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Solutions to the time-dependent Schrödinger equation

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We discuss the extension of the Lewis-Riesenfeld technique of solving the time-dependent Schrödinger equation to cases where the invariant has continuous eigenvalues. The extension is carried out for a general Ermakov system. We also discuss extensions to N-dimensional systems and the calculation of the propagator for a general Ermakov system.

I. INTRODUCTION

Some years ago Lewis and Riesenfeld[1] solved the Schrödinger equation for the time-dependent harmonic oscillator

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega^2(t)q^2\,, \tag{1.1}$$

by making use of an explicitly time-dependent constant of the motion (or time-dependent invariant) I,

$$I=\tfrac{1}{2}(xp-\dot{x}q)^2+\frac{k}{2}\left[\frac{q}{x}\right]^2,\quad k=\text{const} \tag{1.2}$$

where $x(t)$ is a c-number solution to the auxiliary equation

$$\ddot{x}+\omega^2(t)x=k/x^3\,. \tag{1.3}$$

The Lewis invariant (1.2) satisfies

$$\frac{dI}{dt}=\frac{1}{i\hbar}[I,H]+\frac{\partial I}{\partial t}=0\,, \tag{1.4}$$

and has eigenvalues

$$I\psi_n(q,t)=\lambda_n\psi_n(q,t)\,,\qquad \lambda_n=\text{const}\,. \tag{1.5}$$

In a recent paper[2] we extended the Lewis-Riesenfeld solution technique to a general Ermakov system with Hamiltonian

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega^2(t)q^2+\frac{1}{x^2}f\left[\frac{q}{x}\right], \tag{1.6}$$

having Ermakov invariant

$$I=\tfrac{1}{2}(xp-\dot{x}q)^2+\frac{k}{2}\left[\frac{q}{x}\right]^2+f\left[\frac{q}{x}\right], \tag{1.7}$$

where x satisfies the same auxiliary equation (1.3), and $f(q/x)$ is an arbitrary function. The invariant (1.7) also, of course, satisfies (1.4) and (1.5). The Schrödinger equation for the Ermakov Hamiltonian (1.6) is

$$\left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2}+\tfrac{1}{2}\omega^2(t)q^2+\frac{1}{x^2}f\left[\frac{q}{x}\right]\right]\psi(q,t)=i\hbar\frac{\partial\psi}{\partial t}\,. \tag{1.8}$$

In Ref. 2 we proved that the general solution to (1.8) can be written

$$\psi(q,t)=\sum_n c_n e^{i\alpha_n(t)}\psi_n(q,t)\,,\qquad c_n=\langle\psi_n(q,0),\psi(q,0)\rangle e^{-i\alpha_n(0)}\,, \tag{1.9}$$

where

$$\alpha_n(t)=-\frac{\lambda_n}{\hbar}\int^t\frac{dt}{x^2}\,, \tag{1.10}$$

and

$$\psi_n(q,t)=\frac{1}{x^{1/2}}e^{i\dot{x}q^2/(2\hbar x)}\phi_n\left[\frac{q}{x}\right]. \tag{1.11}$$

Here $\phi_n(q/x)=\phi_n(\sigma)$ satisfies the time-independent Schrödinger equation

$$\left[-\frac{\hbar^2}{2}\frac{d^2}{d\sigma^2}+\tfrac{1}{2}k\sigma^2+f(\sigma)\right]\phi_n(\sigma)=\lambda_n\phi_n(\sigma)\,, \tag{1.12}$$

and the $\{\phi_n\}$ form a complete, othornormal set

$$\int\phi_n^*(\sigma)\phi_{n'}(\sigma)d\sigma=\delta_{nn'}\,. \tag{1.13}$$

The unitary transformation $e^{i\dot{x}q^2/(2\hbar x)}$ maps the eigenvalue equation for I (1.5) into the time-independent Schrödinger equation (1.12).

