# Resolvent method of evaluation of the dynamic conductivity for a strongly interacting electron-phonon system

Akira Suzuki and Dennis Dunn
*J. J. Thomson Physical Laboratory, University of Reading, Whiteknights, Reading RG6 2AF, United Kingdom*
(Received 24 February 1981)

A resolvent superoperator approach is described for the evaluation of the dynamic conductivity of a strongly interacting electron-phonon system, starting from a current-correlation integral formula. The conductivity is expressed in terms of a self-energy superoperator and a self-consistent expression is obtained for this temperature-dependent quantity.

## I. INTRODUCTION

Transport coefficients, such as electrical conductivity, are usually calculated either from a Boltzmann transport equation or in terms of correlation functions. Boltzmann's transport equation has a limited validity[1-5] and is applicable only to the case of weak interactions. A large number of theoretical methods have been developed for the evaluation of the electrical conductivity of solids for various model systems using correlation function techniques.[5-20] The correlation function approach can, in principle, be used for any system however strong the interactions. If we consider an arbitrary system subject to an external electric field, then the Fourier component of the current density $\vec{j}(\vec{k},\omega)$ is related to the Fourier component of the *total* electric field $\vec{E}(\vec{k},\omega)$ by

$$\vec{j}(\vec{k},\omega)=\int d\vec{k}'\,\underline{\sigma}(\vec{k},\vec{k}',\omega)\cdot\vec{E}(\vec{k}',\omega)\ , \tag{1.1}$$

where the nonlocal conductivity tensor is given by[14]

$$\underline{\sigma}(\vec{k},\vec{k}',\omega)=\underline{\sigma}^{\rm ex}(\vec{k},\vec{k}',\omega)-\frac{i}{\epsilon_0\omega}\int\frac{d\vec{k}''}{(2\pi)^3}\frac{\underline{\sigma}^{\rm ex}(\vec{k},\vec{k}'',\omega)[(\omega/c)^2\underline{1}-\vec{k}''\vec{k}'']\underline{\sigma}(\vec{k}'',\vec{k}',\omega)}{[\vec{k}''^2-(\omega/c)^2-i\delta\,{\rm sgn}(\omega)]} \tag{1.2}$$

and $\underline{\sigma}^{\rm ex}$ is the external conductivity. This relates the current density to the *external* electric field, that is, the field produced by external sources and which would exist in the absence of the medium, and is given by Kubo's formula[6]:

$$\underline{\sigma}^{\rm ex}(\vec{k},\vec{k}',\omega)=\lim_{\delta\to 0+}\int_0^\infty dt\,e^{-i(\omega-i\delta)t}\int_0^\beta d\beta_1\,{\rm Tr}[\rho\vec{J}(\vec{k}',-i\hbar\beta_1)\vec{J}(\vec{k},t)]\ , \tag{1.3}$$

where $\rho$ is the equilibrium statistical operator and $\vec{J}(\vec{k},t)$ is the spatial Fourier component of the current density operator. A small imaginary part $-i\delta$ added to the frequency implies adiabatic switching of an external electric field and ensures convergence of the integral at $t=\infty$. The Hamiltonian describing such a system must include the effects of Coulomb interactions either directly or by means of quantized electromagnetic field interactions.

An alternative to this exact approach is to consider an effective medium in which electron-lattice and electron-impurity interactions are screened and in which the external electric field is replaced by the total electric field but all other effects of the Coulomb interactions are neglected. The expression for the conductivity in this effective medium is

$$\underline{\sigma}(\vec{k},\vec{k}',\omega)=\int_0^\infty dt\,e^{-i(\omega-i\delta)t}\int_0^\beta d\beta_1\,{\rm Tr}[\rho\vec{J}(\vec{k}',-i\hbar\beta_1)\vec{J}(\vec{k},t)]\ . \tag{1.4}$$

This looks the same as Eq. (1.3) for $\underline{\sigma}^{\rm ex}$ but in (1.4) the statistical operator and the time-evolution operator involve an *effective* Hamiltonian which excludes Coulomb interactions. A discussion of this approach is given by Toyozawa.[15] If such a system is subject to a spatially slowly varying electric field, then the spatially averaged current density is given by the local conductivity tensor,

$$\underline{\sigma}(\omega)=\Omega^{-1}\int_0^\infty dt\, e^{-i(\omega-i\delta)t} \times\int_0^\beta d\beta_1 \mathrm{Tr}[\rho\vec{J}(-i\hbar\beta_1)\vec{J}(t)]\ , \tag{1.5}$$

where $\vec{J}(t)=\vec{J}(\vec{k}=0,t)$.

