Lower and upper bounds for the eigenvalues of the Fokker-Planck equation in detailed balance

H. Brand, A. Schenzle, and G. Schröder
Fachbereich 7, Physik, University Essen, D 4300 Essen, West Germany
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We derive a variational principle for the eigenvalues of the Fokker-Planck equation in detailed balance, which holds for additive as well as multiplicative stochastic processes and which is not restricted to one dimension. An intermediate theorem for the Fokker-Planck equation is presented for the first time from which lower bounds for the eigenvalues can be extracted. The same theorem can be applied to the master equation in detailed balance. We discuss as a first application the lower and upper bounds for the lowest eigenvalue of a quartic potential in one dimension. To show the applicability of our procedure to more complicated problems we give lower and upper bounds for the linewidth factor and the second eigenvalue of the Fokker-Planck equation for the single-mode laser. In addition, we discuss three models of multiplicative stochastic processes which represent three different classes of behavior. For each model we give stationary results, namely, the probability distribution and the first two nontrivial moments. To characterize the long-time behavior of the relaxation processes involved we calculate lower and upper bounds for the lowest eigenvalue. It is pointed out that, at least in the models presented, there does not exist critical slowing down as a function of the strength of the fluctuations. We conclude that one should be cautious when using the term noise-induced phase transition in connection with multiplicative stochastic processes.

I. INTRODUCTION

In many cases the static and dynamic behavior of macroscopic systems can be characterized by a small number of macroscopic variables. To arrive at these equations one can use, e.g., projector techniques or adiabatic elimination.

If one is interested in a statistical description of a macroscopic system, which is especially important near states which are not globally stable, it proves to be fruitful to work in the framework of the Fokker-Planck or the master equation.

As is well known[1-3] the stationary solution of the Fokker-Planck and master equation can be given in closed form (at least up to quadratures) if the condition of detailed balance holds. Obtaining the time-dependent solution of the Fokker-Planck equation for a specific macroscopic system, however, is a much more complicated problem, because it is necessary to solve an eigenvalue problem for an elliptic partial differential equation.

From a physical point of view the study of the time-dependent solution of the Fokker-Planck equation is important for various reasons. The eigenvalues yield, e.g., the relaxation time, i.e., the time which has to be exceeded if one is interested in observing the equilibrium properties of a specific system. In addition one can extract from the eigenvalues the switching time between the two local extrema of the potential, a question which is of relevance for the construction of devices, e.g., in the fields of nonlinear optics,[4] autocatalytic chemical reactions,[5] and electronics[6] and which therefore attracted considerable attention during the last few years.[7-16]

The study of the time-dependent behavior of the solutions of a Fokker-Planck equation with a nonlinear drift is a long-standing problem which may be traced back at least to the pioneering work of H. A. Kramers,[7] which was generalized and refined through several decades.[7,8,10,15,18,19] Up to now the technique of a WKB-type approximation to study the relaxation time of a bistable potential has been the usual concept. However, as is well known Kramers's approximation breaks down if the potential becomes too shallow in the neighborhood of the minima.

To overcome this difficulty and in order to study problems which are not easily accessible to a Kramers-type approximation—monostable potentials, potential for the phase diffusion of the single-mode laser, etc.— one has to look for a different approach to the problems.

One can study, e.g., exactly soluble models[9,11,20] which simulate the structure of a more complicated problem. This technique, however, suffers from

the fact that there are not many Fokker-Planck equations around whose time-dependent solutions are known analytically.[1,9,21–23] Therefore one is interested in a more easily applicable approach. For *one-dimensional, additive* stochastic processes such a technique, namely, a variational principle, has been used, e.g., by Risken[24–26] during the study of the single-mode laser; more recently it has been used in a more general framework by Larson and Kostin,[16,27] who applied this technique to a bistable, biquadratic one-dimensional potential.

Very recently we have presented without derivation a variational principle for the Fokker-Planck equation in detailed balance which holds for additive as well as for multiplicative stochastic processes and which can be applied to one as well as to multidimensional stochastic problems.[14] This principle which assumes an elegant and intuitive form has been applied to the example of the absorptive optical bistability. In the present paper this variational principle which yields an upper bound for the eignvalues will be derived in detail.

Furthermore we present for the first time an "intermediate theorem" for the Fokker-Planck equation in detailed balance from which one can extract *lower bounds* for the eigenvalues, i.e., for the relaxation rates, a problem that has not been discussed previously in the literature of the Fokker-Planck and master equation. To achieve this aim we will make use of the intermediate theorem which was obtained by Weinstein[28,29] nearly 50 years ago (1934).

In the following sections we apply these methods to various examples in one and two dimensions with additive as well as multiplicative fluctuations. We demonstrate that the procedure can be applied easily to real physical problems like, e.g., the single-mode laser and the Arnold-Horsthemke-Lefever model.[30]

The paper is organized as follows. In Sec. II we present in detail the derivation of the variational principle which yields upper bounds for the eigenvalues and we give the intermediate theorem for the Fokker-Planck equation in detailed balance. Furthermore we show in this section how it becomes possible to extract lower bounds for the eigenvalues from the intermediate theorem and what conditions have to be satisfied to arrive at this aim easily.

In Sec. III we give an application to additive stochastic processes: We study a one-dimensional biquadratic potential and the first two lowest-lying eigenvalue of the single-mode laser. The upper bounds for the eigenvalues of the single-mode laser are compared with the "exact" numerical results of Risken and Vollmer[26] whereas the lower bounds are novel. From these results we deduce upper and lower bounds for the linewidth factor and contrast our findings with those of perturbative calculations.

In addition we show that we can give *rigorous lower bounds* also for the effective relaxation rate of the intensity correlation functions of the single-mode laser.

In Sec. IV our procedure is applied to three models with multiplicative fluctuations which cover three large classes of behavior. In addition we include in our discussion results for the static moments and the stationary probability density of these models. From the stationary as well as from the time-dependent results it becomes possible to shed new light on the question of "noise-induced phase transitions," a phenomenon which, at present is under very vivid discussion in the literature.

In Appendix A we show that the intermediate theorem can also be applied to the master equation in detailed balance thus presenting for the first time the possibility of calculating in a rigorous manner lower bounds for the eigenvalues of the master equation and in Appendix B we generalized the results of Sec. II to normal Fokker-Planck operators and discuss as an example the detuned single-mode laser.

II. GENERAL RESULTS

We start from a set of Langevin equations characterizing the macroscopic variables $\{x_i\}$:

$$\dot{x}_i = f_i(\{x_l\}) + g_{ij}(\{x_l\})\xi_j \ , \tag{2.1}$$

where the functions f_i, g_{ij} can depend on the macroscopic variables. The fluctuating forces ξ_i are characterized by their statistical properties. Equations (2.1) may be derived from microscopic equations by elimination of the irrelevant variables.[31,32] If the g_{ij} are independent of the macroscopic variables we call the stochastic processes (2.1) additive; if they depend on the set $\{x_l\}$ we call the processes (2.1) multiplicative.[22]

In the following we assume the fluctuating forces to be of the Gaussian white noise type, i.e.,

$$\begin{aligned} &\langle \xi_i \rangle = 0 \ , \\ &\langle \xi_i \xi_{j\tau} \rangle = \delta(\tau)\delta_{ij} \ . \end{aligned} \tag{2.2}$$

Equations (2.2) and (2.1) describe, then, a multidimensional Markovian process, and we can convert (2.1) into the stochastically equivalent Fokker-Planck equation

$$\dot{P}(\{x_l\},t)=-\partial_i[K_i(\{x_l\})P] + \partial_i\partial_j(K_{ij}P) , \tag{2.3}$$

where the drift vector $K_i(\{x_l\})$ is related to the quantities in Eqs. (2.1) via

$$K_i(\{x_l\})=f_i(\{x_l\})+\tfrac{1}{2}g_{mj}(\partial_m g_{ij}) \tag{2.4}$$

and the positive semidefinite, symmetric diffusion matrix is related to the matrix $g_{ij}(\{x_K\})$ via

$$K_{ij}(\{x_k\})=g_{il}(\{x_k\})g_{jl}(\{x_k\}) . \tag{2.5}$$

In the following, summation over repeated indices is always implied if not stated otherwise.