The interest in the above solution arises from the fact that every potential, $V(\sigma)=\tfrac{1}{2}k\sigma^2+f(\sigma)$, for which we solve (1.12), can be used to generate a solution to the time-dependent Schrödinger equation (1.8). Thus, we have a class of exactly solvable, time-dependent Schrödinger equations depending on the arbitrary function $f(q/x)$.

The work of Lewis and Riesenfeld and our work in Ref. 2 assumes that the eigenvalue spectrum for the invariant I is discrete—Eq. (1.13) contains the Kronecker delta. In general, however, the spectrum of I from Eq. (1.12) will possess both discrete and continuous eigenvalues. Thus, we must investigate the Lewis-Riesenfeld theory in the case of continuous eigenvalues, the case of both discrete and continuous eigenvalues can then be obtained by superposition.

II. CONTINUOUS EIGENVALUES

Here we again start from the Ermakov Hamiltonian

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega^2(t)q^2+\frac{1}{x^2}f\left[\frac{q}{x}\right], \tag{2.1}$$

with

$$\ddot{x}+\omega^2(t)x=k/x^3, \tag{2.2}$$

and

$$I=\tfrac{1}{2}(xp-\dot{x}q)^2+\frac{k}{2}\left[\frac{q}{x}\right]^2+f\left[\frac{q}{x}\right]. \tag{2.3}$$

We assume a continuous eigenvalue spectrum for I:

$$I\psi_\lambda(q,t)=\lambda\psi_\lambda(q,t), \quad \lambda=\text{const}. \tag{2.4}$$

The Schrödinger equation is Eq. (1.8). We expand the solution in terms of the eigenfunctions $\psi_\lambda(q,t)$

$$\psi(q,t)=\int c(\lambda)e^{i\alpha(\lambda,t)}\psi_\lambda(q,t)d\lambda, \quad c(\lambda)=\text{const}. \tag{2.5}$$

The calculations are similar but not identical to the discrete case summarized above. The result is that the solution (2.5) is valid if

$$\psi_\lambda(q,t)=\frac{1}{x^{1/2}}e^{i\dot{x}q^2/(2\hbar x)}\phi_\lambda(q/x), \tag{2.6}$$

$$\alpha(\lambda,t)=-\frac{\lambda}{\hbar}\int^t\frac{dt}{x^2}, \tag{2.7}$$

$$\left[-\frac{\hbar^2}{2}\frac{d^2}{d\sigma^2}+\tfrac{1}{2}k\sigma^2+f(\sigma)\right]\phi_\lambda(\sigma)=\lambda\phi_\lambda(\sigma), \tag{2.8}$$

$$\int\phi_\lambda^*(\sigma)\phi_{\lambda'}(\sigma)d\sigma=\delta(\lambda-\lambda'), \tag{2.9}$$

and

$$c(\lambda)=\langle\psi_\lambda(q,0),\psi(q,0)\rangle e^{-i\alpha(\lambda,0)}. \tag{2.10}$$

Next we present an explicit example of the use of the continuous spectrum theory.

III. EXAMPLE

We consider the original Lewis-Riesenfeld problem of the time-dependent harmonic oscillator, $f=0$

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega^2(t)q^2. \tag{3.1}$$

Lewis and Riesenfeld chose $k=1$ in the invariant (1.2) which implies the eigenvalue problem is discrete—the eigenfunctions $\phi_n(\sigma)$ are related to the Hermite polynomials. We now take $k=0$ in (1.2), which gives

$$I=\tfrac{1}{2}(xp-\dot{x}q)^2, \tag{3.2}$$

$$\ddot{x}+\omega^2(t)x=0. \tag{3.3}$$

The eigenvalue problem is

$$-\frac{\hbar^2}{2}\frac{d^2\phi_\lambda(\sigma)}{d\sigma^2}=\frac{\hbar^2\lambda^2}{2}\phi_\lambda(\sigma), \tag{3.4}$$

with solution

$$\phi_\lambda(\sigma)=\frac{1}{\sqrt{2\pi}}e^{i\lambda\sigma}, \quad -\infty<\lambda<\infty. \tag{3.5}$$