It is this latter approach that we shall take as a starting point. Expressions (1.4) and (1.5) are not, as many people claim, exact but should nevertheless provide a very good approximation for the conductivity even for strong electron-phonon or electron-impurity interactions.

In this paper we will show a detailed method of calculation for the frequency-dependent conductivity of an electron-phonon system, starting from the current correlation formula [Eq. (1.5)]. Our method is based on the resolvent formalism[5,11,17–21] applied to quantum-transport theory. Although one has to work in a superspace of operators, the advantage of working with the resolvent rather than conventional Green's functions[8,10] is that the transport problem is reduced to a study of an effective one-electron operator. Therefore, the difficulties of simultaneously solving both Dyson and Bethe-Salpeter equations in the thermodynamic Green's function approach are thereby avoided. A similar method has been presented earlier by several authors.[5,17–22] However, our method does not heavily depend on the strength of the interaction. Therefore, the results obtained could be applied to the strong-interaction case unlike the previous theories[5,16–22] which are based on the perturbation expansion of the resolvent superoperators. We employ a factorization approximation for the equilibrium statistical operator [Eq. (2.11)] and calculate the current correlation function using the grand canonical distribution for the electrons and the canonical distribution for phonons. Thus, any collision process between an electron and phonons is assumed to take place in the average field of the phonons. The effect of such a field is to induce perturbed single-particle energies and to introduce lifetimes for the electron states. The lifetime broadening due to the interactions is, for example, responsible for the spectral broadening of line shapes and can be studied theoretically by examining the real part of the conductivity tensor.[20–22] We relate the conductivity to a self-energy superoperator and hence it is important to provide a general expression for this temperature-dependent self-energy which is applicable to the strong interaction case. In this paper we show how to do this in a self-consistent manner. The theory is developed independently of the single-particle representation (momentum, Landau, or other), and hence it can be applied irrespective of the system studied. In particular it is valid for systems subject to a constant magnetic field and the differences due to $\vec{B}$ arise only at the final stage of calculation. Therefore, it can be applied to the study of resonance problems[20–24] such as cyclotron resonance ($\omega=\omega_c$), phonon-assisted cyclotron resonance ($\omega=N\omega_c\pm\omega_L$), and magneto-phonon resonance ($\omega\to 0,\omega_L=N\omega_c$).

In Sec. II, the general theoretical method of the evaluation of the frequency-dependent conductivity of an electron-phonon system is outlined. The one-electron resolvent superoperator is given in terms of a simpler effective one-electron resolvent $R_z^D$. This depends on the self-energy superoperator $g_z^D$ which is defined at the outset unlike previous theories[5,17–22] which are based on the perturbation expansion of the resolvent superoperators and hence depend heavily on the potential strength. In Sec. III, the general technique[5,25] for dealing with superoperators is briefly presented. The general expression for the temperature-dependent self-energy superoperator is given. This expression [Eq. (3.20)] is similar to expressions [Eqs. (2.15)–(2.19)] by Prasad[21] who used a somewhat similar approach.

## II. FORMAL PRELIMINARIES

### A. The model Hamiltonian and the conductivity tensor

We consider an electron-phonon system characterized by the following time-independent Hamiltonian:

$$H=\sum_j h^{(j)}+H_{\mathrm{ph}}\ , \tag{2.1}$$

$$h^{(j)}\equiv h_0^{(j)}+\lambda V \equiv h_0^{(j)}+\lambda\sum_{\vec{q}}(\gamma_{\vec{q}}^{(j)}b_{\vec{q}}+\gamma_{\vec{q}}^{\dagger(j)}b_{\vec{q}}^\dagger)\ , \tag{2.2}$$

$$H_{\mathrm{ph}}=\sum_{\vec{q}}(b_{\vec{q}}^\dagger b_{\vec{q}}+\tfrac{1}{2})\hbar\omega_{\vec{q}}\ , \tag{2.3}$$

where $h_0$ is the single-particle energy operator, $V=\sum(\gamma_{\vec{q}}b_{\vec{q}}+\gamma_{\vec{q}}^\dagger b_{\vec{q}}^\dagger)$ is the interaction potential between an electron and a phonon with momentum $\vec{q}$, $\lambda$ is a parameter to indicate the order in the expansion, which is set equal to 1 later on, $b_{\vec{q}}$ and $b_{\vec{q}}^\dagger$ are, respectively, the phonon annihilation and creation operators, $\gamma_{\vec{q}}$ is the *screened* interaction