Choosing the ansatz

$$P(\{x_k\},t)=\sum\!\!\!\!\!\!\int P_n(\{x_k\})e^{-\lambda_n t} \tag{2.6}$$

the time-dependent parabolic Fokker-Planck equation may be transformed into an elliptic partial differential equation

$$\lambda_n P_n = L^{\mathrm{FP}} P_n , \tag{2.7}$$

where

$$L^{\mathrm{FP}}=-\partial_i(K_i(\{x_k\})-\partial_j[K_{ij}(\{x_k\})]) ,$$

and where the eigenvalues λ_n can form a discrete as well as a continuous spectrum.[23] Equation (2.7) supplemented by suitable boundary conditions (e.g., natural boundary conditions) constitutes an eigenvalue problem which is in general not self-adjoint.

There exist no general methods to calculate the eigenvalues for the most general case of a problem that is not Hermitian. We confine ourselves therefore throughout the rest of the present paper to the case where Eq. (2.7) together with appropriate boundary conditions can be cast into self-adjoint form.

As is well known[1] this can be done if the vector of the stationary probability current $\vec{\mathcal{F}}$ is identical to zero:

$$\mathcal{F}_i \equiv K_i P_0 - \partial_j(K_{ij}P_0)=0 .$$

Stratonovich[1] called the condition

$$\partial_m[A_{li}(\partial_j K_{ij}-2K_i)] = \partial_l[A_{mi}(\partial_j K_{ij}-2K_i)] , \tag{2.8}$$

where

$$A_{li}K_{ij}=\delta_{lj}$$

the potential condition, and it is straightforward to check that the restriction (2.8) is identical to the condition of detailed balance,[31,33,34] or generalized detailed balance [3] if the so-called reversible drift vanishes. If the reversible drift is not identical to zero, Eq. (2.7) cannot be cast into self-adjoint form.[34]

Introducing

$$L=P_0^{-1/2}L^{\mathrm{FP}}P_0^{1/2} , \tag{2.9}$$

$$P_0^{-1/2}P_n=\varphi_n , \tag{2.10}$$

where P_0 is the stationary probability density Eq. (2.7) can be rewritten in the form

$$L\varphi_n=\lambda_n\varphi_n , \tag{2.11}$$

where

$$L\equiv L^{+} \tag{2.12}$$

and[34]

$$L=\partial_i(K_{ij}(\{x_k\})\partial_j) - P_0^{-1/2}[\partial_i K_{ij}(\{x_k\})\partial_j P_0^{1/2}] . \tag{2.13}$$

To derive the orthogonality properties of the eigenfunctions P_n we introduce the abbreviations

$$G_n=\frac{P_n}{P_0}=\frac{\varphi_n}{P_0^{1/2}} . \tag{2.14}$$

Next we consider Eq. (2.7) for two different eigenvalues λ_n and λ_m:

$$L^{\mathrm{FP}}P_n=\lambda_n P_n , \tag{2.15}$$

$$L^{\mathrm{FP}}P_m^{*}=\lambda_m P_m^{*} . \tag{2.16}$$

Then we multiply Eq. (2.15) by P_m^{*}/P_0 and Eq. (2.16) by P_n/P_0, and obtain, after integration over the whole phase space for the difference ($d\tau$ denotes the volume element in phase space),

$$\int d\tau\left[\frac{P_m^{*}}{P_0}L^{\mathrm{FP}}P_n-\frac{P_n}{P_0}L^{\mathrm{FP}}P_m^{*}\right]=(\lambda_n-\lambda_m)\int d\tau\frac{P_nP_m^{*}}{P_0} . \tag{2.17}$$

Making use of the explicit form of L^{FP} and taking into account condition (2.8) we have

$$(\lambda_n-\lambda_m)\int d\tau\frac{P_nP_m^*}{P_0}=\int d\tau[G_m^*\partial_i(-1/2K_{ij}P_0\partial_jG_n)-G_n\partial_i(-\tfrac{1}{2}K_{ij}P_0\partial_jG_m^*)]$$
$$\equiv\int d\tau[G_m^*\partial_i\mathcal{F}_i(P_n)-G_n\partial_i\mathcal{F}_i(P_m^*)]\ . \tag{2.18}$$

Integrating by parts we obtain

$$\int d\tau[(\partial_iG_m^*)K_{ij}P_0(\partial_jG_n)-(\partial_iG_n)K_{ij}P_0(\partial_jG_m^*)]=(\lambda_n-\lambda_m)\int d\tau\frac{P_nP_m^*}{P_0} \tag{2.19}$$

provided that the surface integral

$$0=\int_\Sigma d\sigma_i[G_m^*\mathcal{F}_i(P_n)-G_n\mathcal{F}_i(P_m^*)] \tag{2.20}$$

vanishes identically ($d\sigma_i$ denotes the surface element for the integration over the surface Σ).

Let us consider some special cases of the general condition (2.20).

(i) $\mathcal{F}_i(P_n)=0$ on Σ, i.e., the probability current vanishes for all eigenfunctions on the surface which constitutes the boundary of phase space, or, to phrase it differently, probability does not leave the domain of phase space under investigation, a condition which is suggestive and quite physical.

(ii) $G_m=P_m/P_0$ vanishes on the boundary surface, i.e., the eigenfunctions divided by the stationary probability distribution vanish there.

If condition (2.20) is satisfied we conclude immediately from Eq. (2.19) that eigenfunctions corresponding to different eigenvalues are orthogonal with weight function P_o^{-1}:

$$\int\frac{P_nP_m^*}{P_0}d\tau=0\ . \tag{2.20a}$$

For one dimension our result reduces to that given previously by R. L. Stratonovich.[1] If we assume that the eigenfunctions are normalized,

$$\int\frac{|P_m|^2}{P_0}d\tau=\int|\varphi_n|^2d\tau$$
$$=\int P_0|G_n|^2d\tau=1\ , \tag{2.21}$$

we obtain an orthonormal set of eigenfunctions.

As is well known a self-adjoint eigenvalue problem can be cast into the form of a variational principle of the Rayleigh-Ritz type. With the help of Eqs. (2.11), (2.15), and (2.19) the variational principle assumes the following form:

$$\lambda_n\le\frac{\int P_0K_{ij}(\partial_iG_n)(\partial_jG_n^*)d\tau}{\int P_0|G_n|^2d\tau}$$
$$=\frac{\langle K_{ij}(\partial_iG_n)(\partial_jG_n^*)\rangle}{\langle|G_n|^2\rangle}\ , \tag{2.22}$$

where the angular brackets denote stationary averages. Of course, the variational principle (2.22) can be rewritten as well for the sets $\{\varphi_n\}$ and $\{P_n\}$ but then (2.22) would lose its suggestive form and its elegance. From Eq. (2.22) it is immediately clear that the stationary state $G_0=1$ is associated with eigenvalue $\lambda_0=0$. We notice, that the right-hand side of Eq. (2.22) is a positive semidefinite quadratic form, because the diffusion matrix K_{ij} defined in Eq. (2.5) is positive semidefinite and the stationary solution P_0 ($\{x_k\}$) is a nonnegative function. From the fact that the variational expression in Eq. (2.22) assumes its minima λ_n, one concludes that the eigenvalue spectrum of the corresponding Fokker-Planck problem is nonnegative. While calculating an upper bound for the eigenvalue λ_n the chosen test function $P_n=G_nP_0$ has to satisfy, of course, the orthogonality conditions (2.20a) for P_i where $i=0,1,\ldots n-1$. It seems worthwhile to notice that inequality (2.22) has been given previously for the case of a one-dimensional additive stochastic process for the lowest-lying eigenvalue by Larson and Kostin.[16]

