r}'} V(r')\phi(r')} . \tag{22}$$

These relations [(20) and (22)] allow us to write the complete expression for the total wave function $\psi(r)$.

The bound states correspond to the poles of $\langle q \mid T \mid q \rangle$ or the zeros of the denominator in Eq. (22), i.e.,

$$\lambda \int d^n r\, e^{-i\vec{q}\cdot\vec{r}} V(r)\phi(r) = 1 , \tag{23}$$

and their wave functions are given by $\phi(r)$ which satisfies the inhomogeneous integral equation (20). It will be shown that the bound-state wave functions in one and two dimensions can be obtained iteratively from Eq. (20), and their energies will be determined from condition (23). In three dimensions, though there are, in general, no bound states in the weak-coupling limit, Eq. (20) may be useful in obtaining low-energy states.

## III. BOUND STATES IN ONE DIMENSION

The Green's function in one dimension is

$$g(|x|)=\frac{1}{iq}e^{iq|x|}\ , \tag{24}$$

where $q=(2E)^{1/2}$. However, we are still not in a position to expand either the Green's function or the bound-state wave function $\phi(x)$ in powers of $\lambda$ since the coefficients will be unbounded functions. To overcome this difficulty, we note that $\phi(x)\sim e^{iq|x|}$ for $|x|\to\infty$, and therefore one may consider instead an expansion for

$$t(x)=iqe^{-iq|x|}\phi(x)\ . \tag{25}$$

Substituting this expression in Eq. (20), we get

$$t(x)=\int dx' e^{iq(|x-x'|-|x|+x')}\frac{V(x')}{V_0}+\frac{\lambda}{iq}\int dx'\left[e^{iq(|x-x'|-|x|+x')}-\int dx'' e^{iq(|x-x''|-|x|+x'')}\frac{V(x'')}{V_0}\right] \times e^{iq(|x'|-x')}V(x')t(x')\ , \tag{26}$$

where

$$V_0=\int dx\, V(x)\ . \tag{27}$$

Perturbative solutions can now be obtained by iteration. Iterating to first order in $\lambda$ and retaining terms of order $(iq)^2$ and $iq\lambda$, we get

$$\begin{aligned} t(x)=1&+iq\int dx'(|x-x'|-|x|+x')\frac{V(x')}{V_0}+\frac{(iq)^2}{2}\int dx'(|x-x'|-|x|+x')^2\frac{V(x')}{V_0} \\ &+\lambda(iq)\int dx'\left[(|x-x'|-|x|+x')-\int dx''(|x-x''|-|x|+x'')\frac{V(x'')}{V_0}\right] \\ &\times V(x')\left[(|x'|-x')+\int dx'''(|x'-x'''|-|x'|+x''')\frac{V(x''')}{V_0}\right]+\cdots\ . \end{aligned} \tag{28}$$

The first two terms can be used in Eqs. (25) and (23) to get

$$iq=\lambda V_0+\lambda^2\int dx\, dx' V(x)|x-x'|V(x')+O(\lambda^3)\ , \tag{29}$$

where $iq=-(-2E)^{1/2}$, in terms of which the expression for $t(x)$ becomes

$$\begin{aligned} t(x)=1&+\lambda\int dx'(|x-x'|-|x|+x')V(x')+\tfrac{1}{2}\lambda^2 V_0\int dx'(|x-x'|-|x|+x')^2 V(x') \\ &+\lambda^2\int dx'(|x-x'|-|x|+x')V(x')\int dx''(|x'-x''|-x'+x'')V(x'')+O(\lambda^3)\ . \end{aligned} \tag{30}$$

The corresponding wave function is obtained from Eq. (25):

$$\phi(x)=\frac{1}{iq}e^{iq|x|}t(x)\ . \tag{31}$$

It is observed that for short-range potentials, the coefficients of $\lambda^n$ are bounded functions.

The solutions for $t(x)$ given in Eq. (30) require modifications for potentials with long range. We will consider only the special case of

$$V(x)\to -a|x|^{-2}\ \text{for}\ |x|\to\infty\ . \tag{32}$$

Going back to Eq. (26), the leading term for $t(x)$ is

$$t(x)=\int_{-\infty}^{\infty} dx' e^{iq(|x-x'|-|x|+x')}\frac{V(x')}{V_0}+O(\lambda)\ , \tag{33}$$

which leads to[6]

$$t(x)=1+iq\int_{-1/\lambda}^{1/\lambda}(|x-x'|-|x|+x')\frac{V(x')}{V_0}dx'+O(\lambda)\ . \tag{34}$$

The corresponding energy is given by $iq=-(-2E)^{1/2}$,

$$iq=\lambda V_0+4\lambda^2(\ln\lambda)a\int dx\ V(x)+O(\lambda^2)\ . \tag{35}$$

This agrees with the result obtained by Blankenbecler *et al.*[3]