(one-electron) operator, which is defined in terms of the matrix elements of electron states. The energy operator $h_0$ may contain a magnetic field $\vec{B}$.

To evaluate the frequency-dependent conductivity, we start from the current correlation integral formula [Eq. (1.5)][6]

$$\sigma_{rs}(\omega)=\Omega^{-1}\int_0^\infty dt\, e^{-i(\omega-i\delta)t} \times\int_0^\beta d\beta_1 \mathrm{Tr}[\rho J_s(-i\hbar\beta_1)J_r(t)] , \tag{2.4}$$

where $\omega$ is the frequency of the external field, $\delta$ is a positive number, $\Omega$ the volume, and $\rho$ is the grand canonical operator

$$\rho=\rho(H-\zeta N)=e^{\beta(\zeta N-H)}/\mathrm{Tr}(e^{\beta(\zeta N-H)}) . \tag{2.5}$$

$\beta\equiv(k_BT)^{-1}$ is a reciprocal temperature, $\zeta$ the chemical potential, $N$ the electron number operator, and $J_r(t)$ is the $r$ component of a total current operator in Heisenberg picture:

$$J_r(t)=e^{iHt/\hbar}J_re^{-iHt/\hbar},\quad J_r=\sum_j j_r^{(j)} , \tag{2.6}$$

where $\vec{j}$ is the velocity operator multiplied by the electronic charge $-e$. This formula is valid providing $H$ does *not* include Coulomb interactions.

Let us introduce the fictitious Hamiltonian $H'=H-\vec{u}\cdot\vec{J}$ ($\vec{u}$ is a complex-number vector), then further simplification is effected by utilizing the identity[5]:

$$\int_0^\beta d\beta_1\rho(H)J_s(-i\hbar\beta_1)=\lim_{\vec{u}\to0}\frac{\partial}{\partial u_s}\rho(H') . \tag{2.7}$$

Substituting (2.7) into (2.4), Kubo's formula is reduced to a more compact form

$$\sigma_{rs}(\omega)=\lim_{\delta\to0^+}\lim_{\vec{u}\to0}\frac{\partial}{\partial u_s}\Omega^{-1} \times\int_0^\infty dt\, e^{-i(\omega-i\delta)t}\mathrm{Tr}[\rho' J_r(t)] , \tag{2.8}$$

where $\rho'$ is defined by

$$\rho'=\rho(H'-\zeta N) . \tag{2.9}$$

We assume that the phonons are distributed all the time according to the canonical distribution law

$$\rho_{\mathrm{ph}}=\rho_{\mathrm{ph}}(H_{\mathrm{ph}})=e^{-\beta H_{\mathrm{ph}}}/\mathrm{Tr}^{(\mathrm{ph})}(e^{-\beta H_{\mathrm{ph}}}) , \tag{2.10}$$

and further assume that the following factorization is allowed:

$$\rho'=\rho_{\mathrm{ph}}(H_{\mathrm{ph}})\rho\left[\sum h^{(j)}-\zeta N-\vec{u}\cdot\vec{J}\right] . \tag{2.11}$$

This is equivalent to the assumption that the phonons are part of the heat bath for the system and is likely to be valid if the system has a low density of electrons such as in a semiconductor. The approximation is almost certainly invalid for a metal. The Kubo formula can then be expressed in terms of a single-particle trace (denoted by Tr):

$$\sigma_{rs}(\omega)=\lim_{\delta\to0^+}\lim_{\vec{u}\to0}\frac{\partial}{\partial u_s}\Omega^{-1}\int_0^\infty dt\, e^{-i(\omega-i\delta)t}\mathrm{Tr}\left[\left\langle\exp\left[-i(h+H_{\mathrm{ph}})\frac{t}{\hbar}\right]n'\exp\left[i(h+H_{\mathrm{ph}})\frac{t}{\hbar}\right]\right\rangle_{\mathrm{ph}}j_r\right] , \tag{2.12}$$

where the angular brackets, $\langle\cdots\rangle_{\mathrm{ph}}$, denote the averaging over the phonon states, $n'$ is the Fermi operator

$$n'\equiv(e^{\beta(h-\vec{u}\cdot\vec{j}-\zeta)}+1)^{-1},\quad h\equiv h_0+\lambda V . \tag{2.13}$$