For the one-dimensional case of an additive stochastic process Risken considered a variational principle for a special example, namely, for the single-mode laser. However, he obtained a rather complicated expression because he did not introduce the functions G_n. Furthermore it seems worthwhile to notice that inequality (2.22) is not restricted to additive processes. It may also be applied to multiplicative stochastic processes—we will show this explicitly in Sec. IV—and even to processes where K_{ij} acquires contributions from additive and from multiplicative noise. In addition it should be stressed that inequality (2.22) can even be applied if some of the diagonal elements of the diffusion matrix K_{ij} vanish as is the case, e.g., for the Kramers equation.

Now we present the extended version of Weinstein's intermediate theorem suitable for the Fokker-Planck equations that can be cast into self-adjoint form. We assume the following conditions to hold:

(i) $L\varphi_n=\lambda_n\varphi_n$, where $L\equiv L^+$ [cf. Eqs. (2.11) and (2.12)];

(ii) appropriate boundary conditions for P_n and thus for $\varphi_n P_0^{1/2}$ and for $G_n P_0$ (e.g., natural boundary conditions);

(iii) existence of an arbitrary function $u(\{x_K\})\neq 0$ which is continuously differentiable two times which satisfies the same boundary conditions as $G_n P_0^{1/2}$.

In addition, we introduce the abbreviations

$$\alpha=\frac{\int u^* L u d\tau}{\int |u|^2 d\tau} \tag{2.23}$$

and

$$\beta^2=\frac{\int (Lu^*)(Lu)d\tau}{\int |u|^2 d\tau} . \tag{2.24}$$

Then we have

$$\beta^2\geq\alpha^2 \tag{2.25}$$

and there exists at least one eigenvalue of problem (i) between

$$\alpha-(\beta^2-\alpha^2)^{1/2}$$

and (2.26)

$$\alpha+(\beta^2-\alpha^2)^{1/2} .$$

The proof parallels that of Weinstein [28] who was interested in the corresponding quantum-mechanical problem. For an ordinary differential equation all details have been worked out by Kamke.[29] Of course, the same procedure can be carried out for the master equation in detailed balance [2] and we describe this in Appendix A.

For further reference we rewrite β^2 in terms of G_n. We have then

$$\beta^2=\frac{\int P_0^{-1}|L^{\mathrm{FP}}P_0 G_n|^2}{\int P_0 |G_n|^2} \tag{2.27}$$

or

$$\beta^2=\frac{\int P_0^{-1}|\partial_i(K_{ij}P_0\partial_j G_n)|^2}{\int P_0 |G_n|^2} . \tag{2.28}$$

For a one-dimensional additive stochastic process we have the simpler expression

$$\beta^2=\left[\frac{Q}{2}\right]^2\frac{\int P_0^{-1}[\partial_x(P_0\partial_x G_n)]^2}{\int P_0 G_n^2} . \tag{2.29}$$

What can be done to apply (2.26) to a specific problem and to extract useful information? (In the following we assume for simplicity that the part of the eigenvalue spectrum which is of interest is discrete.)

The first step will then usually be the calculation of upper bounds for the excited states via the variational principle (2.22) and we assume this to be done. Then we have a discrete series of upper bounds for the eigenvaules

$$0\equiv\lambda_0\leq\lambda_1^{\mathrm{up}}\leq\lambda_2^{\mathrm{up}}\leq\cdots .$$

We now insert the approximate eigenfunctions φ_n— evaluated via the generalized Ritz principle — into Eq. (2.26) and obtain a value $\lambda_n^{\mathrm{low}}\leq\lambda_n^{\mathrm{up}}$. The doman defined by these two values then contains at least one eigenvalue of the Fokker-Planck problem. If the domains are not overlapping and there exists a limiting case, which allows us to assign the domains to specific eigenvalues, we obtain rigorous upper and lower bounds. The intermediate theorem can be carried out most easily for problems with well-separated eigenvalues but care has to be taken in choosing the variational test functions when the eigenvalues approach each others. When the eigenvalue spectrum contains a "continuous band" of eigenvalues, it is, however, impossible to extract information from the intermediate theorem.

In many applications of the Fokker-Planck equation the eigenvalue spectrum contains a discrete branch with well-separated eigenvalues and the methods proposed here allow us to find rigorous upper and lower bounds. It will become clear in the following sections, that these methods are not restricted to the lowest eigenvalues but can give bounds for higher eigenvalues as well. In the applications which follow in Secs. III and IV, emphasis has been laid on showing that the procedure works in practice and does not require time-consuming, extensive numerical calculations. Nevertheless we will find in many examples that it is possible to calculate lower *and* upper bounds for the eigenvalues by the methods presented here to a quite surprising accuracy. If one can only calculate upper bounds it is much more difficult to estimate the accuracy of the procedure.

III. ADDITIVE STOCHASTIC PROCESSES

As a first application of the general results derived in Sec. II we investigate the lowest-lying eigenvalue of a biquadratic one-dimensional poten-

tial. The corresponding Langevin equation takes the form

$$\dot{x}=ax-x^3+\xi\ , \tag{3.1}$$

where ξ is assumed to be Gaussian white noise and $b>0$:

$$\langle\xi\rangle=0,\ \ \langle\xi\xi_\tau\rangle=Q\delta(\tau)\ . \tag{3.2}$$

The corresponding Fokker-Planck equation reads

$$\frac{\partial P}{\partial t}=-\partial_x[(ax-x^3)P]+\frac{Q}{2}\partial_{xx}P\ . \tag{3.3}$$

For the stationary solution we have

$$P_0=\mathcal{N}\exp\left[-\frac{a}{Q}x^2-1/2\frac{x^4}{Q}\right]\ . \tag{3.4}$$

As a first simple test function G_1, satisfying obviously the orthogonality condition

$$\int P_0G_1dx=0\ , \tag{3.5}$$

we choose

$$G_1(x)\equiv x\ , \tag{3.6}$$

and we obtain from (2.22) and (2.29),

$$\lambda_1\leq\frac{Q}{2}\frac{\int P_o dx}{\int P_0x^2dx}=\frac{Q}{2}\frac{1}{\langle x^2\rangle}\ , \tag{3.7}$$

$$\beta^2=\left[\frac{Q}{2}\right]^2\frac{\int P_0^{-1}(\partial_xP_0)^2dx}{\int P_0x^2dx}\ . \tag{3.8}$$

The upper and lower bounds for the choice (3.6) are plotted in Fig. 1 as dashed lines. This result can be improved easily as indicated in Fig. 1 using the somewhat more refined test function

$$\tilde{G}_1(x)=\tanh(cx) \tag{3.9}$$

which contains one variational parameter c. The parameter c was determined by minimizing the upper bound. Of course, it is also possible to fix c by minimizing $(\beta^2-\alpha^2)^{1/2}$, but, as is well known the results for the optimum values are the same.

It is immediately seen from Fig. 1 that the trivial test function G_1 yields reasonable lower bounds for small values of a, whereas the difference between lower and upper bounds become larger for increasing a. For the region $-4\leq a\leq 2$ the second test function $\tilde{G}_1$ gives excellent upper and lower bounds, e.g., for $a=0$ the difference between the bound is only 2%.

