Precision spectroscopy of a charged particle in an imperfect Penning trap

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The phenomenal accuracies achieved for the spectroscopy of single charged particles suspended in Penning traps has prompted this study of the imperfect Penning trap. The principal result is a new prescription for the cyclotron frequency in terms of the observable eigenfrequencies of the imperfect trap. The new prescription is completely insensitive to a misalignment of the magnetic field direction with the axis of the Penning electrodes, and it is also insensitive to the most significant imperfections in the electrostatic potential. These systematic effects can therefore be completely circumvented in measurements of the anomalous magnetic moments of the electron and positron, and also in experiments on protons and heavier ions where the effects are much larger.

A single, essentially free electron was first trapped[1] in a Penning trap at the University of Washington in 1973. Subsequent refinements[2] produced a measurement of the magnetic moment anomaly of the electron[3] and positron[4] to an accuracy of 5×10^{-8}. The measurements of the anomalies provide a rigorous test of quantum electrodynamics. Comparison of the magnetic moments of the e^- and e^+ provide a test of *CPT* invariance with an unprecedented precision. Experiments are now underway whose ultimate goals are to trap a single proton[5] and to improve the e^- and e^+ anomaly measurements to 10^{-9} or better.[6] There is a prospect of measuring the cyclotron frequencies of heavier ions to accuracies of 10^{-12}.

This unprecedented accuracy has led us to study the small systematic corrections which result from imperfections in the Penning trap. Our major result is a prescription for determining the cyclotron frequency of the charged particle in terms of the observable eigenfrequencies of the imperfect trap. The new prescription is completely insensitive to what is perhaps the major imperfection of a real Penning trap—the misalignment of the magnetic field direction with the symmetry axis of the trap electrodes. The new prescription is also insensitive to the lowest-order imperfections in the electrostatic potential. Higher-order electrostatic imperfections are intrinsically less important and can either be tuned out[7] or are very small because of the high degree of symmetry maintained in precision Penning traps. We thus show that a Penning trap which is both misaligned internally and with respect to the magnetic field direction, whose electrodes depart from ideal geometry, with the surfaces of these electrodes having patch effects, can still be used to measure the cyclotron frequency of a charged particle to extremely high precision. Such systematic effects can thereby be eliminated in the determination of the anomalous magnetic moments of the electron[3] and positron[4] and in the determination of the electron-proton mass ratio.[5]

Our results do, however, assume that the magnetic field is perfectly homogeneous. A homogeneous field is realized to great extent in an experiment underway by using a state-of-the-art NMR magnet and carefully selecting the electrode materials.[6] Experiments to date have relied upon sideband cooling of the charged particle to the center of a magnetic bottle.[3] The treatment of magnetic field inhomogeneities requires a method which differs from that employed here.

Consider a particle of charge q and mass m which is bound in a Penning trap. The electrodes produce, to a good approximation, the ideal potential

$$V=\text{const}[z^2-\tfrac{1}{2}(x^2+y^2)] \quad . \tag{1}$$

Imperfections in the electrodes contribute a small additional potential which can be expanded in an infinite series of spherical harmonics $Y_l^m(\theta,\phi)$ multiplied by $(r/z_0)^l$. Here the radial coordinate r is scaled by a characteristic trap dimension z_0 so that the numerical coefficients in this series are of order unity. For the charged particle r/z_0 is typically smaller than 10^{-2}. In this region the series converges rapidly and terms with $l\geq6$ can be neglected. The constant contribution with $l=0$ is not observable and can be ignored. Spatially uniform electric fields represented by $l=1$ are normally very small because the Penning electrodes are constructed to be symmetric under the inversion $\vec{r}\to-\vec{r}$. Moreover, this constant electric field also has no observable effect, for we may place the coordinate origin at the (metastable) equilibrium point of the potential, thereby removing any terms which are linear in the coordinates. The inversion symmetry makes the $l=5$ contribution negligible. The terms which behave as $(r/z_0)^3$ and $(r/z_0)^4$ produce anharmonicity. The $l=4$ contributions dominate because of the inversion symmetry. Both anharmonicities can be tuned out to a great extent by the use of compensation electrodes.[7]

The potential we must thus consider for the imperfect Penning trap is entirely harmonic. It consists of the ideal potential of Eq. (1) plus harmonic imperfections because of departures from ideal geometry in the trap electrodes (whose surfaces may also have patch effects) and misalignments of these electrodes with respect to each other. To exploit an analogy with the familiar inertia tensor, we write the potential as

$$V=\frac{1}{2}\sum_{k,l=1}^{3}A_{kl}x_kx_l \quad , \tag{2}$$

where $x_1=x$, $x_2=y$, $x_3=z$, and A_{kl} is a symmetric matrix. Laplace's equation requires that A_{kl} be traceless. A rotation of the coordinate axes produces the transformation $A\rightarrow RAR^{-1}$, where R is an orthogonal matrix. We use such a rotation to work in the "principal-axis coordinate system" where RAR^{-1} is diagonal but still, of course, traceless. In this coordinate system the potential energy U has the form

$$U=\frac{1}{2}m\omega_z^2[z^2-\frac{1}{2}(x^2+y^2)-\frac{1}{2}\epsilon(x^2-y^2)] \quad . \tag{3}$$

We see that the harmonic imperfections are represented in a completely general way by the single asymmetry parameter ϵ. Projections of equipotentials upon the xy plane are thus elliptical. For small ϵ this asymmetry parameter is the fractional difference in length of the principal axes of these ellipses.