This can be written in the form

$$\sigma_{rs}(\omega)=-i\hbar\lim_{\delta\to0+}\Omega^{-1}\mathrm{tr}[\Psi_s(-\hbar\omega+i\hbar\delta)j_r] , \tag{2.14}$$

where

$$\Psi_s(z)\equiv\lim_{\vec{u}\to0}\frac{\partial}{\partial u_s}n_z'\equiv\lim_{\vec{u}\to0}\left\langle i\int_0^\infty dt\, e^{izt}\frac{\partial}{\partial u_s}\exp[-i(\mathcal{h}+\mathscr{H}_{\mathrm{ph}})t]n'\right\rangle_{\mathrm{ph}}$$

$$=\lim_{\vec{u}\to0}\frac{\partial}{\partial u_s}\langle(\mathcal{h}_0+\mathscr{H}_{\mathrm{ph}}+\lambda\mathscr{V}-z)^{-1}n'\rangle_{\mathrm{ph}} . \tag{2.15}$$

A script letter denotes a superoperator which generates a commutator upon acting on an ordinary quantum operator such that

$$\mathscr{A}B \equiv [A,B] \equiv AB - BA \ . \tag{2.16}$$

As can be seen from (2.14) and (2.15), evaluation of $\sigma_{rs}(\omega)$ is reduced to a calculation of the phonon-averaged quantal density operator $n_z'$ defined in the Laplace transformed space, viz.,

$$\begin{aligned} n_z' &\equiv \langle (\mathscr{h}_0 + \mathscr{H}_{\rm ph} + \lambda\mathscr{V} - z)^{-1} n' \rangle_{\rm ph} \\ &\equiv \langle R_z n' \rangle_{\rm ph} \ , \end{aligned} \tag{2.17}$$

where the resolvent superoperator is defined by

$$R_z \equiv (\mathscr{h}_0 + \mathscr{H}_{\rm ph} + \lambda\mathscr{V} - z)^{-1} \ . \tag{2.18}$$

### B. Evaluation of $n_z'$

In order to evaluate $n_z'$, let us consider the following identity:

$$n_z' = \langle R_z n' \rangle_{\rm ph} = \langle R_z^D n' \rangle_{\rm ph} + \langle (R_z - R_z^D) n' \rangle_{\rm ph} \ , \tag{2.19}$$

where the effective one-electron resolvent superoperator $R_z^D$ and the self-energy superoperator $g_z^D$ are, respectively, defined by

$$R_z^D = (\mathscr{h}_0 + \mathscr{H}_{\rm ph} + g_z^D - z)^{-1} \ , \tag{2.20}$$

$$g_z^D = -\lambda^2 \langle \mathscr{V} R_z \mathscr{V} \rangle_{\rm ph} \ . \tag{2.21}$$

From (2.18) and (2.20) we obtain

$$R_z - R_z^D = R_z^D (g_z^D - \lambda\mathscr{V}) R_z \ . \tag{2.22}$$

Substituting (2.22) into (2.19), we obtained (2.19) as

$$\begin{aligned} n_z' &\equiv \langle R_z n' \rangle_{\rm ph} \\ &\equiv \langle R_z^D n' \rangle_{\rm ph} + \langle R_z^D (g_z^D - \lambda\mathscr{V}) R_z n' \rangle_{\rm ph} \\ &= (\mathscr{h}_0 + g_z^D - z)^{-1} \hat{X} \ , \end{aligned} \tag{2.23}$$

where

$$\hat{X} = \langle n' \rangle_{\rm ph} + \langle (g_z^D - \lambda\mathscr{V}) R_z n' \rangle_{\rm ph} \ . \tag{2.24}$$