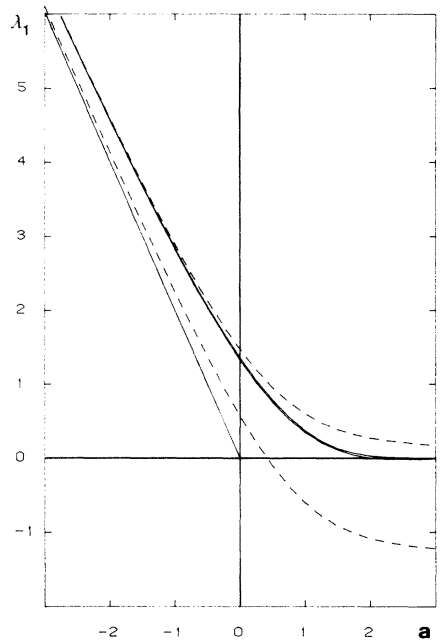

FIG. 1. Lower and upper bounds for the first excited state of the one-dimensional biquadratic potential. The straight line indicates the asymptotic behavior for $a\rightarrow-\infty$. We have plotted the upper and lower bounds for $\tilde{G}_1(x)=\tanh(cx)$ as a solid line and the corresponding results for $G_1(x)=x$ as a dashed line.

From Fig. 1 it can be concluded that a small improvement for the upper bound can give rise to a large impovement of the lower bound. As will become clear in the following examples this is generally the case and can be traced back to our procedure of constructing the lower bound.

In addition it seems worthwhile to note that the results of Dekker *et al.*, which have been obtained via direct numerical integration of the underlying differential equation are situated between the lower and upper bounds presented here. We note in passing that the conjecture by Dekker *et al.* that a variational principle cannot give accurate results for the eigenvalues does not hold. As is well known, the accuracy of a variational result is not limited in principle because it can always be improved by choosing a more appropriate test function. This is immediately seen from the first example discussed here and from all examples which are considered in what follows. For completeness we have included in Fig. 1 the asymptotic value of λ_1 for $a\rightarrow-\infty$.

To test the power of the method we examine as the next example the second excited state of the Fokker-Planck equation (3.3) in the monostable as well as bistable regime.

We take for the test function G_2 the ansatz

$$G_2=\tau-e^{-dx^2}. \tag{3.10}$$

This choice can be motivated as follows. In order to approximate the second eigenfunction P_2 the test function has to be symmetric, thereby satisfying orthogonality on the first eigenfunction P_1, and it has to contain two internal zeros. The parameter τ is used to guarantee orthogonality to the stationary state,

$$\int G_2 P_0 dx=0, \tag{3.11}$$

while d is used as variational parameter.

The results for the lower and upper bound are plotted in Fig. 2 and we see that $\lambda(a)$ has a minimum near $a\approx 1.5$. For a large ($\gtrsim 4$), i.e., in the bistable regime, the difference between the upper and lower bound is less than 10% and even near threshold the difference never reaches 30%. This is a quite satisfactory result keeping in mind that we have made use only of the simple test function (3.10) in order to minimize the numerical calculations. Furthermore this result shows that the method presented in Sec. II can be applied not only to the first excited state but to higher excited states as well, even in the bistable domain. The asymptotic values for $a\rightarrow\pm\infty$ have been included in Fig. 2 as a guide to the eye.

Now we consider an example which provides a more sensitive test of our procedure, namely, the linewidth factor of the single-mode laser, which has been investigated by various methods (Refs. 24–26 and 35 and references cited in Ref. 35).

The Langevin equation for the single-mode laser reads[25]

$$\dot{x}^+=ax^+-x^+|x|^2+\xi. \tag{3.12}$$

For the associated Fokker-Planck equation we obtain, with $x^+=re^{i\varphi}$,

$$\frac{\partial P}{\partial t}=\frac{1}{r}\partial r[r(ar-r^3)P]+\frac{1}{r}\partial_r(r\partial_r P)+\frac{1}{r^2}\partial_\varphi\partial_\varphi P. \tag{3.13}$$

The stationary solution of (3.13) is

$$P_0=\mathcal{N}\exp\left[-\frac{a}{2}r^2-\frac{r^4}{4}\right], \tag{3.14}$$

where $\mathcal{N}$ is the normalization factor. As a suitable test function to describe the phase diffusion we use

$$G_{10}(r,\varphi)=e^{i\varphi}\tanh(dr), \tag{3.15}$$

$G_{10}(r,\varphi)$ is chosen to satisfy the appropriate orthogonality condition. We have plotted in Fig. 3 the

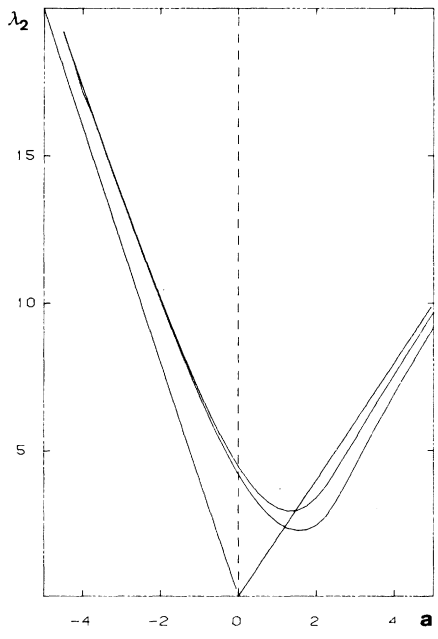

FIG. 2. The upper and lower bounds for the second excited state [using $G_2(x)=\tau-e^{-dx^2}$ as a test function] are plotted as a solid line and the asymptotic values for $a\rightarrow\pm\infty$ are given by straight lines.

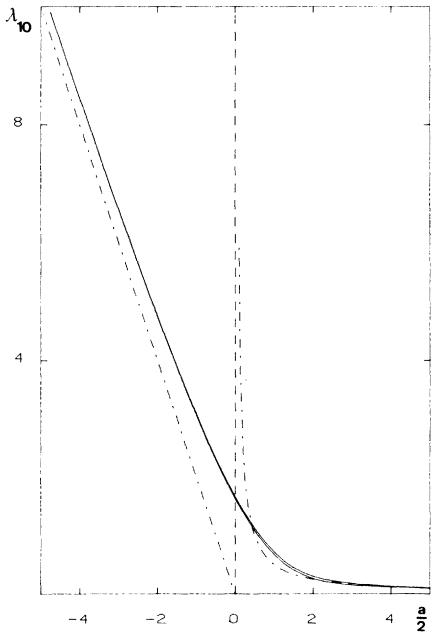

FIG. 3. Lower and upper bounds for λ_{10} of the single-mode laser; the asymptotic values are indicated as dashed-dotted lines.

upper and lower bound for λ_{10} as a function of the pump parameter a for $-10 \leq a \leq 10$. We see, that the asymptotic behavior [$\lambda_{10}=1/|a|$ above threshold ($a \to \infty$), and $\lambda_{10}=|a|$ below threshold ($a \to -\infty$)] is approached more rapidly above threshold than below. To display the quality of the results for λ_{10} [obtained for (3.15)] in more detail we show in Fig. 4 the linewidth factor $\tilde{\alpha}$ which is associated with the eigenvalue λ_{10} via

$$\tilde{\alpha}=\lambda_{10}\langle r^2\rangle , \tag{3.16}$$

where $\langle r^2\rangle$ is the static moment

$$\langle r^2\rangle=\int r^2 P_0 d\tau . \tag{3.17}$$

From Fig. 4 we conclude that the asymptotic values for $\tilde{\alpha}$ ($\tilde{\alpha}=2$ below threshold, $\tilde{\alpha}=1$ above threshold) are reached rapidly for large values of $|a|$. In the threshold region the differences between the upper and lower bound is always less than 15% even though we have chosen an elementary test function with a single variational parameter.