## IV. BOUND STATES IN TWO DIMENSIONS

The Green's function in two dimensions is given by[7]

$$g(|\vec{r}_i-\vec{r}_j|)=-\frac{1}{\pi}K_0(2x^{1/2})$$

$$=\frac{1}{\pi}\left[\frac{1}{2}(\ln x)\sum_{n=0}^{\infty}\frac{x^n}{(n!)^2}-\sum_{n=0}^{\infty}\frac{x^n}{(n!)^2}\psi(n+1)\right]\ , \tag{36}$$

where $K_0$ is the Bessel function of the third kind for an imaginary variable,[9]

$$x=\frac{1}{2}|E|(\vec{r}_i-\vec{r}_j)^2\ ,$$
$$\psi(x)=\frac{1}{\Gamma(x)}\frac{d}{dx}\Gamma(x)\ . \tag{37}$$

In order to simplify $g(R), R=|\vec{r}_i-\vec{r}_j|$, one notes the important property[1,4] that $E(\lambda)$ is exponentially small for $\lambda\to 0$, i.e., $E(\lambda)\sim\exp(-1/\lambda)$. We then write

$$g(R)=\frac{1}{\pi}e^{-\epsilon R}[\ln(\tfrac{1}{2}\epsilon R)-\psi(1)]-\frac{1}{\pi}\{e^{-\epsilon R}[\ln(\tfrac{1}{2}\epsilon R)-\psi(1)]+K_0(\epsilon R)\}\ , \tag{38}$$

where $\epsilon=(2|E|)^{1/2}$, and $-\psi(1)=0.5772$ is the Euler constant. Now consider the case where $r_j$ is finite. Then the second term is of the order of $\exp(-1/\lambda)$ for $r_i\leq 1$. For $r_i\geq 1$, since the derivative of this term is bounded, we can replace $R$ by $r_i$ with errors of order $\exp(-1/\lambda)$. Therefore for $r_j$ finite, we can write

$$g(R)=\frac{1}{\pi}e^{-\epsilon R}[\ln(\tfrac{1}{2}\epsilon R)-\psi(1)]-\frac{1}{\pi}\{e^{-\epsilon r_i}[\ln(\tfrac{1}{2}\epsilon r_i)-\psi(1)]+K_0(\epsilon r_i)\}+O(e^{-1/\lambda})\ . \tag{39}$$

We can similarly replace $R$ by $r_i$ in the exponent of the first term, introducing errors of order $\exp(-1/\lambda)$, so that we finally get for $r_j$ finite,

$$g(R)=\frac{1}{\pi}e^{-\epsilon r_i}\ln\left[\frac{R}{r_i}\right]-\frac{1}{\pi}K_0(\epsilon r_i)+O(e^{-1/\lambda})\ . \tag{40}$$

In order that the expansion of the wave function is in terms bounded functions, we take out the asymptotic behavior of $\phi(r)$ by defining

$$t(r)=e^{\epsilon r}\phi(r)\ . \tag{41}$$

Substituting Eqs. (40) and (41) into Eq. (20), one gets

$$t(r_1)=-\frac{1}{\pi}[\ln r_1+e^{\epsilon r_1}K_0(\epsilon r_1)]+\frac{1}{\pi}\int d^2r_2(\ln|\vec{r}_1-\vec{r}_2|)\frac{V(r_2)}{V_0}$$
$$+\frac{\lambda}{\pi}\int d^2r_2\left[\ln\frac{|\vec{r}_1-\vec{r}_2|}{r_1}-\int d^2r_3\left[\ln\frac{|\vec{r}_1-\vec{r}_3|}{r_1}\right]\frac{V(r_3)}{V_0}\right]V(r_2)t(r_2)+O(e^{-1/\lambda})\ . \tag{42}$$

We iterate the equation, and omitting terms of the order $\exp(-1/\lambda)$, get the asymptotic series

$$t(r_1)=-\frac{1}{\pi}[\ln r_1+e^{\epsilon r_1}K_0(\epsilon r_1)]+P(r_1)+\lambda\int d^2r_2Q(r_1,r_2)P(r_2)$$
$$+\lambda^2\int d^2r_2Q(r_1,r_2)\int d^2r_3Q(r_2,r_3)P(r_3)+\cdots+\lambda^n\int d^2r_2Q(r_1,r_2)\cdots$$
$$\times\int d^2r_{n+1}Q(r_n,r_{n+1})P(r_{n+1})+\cdots\,, \tag{43}$$

where

$$P(r_n)=\frac{1}{\pi}\int d^2r''(\ln|\vec{r}_n-\vec{r}\,''|)\frac{V(r'')}{V_0}\,,$$
$$Q(r_n,r_{n+1})=\frac{1}{\pi}\left[\ln|\vec{r}_n-\vec{r}_{n+1}|-\int d^2r'(\ln|\vec{r}_n-\vec{r}\,'|)\frac{V(r')}{V_0}\right]V(r_{n+1})\,. \tag{44}$$

The bound-state wave function can then be obtained from Eq. (41). It is important to observe that the functions which are coefficients of $\lambda^n$, $n>0$, are not only bounded but go to zero as $r_1\to\infty$. This expression for $t(r)$ can be used in Eqs. (41) and (23), leading to the asymptotic series in Eq. (9) for $\ln(-E)$.

Finally we remark that, in general, there are no bound states in the weak-coupling limit for the three-dimensional problem. However, the inhomogeneous integral equation (20) along with the bound-state condition (23), may be quite useful in obtaining solutions for moderate values of $\lambda$ since the kernel is a difference of two terms and therefore the iterative series may be expected to converge fairly rapidly.

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