The proof that Eq. (2.23) is a consequence of the previous line is given in the Appendix. We expect that for most systems,

$$\hat{X} \approx \langle n' \rangle_{\rm ph} \tag{2.25}$$

since the terms $\langle g_z^D R_z n' \rangle_{\rm ph}$ and $\lambda \langle \mathscr{V} R_z n' \rangle_{\rm ph}$ tend to cancel. In fact, we can show that at high frequencies $\langle (g_z^D - \lambda\mathscr{V}) R_z n' \rangle_{\rm ph} \sim 1/z^2$ whereas, of course $\langle n' \rangle_{\rm ph}$ is independent of $z$. If in fact (2.25) is correct, then the factorization (2.23) succeeds in separating the dynamic and static properties: $(\mathscr{h}_0 + g_z^D - z)^{-1}$ represents the dynamic properties and $\langle n' \rangle_{\rm ph}$ is purely static. The approximation procedures applicable to dynamic and to static problems are usually quite distinct. We are concerned in this paper mainly with dynamic properties of the system and so we shall assume in general that $\langle n' \rangle_{\rm ph}$ is known. For weakly interacting systems, however, we can determine $\langle n' \rangle_{\rm ph}$ by means of a perturbation expansion of $n'$,

$$\begin{aligned} n' &= \frac{i}{2\pi} \int d\mathscr{I} \, f(\mathscr{I}) (h_0 + \lambda V - \vec{\mathrm{u}} \cdot \vec{\mathrm{j}} - \mathscr{I})^{-1} \\ &= \frac{i}{2\pi} \int d\mathscr{I} \, f(\mathscr{I}) [G^0(\mathscr{I}) - \lambda G^0(\mathscr{I}) V G^0(\mathscr{I}) + \lambda^2 G^0(\mathscr{I}) V G^0(\mathscr{I}) V G^0(\mathscr{I}) \\ &\qquad - \lambda^3 G^0(\mathscr{I}) V G^0(\mathscr{I}) V G^0(\mathscr{I}) V G^0(\mathscr{I}) + O(\lambda^4)] \ , \end{aligned} \tag{2.26}$$

where $f(\mathscr{I})$ and $G^0(\mathscr{I})$ are, respectively, defined by

$$f(\mathscr{I}) \equiv (e^{\beta(\mathscr{I} - \zeta)} + 1)^{-1} \ , \tag{2.27}$$

$$G^0(\mathscr{I}) \equiv (h_0 - \vec{\mathrm{u}} \cdot \vec{\mathrm{j}} - \mathscr{I})^{-1} \ . \tag{2.28}$$

What is required next is some systematic method of evaluating $g_z^D$ [Eq. (2.21)]. Such a method can be devised by expressing $R_z$ as in infinite series; by successive iterations using (2.21) and (2.22), we obtain

$$R_z = R_z^D \sum_{n=0}^{\infty} [(g_z^D - \lambda\mathscr{V})R_z^D]^n$$

$$= R_z^D - \lambda R_z^D\mathscr{V}R_z^D + \lambda^2(R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D - R_z^D\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D)$$

$$-\lambda^3(R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D - R_z^D\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D\mathscr{V}R_z^D - R_z^D\mathscr{V}R_z^D\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D)$$

$$+\lambda^4(R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D - R_z^D\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D$$

$$-R_z^D\mathscr{V}R_z^D\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D\mathscr{V}R_z^D - R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D$$

$$-R_z^D\langle\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D + R_z^D\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D) + O(\lambda^5) . \tag{2.29}$$

Substituting (2.29) into (2.21), we obtain $g_z^D$ as

$$g_z^D \equiv -\lambda^2\langle\mathscr{V}R_z\mathscr{V}\rangle_{\rm ph}$$

$$= -\lambda^2\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph} - \lambda^4(\langle\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph} - \langle\mathscr{V}R_z^D\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D\mathscr{V}\rangle_{\rm ph})$$

$$-\lambda^6(\langle\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph} - \langle\mathscr{V}R_z^D\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}$$

$$-\langle\mathscr{V}R_z^D\mathscr{V}R_z^D\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph} - \langle\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D\mathscr{V}\rangle_{\rm ph}$$

$$-\langle\mathscr{V}R_z^D\langle\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D\mathscr{V}\rangle_{\rm ph}$$

$$+\langle\mathscr{V}R_z^D\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D\mathscr{V}\rangle_{\rm ph}) + O(\lambda^8) . \tag{2.30}$$