We wish to notice that any other approach which is used to calculate the linewidth factor of the single-mode laser has to yield values which lie between the upper and lower bounds presented here and this must be true for all pump parameters.

Various approximation schemes which deal with this problem have been discussed recently by Ziegler and Horner,[35] who compared their results with those discussed previously by various authors.[25,36] When comparing Fig. 3 of Ref. 35 with the results presented here it can be immediately concluded that none of the *approximate methods* presented so far yield values which lie inside the admissible domain given in the present paper for all values of the pump parameter.

In closing the discussion of λ_{10} we wish to mention that this eigenvalue has been calculated by Risken and Vollmer via direct numerical integration of the corresponding differential equation giving values inside the domain discussed above. Within the admissible range their values are near the upper bounds presented here, a fact which has been found to hold as well for the one-dimensional problem. This leads to the conjecture that the exact eigenvalue always lies closer to the upper bound.

After the investigation of the eigenvalue λ_{10} which dominates the long-time behavior of the amplitude correlation function,[25] we consider the eigenvalue λ_{01} which yields the most important contribution to the intensity correlation function for large times. As a test function in this case we use

$$G_2(x)=(1-\tau x^2)\frac{1}{\cosh^2 ex} , \tag{3.18}$$

where τ is fixed by the appropriate orthogonality condition,

$$\int P_0 G_2 r\, dr\, d\varphi=0 , \tag{3.19}$$

and e serves as variational parameter.

In Fig. 5 we have plotted the results for the upper and upper and lower bound for λ_{01} using Eq. (3.18) and we have indicated the asymptotic

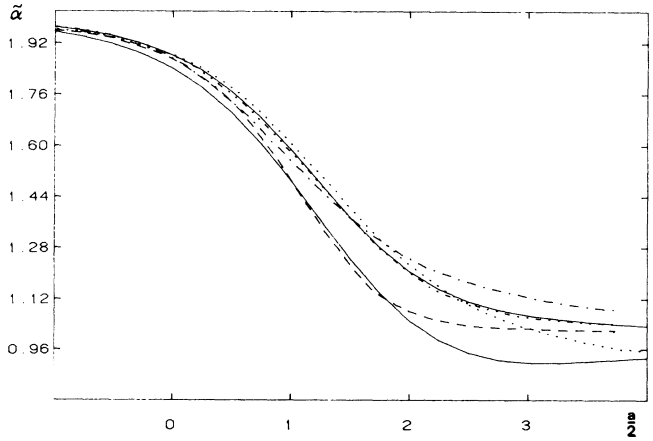

FIG. 4. The linewidth factor as a function of the pump parameter. The upper and lower bounds calculated in the present paper are plotted as solid lines. The numerical integration by Risken and Vollmer is indicated by the double-dotted–dashed line. Three of the approximations (a,e,f) plotted in Fig. 3 of the paper by K. Ziegler and H. Horner are shown.

values as straight lines. In addition we have included in Fig. 5 the upper bound given by Risken.[24] Using the same test function as Risken we also calculated the corresponding lower bound which is also plotted in Fig. 5. For the difference between upper and lower bound [using (3.18)] we find a maximum of less than 25% near $a \approx 2.5$. This result demonstrates once again the applicability of the proposed method even for problems which are not strictly one dimensional.

Furthermore we wish to stress that the presented lower bound for λ_{01} sets a general upper bound for the effective relaxation time (which has been introduced in Ref. 26 of the intensity correlation function or, correspondingly, a general lower bound for the effective relaxation constant.

In summarizing this section we can conclude that we have demonstrated the applicability of the general method of Sec. II to various problems—to a monostable potential as well as to a bistable one in one and two dimensions.

It was shown that even a somewhat clumsy choice of the test function can give reasonable results in some cases [cf., e.g., the function $G(x)=x$ for the monostable quartic potential]. All examples of this section have been additive stochastic processes. In the next section we turn to the discussion of some models for multiplicative stochastic processes, i.e., processes with a diffusion matrix which depends on the stochastic variable.

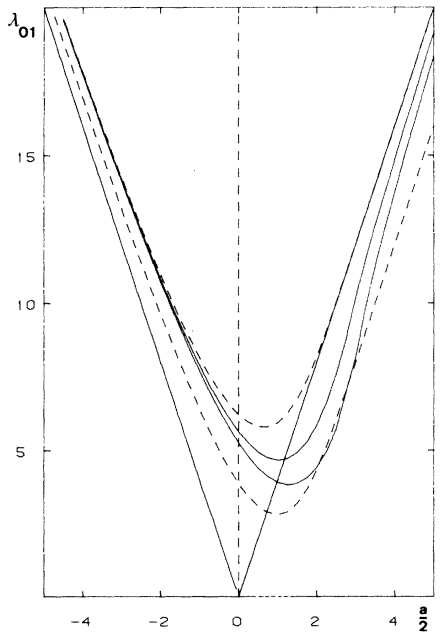

FIG. 5. Lower and upper bounds for λ_{01}. We have plotted the results for $G_2(x)=1-gx^2$ as dashed lines and the results for $G_2(x)=(1-\tau x^2)1/\cosh^2 ex$ as solid lines. The asymptotic values are indicated as straight solid lines.

IV. MULTIPLICATIVE STOCHASTIC PROCESSES

During the last few years multiplicative stochastic processes have attracted a continuously increasing attention concerning their stationary[5,13,21,22,37–44] as well as the dynamic[21–23,42] behavior. In the present section we study in detail three one-dimensional models for multiplicative stochastic processes with respect to their stationary as well as their dynamic behavior. Recently it was conjectured by various authors[38,45] that "noise-induced phase transitions" and "critical slowing down" might occur for multiplicative stochastic processes and it will be one of the purposes of this section to test the validity of the above mentioned conjectures.

As a first model we consider a symmetrized version of a model proposed by Arnold, Horsthemke, and Lefever.[30] The corresponding Langevin equation reads

$$\dot{x} = -x + (1-x^2)\xi \tag{4.1}$$

and the stochastic variable x assumes values in the finite interval $[-1,1]$. For the stationary solution of the Fokker-Planck equation which is stochastically equivalent to the Langevin equation (4.1), one obtains

$$P_0 = \mathcal{N}(1-x^2)^{-1}\exp\left[-\frac{1}{Q}(1-x^2)^{-1}\right], \tag{4.2}$$

and for the most probable values one has

$$\hat{x} = \begin{cases} 0, & Q<1 \\ \pm\left[1-\dfrac{1}{Q}\right]^{1/2}, & Q>1\,. \end{cases} \tag{4.3}$$

Thus one finds that $Q<Q_c=1$ the stationary probability density has one peak centered at $x=0$ whereas for $Q>Q_c$ the stationary solution peaks near $x=\pm(1-1/Q)^{1/2}$ and has a minimum for $x=0$. We have plotted the stationary probability distribution for $Q=$ 0.1, 1, and 20 in Fig. 6 clearly showing the behavior obtained from the qualitative

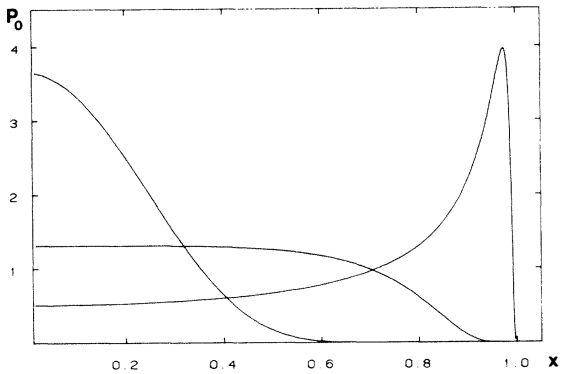

FIG. 6. Stationary probability distribution for the symmetrized version "Arnold-Horsthemke-Lefever (AHL) model" for $Q = 0.1$, 1, and 20, Eq. (4.2).