We have modified the eigenvalue in (3.4) for notational simplicity. The eigenfunctions $\psi_\lambda(q,t)$ have the form

$$\psi_\lambda(q,t)=\frac{1}{x^{1/2}}e^{i\dot{x}q^2/(2\hbar x)}\phi_\lambda(q/x), \tag{3.6}$$

while the phase functions $\alpha(\lambda,t)$ are given by

$$\alpha(\lambda,t)=-\frac{\hbar\lambda^2}{2}\int^t\frac{dt}{x^2}. \tag{3.7}$$

The solution to the Schrödinger equation is then

$$\psi(q,t)=\frac{1}{\sqrt{2\pi x}}\int c(\lambda)e^{-(\hbar\lambda^2/2)\int^t(dt/x^2)}\times e^{i\dot{x}q^2/(2\hbar x)}e^{i\lambda q/x}d\lambda, \tag{3.8}$$

where

$$c(\lambda)=\langle\psi_\lambda(q,0),\psi(q,0)\rangle e^{-i\alpha(\lambda,0)}. \tag{3.9}$$

The result (3.8) represents the solution to the time-dependent harmonic oscillator Schrödinger equation in terms of a Fourier transform.

The original derivation by Lewis and Riesenfeld ($k=1$) gave the solution in terms of a Hermite transform. These two modes of representing the solution to the time-dependent harmonic oscillator have also been given by Burgan *et al.*[3] The solution by Burgan *et al.* is derived by rescaling the space and time coordinates. The results by Burgan *et al.* are for the N-dimensional time-dependent harmonic oscillator. However, making use of invariants for higher dimensional systems[4] we can extend our arguments to higher dimensions.

IV. DISCUSSION

We have extended the Lewis-Riesenfeld solution technique to cases where the invariant has continuous eigenvalues. As an example we solved the Schrödinger equation for the time-dependent harmonic oscillator for which the invariant has an entirely continuous spectrum. This changes the expansion of the wave function from a discrete Hermite transform to a continuous Fourier transform. Many other examples could be constructed since we have shown how every solution to the time-independent Schrödinger equation (2.8) allows us to generate a solution to a time-dependent Schrödinger equation involving the function f.

Using N-dimensional Lewis-type invariants[4] the results of this paper and Ref. 2 can be extended to certain N-dimensional time-dependent Schrödinger equations. For the N-dimensional time-dependent harmonic oscillator the results agree with those presented by Burgan *et al.*[3] Our techniques are quite different than Burgan *et al.* since these authors do not explicitly use the Lewis-Riesenfeld theory.

As a final point it was proven by Khandekar and Lawande[5] that the Feynman propagator (Green's function) for the solution

$$\psi(q,t)=\sum_{\lambda} c_{\lambda} e^{i\alpha_{\lambda}(t)}\psi_{\lambda}(q,t) \tag{4.1}$$

can be written

$$K(q'',t'';q',t')=\sum_{\lambda} e^{i[\alpha_{\lambda}(t'')-\alpha_{\lambda}(t')]} \times \psi_{\lambda}^{*}(q',t')\psi_{\lambda}(q'',t'') , \tag{4.2}$$

which is the generalization of the usual expansion formula for time-independent Hamiltonians. In (4.2) the sum over λ can contain both discrete and continuous eigenvalues, i.e., sums and integrals. Using the results of Ref. 2 and this paper we can calculate the propagator (4.2) for any Ermakov system. The solution, of course, depends on the function f. Khandekar and Lawande have constructed K for the cases $f=0$, $f=cx^2/q^2$ discussed in Ref. (2). Note also that since $\alpha_{\lambda}=(-\lambda/\hbar)\int^{t}(dt/x^2)$ for all Ermakov systems we only need solve the time-independent Schrödinger equation (2.8) in order to employ (4.2) for the calculation of the propagator.

We plan to pursue extensions of the ideas presented here and in Ref. 2 to more general time-dependent systems.

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