When the magnetic field $\vec{B}$ is aligned perfectly along the z axis, the motion along this axis is uncoupled from the motion in the xy plane. The overall multiplicative constant in Eq. (3) has been chosen such that ω_z is the axial harmonic osciallation frequency in this ideal limit. We shall treat the general case with a misaligned magnetic field given by

$$B_z=B\cos\theta \quad ,$$
$$B_x=B\sin\theta\cos\phi \quad , \tag{4}$$
$$B_y=B\sin\theta\sin\phi \quad ,$$

in the principal-axes coordinate system. The cyclotron frequency ω_c, the frequency in the absence of the Penning electrodes, is given by

$$\omega_c=qB/mc \quad . \tag{5}$$

Three coupled, linear second-order differential equations of motion are easily deduced by computing the Lorentz force and using Newton's second law. Assuming a time dependence of the form $\exp(-i\omega t)$ yields a set of three homogeneous algebraic equations with the determinant

$$F(\omega^2)=\begin{vmatrix}\omega^2+\frac{1}{2}\omega_z^2(1+\epsilon) & -i\omega\omega_c\cos\theta & +i\omega\omega_c\sin\theta\sin\phi\\ i\omega\omega_c\cos\theta & \omega^2+\frac{1}{2}\omega_z^2(1-\epsilon) & -i\omega\omega_c\sin\theta\cos\phi\\ -i\omega\omega_c\sin\theta\sin\phi & i\omega\omega_c\sin\theta\cos\phi & \omega^2-\omega_z^2\end{vmatrix} \quad . \tag{6}$$

This set of equations has a solution only if the frequency is an eigenfrequency $\bar{\omega}$ determined by the characteristic equation

$$F(\bar{\omega}^2)=0 \quad . \tag{7}$$

Therefore, the determinant (6) is a cubic polynomial of the form

$$F(\omega^2)=(\omega^2-\bar{\omega}_c^2)(\omega^2-\bar{\omega}_z^2)(\omega^2-\bar{\omega}_m^2) \quad , \tag{8}$$

where $\bar{\omega}_c$, $\bar{\omega}_z$, and $\bar{\omega}_m$ are, respectively, the observable cyclotron, axial, and magnetron frequencies of the misaligned trap. The determinant (6) is easily calculated. Expanding the result in the powers ω^0, ω^2, ω^4 and comparing the coefficients with the corresponding expansion of Eq. (8) we find that[8]

$$\bar{\omega}_c^2\bar{\omega}_z^2\bar{\omega}_m^2=\frac{1}{4}\omega_z^6(1-\epsilon^2) \quad , \tag{9}$$

$$\begin{aligned}\bar{\omega}_c^2\bar{\omega}_z^2+\bar{\omega}_c^2\bar{\omega}_m^2+\bar{\omega}_z^2\bar{\omega}_m^2 &=\omega_c^2\omega_z^2(1-\tfrac{3}{2}\sin^2\theta-\tfrac{1}{2}\epsilon\sin^2\theta\cos2\phi)\\ &\quad-\tfrac{3}{4}\omega_z^4(1+\tfrac{1}{3}\epsilon^2) \quad ,\end{aligned} \tag{10}$$

and

$$\bar{\omega}_c^2+\bar{\omega}_z^2+\bar{\omega}_m^2=\omega_c^2 \quad . \tag{11}$$

Equation (11) is our principal result. It gives an exact prescription for obtaining the cyclotron frequency ω_c in the absence of the Penning electrodes in terms of the measurable eigenfrequencies of an imperfect trap. This new prescription is completely independent of the misalignment angles (θ,ϕ) and the distortion parameter ϵ. As mentioned earlier, of the potential electrostatic imperfections, only those which behave as $(r/z_0)^3$ and $(r/z_0)^4$ could affect the prescription, but these anharmonic terms can be tuned out.