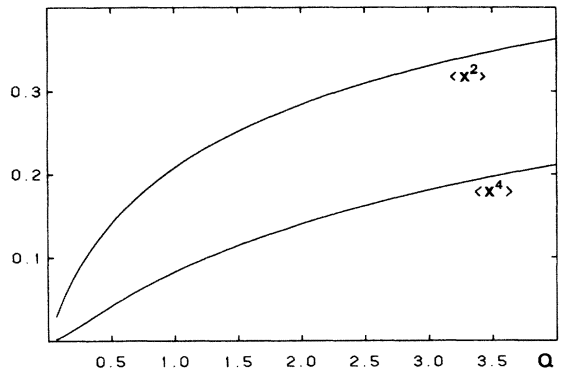

FIG. 7. The moments $\langle x^2\rangle$ and $\langle x^4\rangle$ for the symmetrized AHL model as a function of Q.

discussion above. From this behavior of P_0 it was concluded [30] that a noise-induced phase transition occurs at $Q_c = 1$. To check this hypothesis we have calculated the moments of the stationary probability distribution:

$$M_{2n} = \langle x^{2n}\rangle = \mathcal{N}\int_0^1 \frac{x^{2n}}{1-x^2}\exp\left[-\frac{1}{Q}(1-x^2)^{-1}\right]dx$$

and we have plotted the results for M_2 and M_4 in Fig. 7 as a function of Q ranging from $Q = 0.2$ to $Q = 4$ including $Q_c = 1$. As is obvious from Fig. 7 both moments $\langle x^2\rangle$ and $\langle x^4\rangle$ increase monotonically for increasing values of Q and nothing peculiar happens near $Q_c = 1$, i.e., we obtain no indication for the occurrence of a phase-transition–type behavior near Q_c in the moments. Therefore we study the lowest-lying eigenvalue with the aid of the general techniques presented in Sec. II.

For the uppper bound of the first eigenvalue we have

$$\lambda_1 \le \frac{Q}{2}\frac{\int_0^1 P_0 K(x)(\partial_x G_1)^2 dx}{\int_0^1 P_0 G_1^2 dx} \tag{4.4}$$

and for the quantity β^2 defined in (2.29),

$$\beta^2 = \frac{Q^2}{4}\frac{\int_0^1 dx\, P_0\left[\partial_x K(x)\partial_x G_1 + \frac{K(x)}{P_0}\partial_x P_0 \partial_x G_1 + K(x)\partial_{xx} G_1\right]^2}{\int_0^1 P_0 G_1^2 dx}, \tag{4.5}$$

where $K(x)$ must be identified in the present model with $K(x) \equiv (1-x^2)^2$. As a simple test function G_1 we choose

$$G_1 = x e^{-ax^2} \tag{4.6}$$

with one variational parameter a.

In Fig. 8 we have plotted upper and lower bounds for the eigenvalue λ_1 for $Q = 0.1$ to $Q = 4$ and as is easily checked, both upper and lower bounds increase monotonically as a function of Q and there occurs no especially small value for λ_1 near $Q_c = 1$. That is, even in the regime where the stationary probability distribution shows two peaks the lowest eigenvalue increases contrary to the case

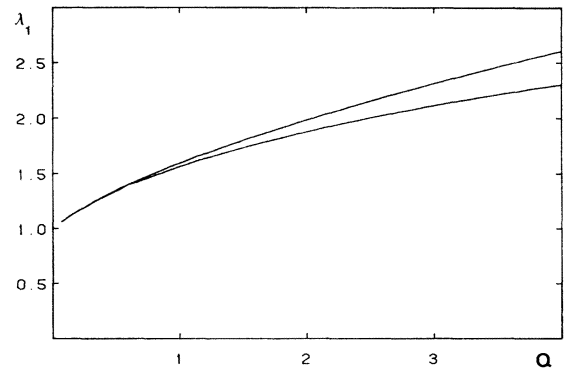

FIG. 8. Upper and lower bounds for λ_1 of the symmetrized AHL model as a function of Q.

of a bistable potential with additive fluctuations. In the latter case the eigenvalue is known to decrease exponentially well inside the bistable domain.[17]

To summarize our results for the first model, which is restricted to a finite domain, we have found no evidence for a phase-transiton—type behavior near the "critical point" Q_c and nothing similar to "critical slowing down" has been observed.

To check whether these results are specific for a multiplicative stochastic process on a bounded domain we have studied two further examples on an unbounded domain for the stochastic variable.

The first one of these examples is given by the Langevin equation

$$\dot{x} = -dx + \frac{1}{(1+x^2)^{1/2}}\xi , \tag{4.7}$$

and the stochastic variable can assume values in the interval $(-\infty, \infty)$. For the stationary probability distribution of the Fokker-Planck equation associated with (4.7) we have

$$P_0 = \mathcal{N}(1+x^2)^{1/2}\exp\left[-\frac{d}{Q}(1+\tfrac{1}{2}x^2)x^2\right] . \tag{4.8}$$

The extrema of the stationary probability distribution are given by

$$\hat{x} = \begin{cases} 0, & Q < 2d \\ \pm[(Q/2d)^{1/2}-1]^{1/2}, & Q < 2d \end{cases} \tag{4.9}$$

i.e., for $Q > 2d$ the function P_0 shows two peaks, whereas for $Q < 2d$ only one peak occurs centered at $x = 0$. In the terminology of "noise-induced phase transition" this behavior of P_0 near $Q = 2d$ should have observable consequences on the stochastic properties of the model given by Eq. (4.7). To test this hypothesis we have calculated the moments of P_0

$$M_n = \langle x^n \rangle = \mathcal{N}\int_{-\infty}^{\infty}(1+x^2)^{1/2}x^n\exp\left[-\frac{d}{Q}(1+\tfrac{1}{2}x^2)x^2\right] . \tag{4.10}$$

In Fig. 9 we have plotted the stationary probability density as a function of d/Q and in Fig. 10 the moments $\langle x^2 \rangle$ and $\langle x^4 \rangle$ are shown for the same range of d/Q. As is easily checked the moments and the stationary probability density P_0 depend only on the ration d/Q due to the simple structure of the model (4.7).

From inspection of Figs. 9 and 10 we immediately conclude that although the number of peaks of P_0 changes at $Q_c = 2d$ the moments reveal a monotonic behavior over the whole range of Q values and nothing indicates the special role which should be played by $Q = Q_c$.

To study the time-dependent solution of the Fokker-Planck equation associated with Eq. (4.7) we used as above the techniques described in Sec. II and we have obtained for the lowest-lying eigenvalue, which dominates the long-time behavior, the results plotted in Fig. 11 for $2.5 \gtrsim d/Q \gtrsim 0.25$. As is immediately checked upper and lower bounds increase with increasing Q monotonically and no

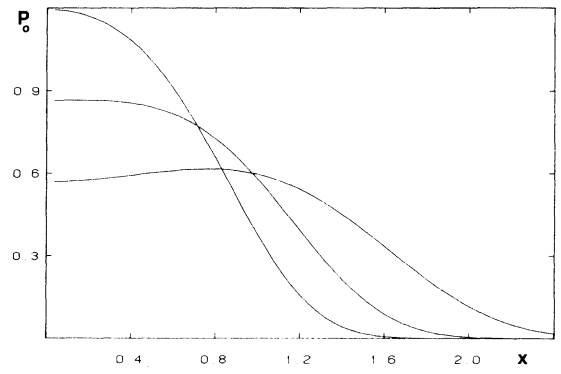

FIG. 9. The stationary probability distribution for the model introduced in Eq. (4.7) for $Q = 1,2,5$ ($d = 1$).

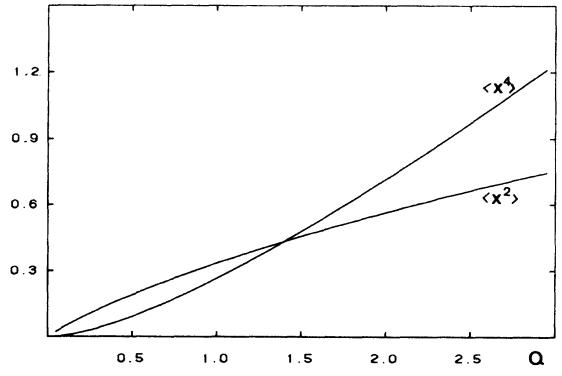

FIG. 10. Moments associated with the process given in Eq. (4.7). $\langle x^2 \rangle$ and $\langle x^4 \rangle$ are plotted as a function of Q ($d = 1$, $Q_c = 2$).

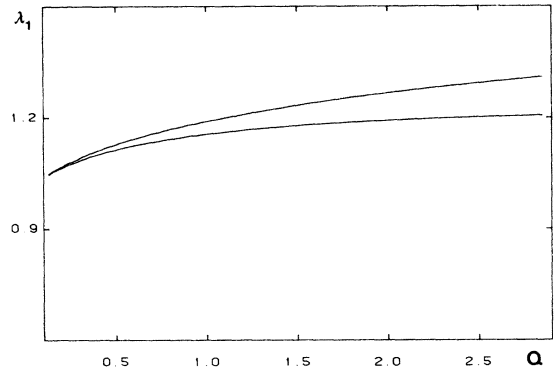

FIG. 11. Upper and lower bound for the lowest-lying eigenvalue λ_1 of the same stochastic process as a function of Q for $d=1$.

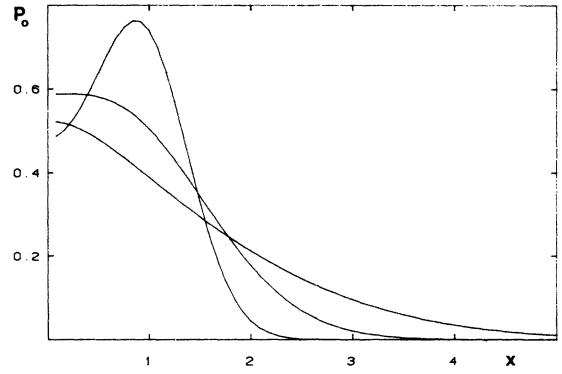

FIG. 12. Stationary probability distribution for $Q = 0.5$, 2, and 8 ($d=1$) for the process introduced in Eq. (4.12).

special features occur near $Q_c/d=2$. In addition it seems worthwhile to note a scaling property for the eigenvalues of the Fokker-Planck equation associated with model (4.7). We obtain

$$\frac{\lambda_n}{Q}=f_n\left[\frac{d}{Q}\right]. \tag{4.11}$$

That is, if one has derived the values of λ_n for one value of the damping constant d for all Q, one can extract via (4.11) the eigenvalues for all $d>0$.

As a third model we wish to consider in this section

$$\dot{x}=dx-x^3+(1+x^2)^{1/2}\xi\,, \tag{4.12}$$

and for the corresponding stationary probability distribution we find

$$P_0=\mathcal{N}(1+x^2)^{(1+d)/Q-1/2}\exp\left[-\frac{x^2}{Q}\right]. \tag{4.13}$$

By construction the number of peaks of P_0 (4.13) decreases for increasing values of Q contrary to the two models studied above. We find

$$x_0=\begin{cases}0, & Q>2d\\ \pm(d-\frac{1}{2}Q)^{1/2}, & Q<2d\end{cases}$$

i.e., two maxima for P_0 at $x_0=\pm(d-\frac{1}{2}Q)^{1/2}$ and one minimum at $x_0=0$ if $Q<2d$, and one maximum at $x_0=0$ if $Q>2d$. This qualitative discussion of the structure of P_0 is supplemented by the plots given in Fig. 12. To simplify the following discussions we choose the special parameter $d=0.3$, i.e., the "critical value" is $Q_c=0.6$. For this value of d we have calculated the moments $\langle x^2\rangle$ and $\langle x^4\rangle$ as a function of Q. The results of these calculations are plotted in Fig. 13 showing an increasing value of the moments for all Q. Concerning the time-dependent behavior we proceed as above and we plot the corresponding bounds for λ_1 in Fig. 14 showing that λ_1 increases monotonically with increasing Q. At $Q_c=2d$ the plot shows no special properties like "critical slowing down" or anything related.

To summarize the results of Sec. IV, it should be noted that we have studied three different models which are simple examples of three large classes of multiplicative stochastic processes. The first example is confined to a bounded domain (for the stochastic variable) and the corresponding stationary probability distribution splits into two peaks for a certain value of Q with increasing Q. In the second class the same type of behavior occurs; the stochastic variable, however, is allowed to assume values on an unbounded domain and our second example [Eq. (4.7)] is a simple representative of this class. Another large class of multipli-

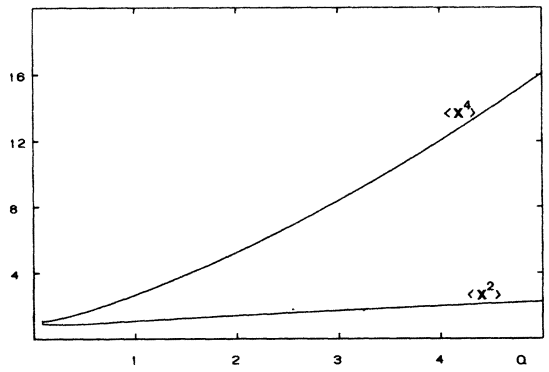

FIG. 13. Static moments $\langle x^2\rangle$ and $\langle x^4\rangle$ associated with Eqs. (4.12) and (4.13) for $d=1$ as a function of Q.

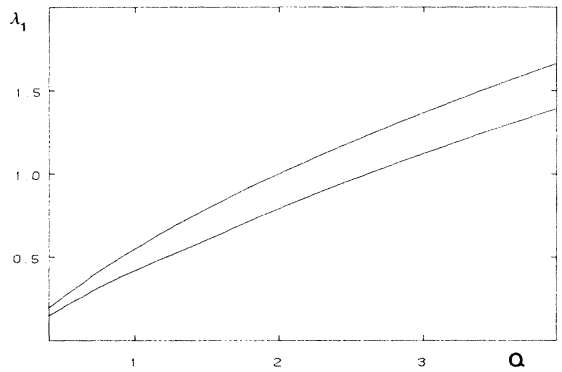

FIG. 14. Upper and lower bounds for the first excited state as a function of Q.

cative stochastic processes one might consider consists of those processes in which the number of peaks of the stationary probability distribution decreases while increasing the strength of the fluctuations Q, and for this class we have studied as an example Eq. (4.12).