In present and proposed single-particle spectroscopy experiments, the cyclotron frequency ω_c is determined by the measurement of $\bar{\omega}_c$ to great precision with $\bar{\omega}_z$ and $\bar{\omega}_m$ measured with lesser accuracy. In some cases $\bar{\omega}_m$ need not be measured. This is possible because typically the Penning electrodes contribute only a weak perturbation to the cyclotron motion of the charged particle in a strong magnetic field, giving

$$\bar{\omega}_c^2 >> \bar{\omega}_z^2 >> \bar{\omega}_m^2 \ . \tag{12}$$

In this case, a simple expansion of Eq. (11) suffices. Recall that the axial motion is uncoupled for a perfect trap. In this limit $\bar{\omega}_z = \omega_z$, and Eq. (9) gives $\bar{\omega}_m = \tilde{\omega}_m$, where

$$\tilde{\omega}_m \equiv \bar{\omega}_z^2 / 2\bar{\omega}_c \ . \tag{13}$$

[This result yields the second inequality in Eq. (12) from the first inequality.] Thus we add and subtract $\tilde{\omega}_m^2$ on the left-hand side of Eq. (11) and expand in the small quantity $(\bar{\omega}_m^2 - \tilde{\omega}_m^2)$ to derive

$$\frac{\omega_c}{\bar{\omega}_c} \simeq 1 + \frac{1}{2}\left(\frac{\bar{\omega}_z}{\bar{\omega}_c}\right)^2 + \frac{1}{8}\left(\frac{\bar{\omega}_z}{\bar{\omega}_c}\right)^4 \left[\left(\frac{\bar{\omega}_m}{\tilde{\omega}_m}\right)^2 - 1\right] \ . \tag{14}$$

The first correction in Eq. (14) is due to the slight weakening of the cyclotron motion centripetal force by the "hill" of the electrostatic potential of Eq. (3). It is precisely the well-known correction obtained for an ideal Penning trap except that the observed frequencies are used. We find that the correction for trap imperfections enters in at the much higher order $(\bar{\omega}_z/\bar{\omega}_c)^4$. It suffices to work to leading order in the small ratio $(\bar{\omega}_z/\bar{\omega}_c)^2$ to compute this second correction in Eq. (14). In this order $\bar{\omega}_c^2 \simeq \omega_c^2$, and Eq. (10) reduces to the statement

$$\bar{\omega}_z^2 \simeq \omega_z^2 [1 - \tfrac{3}{2} \sin^2\theta (1 + \tfrac{1}{3}\epsilon \cos 2\phi)] \ . \tag{15}$$

Equation (9) now informs us that

$$\bar{\omega}_m \simeq \tilde{\omega}_m (1 - \epsilon^2)^{1/2} [1 - \tfrac{3}{2} \sin^2\theta (1 + \tfrac{1}{3}\epsilon \cos 2\phi)]^{-3/2} \ , \tag{16}$$

and thus for small imperfections with $\theta << 1$ and $|\epsilon| << 1$,

$$\frac{\omega_c}{\bar{\omega}_c} = 1 + \frac{1}{2}\left(\frac{\bar{\omega}_z}{\bar{\omega}_c}\right)^2 + \frac{9}{16}\left(\frac{\bar{\omega}_z}{\bar{\omega}_c}\right)^4 (\theta^2 - \tfrac{2}{9}\epsilon^2) \ . \tag{17}$$

Either Eq. (15) or (16) can be used to study the imperfections described by θ, ϕ, and ϵ and to thereby align Penning traps. The magnetron frequency $\bar{\omega}_m$ has been observed in a variety of traps[9] to be larger than $\tilde{\omega}_m$. This is in accord with Eq. (16) and the expectation that the angular misalignment of the magnetic field, θ, is larger than the asymmetry in the Penning electrodes, $|\epsilon|$.

The imperfection corrections will probably be completely negligible for trapped electron experiments, with the consequence that the magnetron frequency $\bar{\omega}_m$ need not be measured. For example, in a current[6] experiment, $\bar{\omega}_c/2\pi = 164$ GHz and $\bar{\omega}_z/2\pi = 60$ MHz, and the ratio $(\bar{\omega}_z/\bar{\omega}_c)^4$ is approximately 10^{-14}. Even for an angular misalignment as large as $\theta = 1°$ or for an electrode asymmetry as large as $|\epsilon| = 1\%$, the net correction in Eq. (17) is only on the order of 10^{-18}.

The imperfection corrections become much more significant for more massive particles, which have much lower cyclotron frequencies. For example, the frequency ratio $(\bar{\omega}_z/\bar{\omega}_c)^4$ is 3×10^{-4} for the recent proton experiment,[5] where $\bar{\omega}_c/2\pi = 76$ MHz and $\bar{\omega}_z/2\pi = 10$ MHz. In this case, for $\theta = 1°$ or $|\epsilon| = 1\%$, the second correction in Eq. (17) is about 5×10^{-8}, an important correction compared to the goal of an accuracy of 10^{-9}. Our prescription for the cyclotron frequency in terms of the measurable eigenfrequencies given in Eq. (11) should thus be very useful for such experiments.

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[8]We consider only the case of experimental interest where Eq. (7) has three positive real roots. For very large θ the motion is unstable. This is evidenced by Eq. (10), whose right-hand side becomes negative for large θ.

[9]This has been observed in the molybdenum traps of Refs. 3 and 5 [R. S. Van Dyck (private communication)] and in the copper trap used in Ref. 6 [G. Gabrielse (unpublished)].