As a result of the study of these processes we have found that no spectacular features are observed for the moments $\langle x^n \rangle$ or the lowest-lying eigenvalues near the point at which the number of extrema of the stationary probability distribution changes. From these results we arrive at the conclusion that there exists, at least for the models studied above, no phenomenon like critical slowing down at the points $Q=Q_c$. Furthermore we feel that there is no reason to coin the term "noise-induced phase transition" for an effect which is only related to the details of the stationary probability distribution. Observable physical quantities like stationary moments or relaxation times, however, do not show any peculiarities at the point $Q=Q_c$.

V. CONCLUSION

In this paper, we have presented a general approach to characterize the discrete eigenvalues of a Fokker-Planck problem, satisfying the "potential conditions," in terms of rigorous upper and lower bounds.

The upper bound was obtained by a variational method analogous to the Rayleigh-Ritz method. It was shown that the variational principle assumes a rather elegant and simple form for additive as well as multiplicative stochastic processes.

The lower bound has been derived by using the intermediate theorem of Weinstein which can be applied to the Fokker-Planck equation. By successive improvement of the variational test function the upper bound can be decreased while the lower bound increases simultaneously.

In the main part of the paper, we have applied these methods to various stochastic processes in order to demonstrate the practical applicability of the present approach.

For the case of additive noise we compare the upper and lower bounds for the low-lying eigenvalues for a bistable potential and for the single-mode laser with the results derived by alternative methods, or numerically, by various authors.

To discuss the question of multiplicative noise we have studied a class of a models for which some authors recently predicted a new class of phase transitions and critical slowing down. In addition we have shown in the present paper that for various other models discussed here, the upper and lower bounds for the lowest-lying nontrivial eigenvalues are not consistent with the prediction of critical slowing down. The time-dependent properties of the models studied are found to have little if anything in common with the traditonal notion of phase transitions and thus the concept of "noise-induced phase transitions" appears to be ill defined and a source of confusion.

APPENDIX A: INTERMEDIATE THEOREM FOR THE EIGENVALUES OF THE MASTER EQUATION IN DETAILED BALANCE

We briefly sketch the application of the intermediate theorem to the master equation (for the details of notation we refer to Ref. 2). The master equation reads

$$\dot{P}(m)=L_{mn}P(n) , \tag{A1}$$

where

$$L_{mn}=W(m,n)-\delta_{mn}\sum_l W(l,n) . \tag{A2}$$

Using the separation ansatz

$$P(m)=e^{-\lambda t}\varphi_m , \tag{A3}$$

we have

$$L_{mn}\varphi_n^\alpha=-\lambda_\alpha\varphi_m^\alpha . \tag{A4}$$

Because L_{mn} is in general not self-adjoint we have to consider also the adjoint problem

$$\chi_m^\alpha L_{mn}=-\lambda_\alpha^*\chi_n^\alpha . \tag{A5}$$

If detailed balance holds, i.e.,

$$W(m,n)P(n)=W(n,m)P(m) \tag{A6}$$

a variational principle can be associated with Eqs. (A4) and (A5) (Ref. 2) which takes the form

$$-\lambda_\alpha \leq \frac{(\chi_m^\alpha L_{mn}\varphi_n^\alpha)}{(\chi_m^\alpha \varphi_n^\alpha)} , \tag{A7}$$

where $\{\chi_m\}$, $\{\varphi_n\}$ are assumed to form a biorthogonal set of eigenvectors. To carry over the intermediate theorem to the master equation it is more convenient to write Eq. (A7) in the form

$$-\lambda_\alpha \leq \frac{(\tilde{\chi}_m \tilde{L}^s_{mn}\tilde{\chi}_n)}{(\tilde{\chi}_m\tilde{\chi}_m)} , \tag{A8}$$

where the $\{\tilde{\chi}_m\}$ are orthogonal to all lower eigenfunctions and where

$$\tilde{L}^s_{nm}=W(n,m)\frac{P^{1/2}(m)}{P^{1/2}(n)} \tag{A9}$$

and

$$\tilde{\chi}_n^\alpha = P_n^{1/2}\chi_n^\alpha . \tag{A10}$$

From Eq. (A8) it is now obvious how one has to proceed to get the intermediate theorem for the master equation. Introducing the abbreviations

$$\alpha=\frac{(\tilde{\chi}_m\tilde{L}^s_{mn}\tilde{\chi}_n)}{(\tilde{\chi}_m\tilde{\chi}_m)} \tag{A11}$$

and

$$\beta^2=\frac{(\tilde{\chi}_m\tilde{L}^s_{mn}\tilde{L}^s_{n0}\tilde{\chi}_0)}{(\tilde{\chi}_m\tilde{\chi}_m)} \tag{A12}$$

we find that $\beta^2 \geq \alpha^2$ and that there exists at least one eigenvalue which lies between

$$\alpha-(\beta^2-\alpha^2)^{1/2}$$

and (A13)

$$\alpha+(\beta^2-\alpha^2)^{1/2} .$$

The extraction of lower bounds for the eigenvalues of the master equation from the upper bound and from the intermediate theorem proceeds via the same path and under the same restrictions as for the cast of the Fokker-Planck equation.

APPENDIX B

In the main part of the present paper we have confined ourselves to Fokker-Planck equations that can be cast into self-adjoint form. It is possible, however, to generalize the results presented in Sec. II to so-called normal operators.

As has been discussed in Sec. II the Fokker-Planck operator L^{FP} and the transformed operator L, [34]

$$L=P_0^{-1/2}L^{\mathrm{FP}}P_0^{1/2} , \tag{A14}$$

are in general non-Hermitian. The operator L can be split into a Hermitian part L_H and an anti-Hermitian part L_A. If the anti-Hermitian part L_A and the Hermitian part L_H commute, i.e.,

$$[L_A,L_H]=0 , \tag{A15}$$

we have the equations

$$L_H\psi_\mu=-\mathrm{Re}(\lambda_\mu)\psi_\mu , \tag{A16}$$

$$L_A\psi_\mu=-i\,\mathrm{Im}(\lambda_\mu)\psi_\mu . \tag{A17}$$

It is now easy to check that all considerations of Sec. II can be carried out for Eq. (A16) and it becomes thus possible to get upper and lower bounds for the real part of the eigenvalues for Fokker-Planck operators that can be cast into normal from (A15). From Eq. (A17) one can then obtain a rough estimate for the imaginary part of the eigenvalues.

As an example we consider the equations ($x^+=u+iv=re^{i\varphi}$)

$$x^+=(a+ib)x^+-|x^+|^2x^++\xi , \tag{A18}$$

where ξ is assumed to be Gaussian white noise. In polar coordiantes Eq. (A18) reads

$$\dot{r}=ar-r^3+\xi , \tag{A19}$$

$$\dot{\varphi}=b+\xi . \tag{A20}$$

For the Hermitian part of the Fokker-Planck operator which is associated with the Langevin equations (A19) and (A20) we find

$$L_H=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+V(r) , \tag{A21}$$

and for the anti-Hermitian part

$$L_A=-b\frac{\partial}{\partial\varphi} , \tag{A22}$$

where

$$V(r)=-\frac{r^6}{4}+\frac{a}{2}r^4+r^2\left(-\frac{a^2}{4}+2\right)+(-a)\ . \tag{A23}$$

From inspection of Eqs. (A21) – (A23) it is obvious that L_H and L_A commute and that the eigenvalues of Eq. (A21) are those of the single-mode laser. Equation (A18) can be derived in a physical context, e.g., from the Maxwell-Bloch equations for two-level atoms under the assumption that there is a small detuning in the equation for the field whereas the polarization has no detuning. Then one obtains via adiabatic elimination of the polarization and the inversion of the two-level atoms, Eq. (A18). If one starts, however, in the Maxwell-Bloch equations with a detuned polarization one obtains via adiabatic elimination a Langevin equation whose corresponding Fokker-Planck equation cannot be cast into normal form.[46]

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