We should emphasize that these series expansions for $R_z$ and $g_z^D$ are *not* the conventional perturbation expansions and that, in general, we expect the convergence to be quite rapid, even for strongly interacting systems. The reason for this is that the effects of the interaction are included in the denominator of $R_z^D$. This also has the effect that the term-by-term divergences experienced in the usual perturbation theory are circumvented. The operator $g_z^D$ defined by (2.21) is similar but *not* identical to the corresponding operator defined in diagramatic terms in the previous theories.[5,17–22] To second order in (2.30) these operators are the same but there are differences in fourth and higher orders.

Using these expressions for $R_z$ and $g_z^D$, we obtain $\langle(g_z^D - \lambda\mathscr{V})R_z n'\rangle_{\rm ph}$ as

$$\langle(g_z^D - \lambda\mathscr{V})R_z n'\rangle_{\rm ph} = -\lambda\langle\mathscr{V}R_z^D n'\rangle_{\rm ph} + \lambda^2(\langle\mathscr{V}R_z^D\mathscr{V}R_z^D n'\rangle_{\rm ph} - \langle\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D n'\rangle_{\rm ph})$$

$$-\lambda^3(\langle\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D n'\rangle_{\rm ph} - \langle\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D\mathscr{V}R_z^D n'\rangle_{\rm ph} - \langle\mathscr{V}R_z^D\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D n'\rangle_{\rm ph})$$

$$+\lambda^4(\langle\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D n'\rangle_{\rm ph} - \langle\langle\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D n'\rangle_{\rm ph}$$

$$-\langle\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D\mathscr{V}R_z^D\mathscr{V}R_z^D n'\rangle_{\rm ph} - \langle\mathscr{V}R_z^D\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D\mathscr{V}R_z^D n'\rangle_{\rm ph}$$

$$-\langle\mathscr{V}R_z^D\mathscr{V}R_z^D\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D n'\rangle_{\rm ph} + \langle\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D n'\rangle_{\rm ph}$$

$$+\langle\langle\mathscr{V}R_z^D\langle\mathscr{V}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D\mathscr{V}\rangle_{\rm ph}R_z^D n'\rangle_{\rm ph}) + O(\lambda^5) . \tag{2.31}$$

In what follows, we shall neglect this term. Let us consider the $g_z^D$ expansion [Eq. (2.30)]. In this expression each term contains implicitly many terms which involve summations over many phonon momenta. However, only a small proportion of all the terms implied in the expansion of $g_z^D$ survives the phonon averaging process. Firstly, any term with an odd number of $V$s is zero since $V$ is a bilinear function of $b_{\vec{q}}$ and $b_{\vec{q}}^\dagger$ [see Eq. (2.2)]. We have already used this to obtain expression (2.30). Secondly, as we have assumed there is no correlation between phonons having different momenta, if we write the interaction operator as

$$V = \sum_{\vec{q}} V_{\vec{q}} , \quad \mathscr{V} = \sum_{\vec{q}} \mathscr{V}_{\vec{q}} , \tag{2.32}$$

then the only terms which are nonzero are those which have an even number of $V_{\vec{q}}, \mathscr{V}_{\vec{q}}$ operators in the phonon averaging process. So that, for example, $\langle \mathscr{V}_{\vec{q}} R_z^D \mathscr{V}_{\vec{q}'} \rangle_{\rm ph}$ is zero unless $\vec{q}=\vec{q}'$. In this way we obtain the internal structure of $g_z^D$ as

$$g_z^D = -\lambda^2 \sum_{\vec{q}} \langle \mathscr{V}_{\vec{q}} R_z^D \mathscr{V}_{\vec{q}} \rangle_{\rm ph} - \lambda^4 \left[ \sum_{\vec{q}} (\langle \mathscr{V}_{\vec{q}} R_z^D \mathscr{V}_{\vec{q}} R_z^D \mathscr{V}_{\vec{q}} R_z^D \mathscr{V}_{\vec{q}} \rangle_{\rm ph} - \langle \mathscr{V}_{\vec{q}} R_z^D \langle \mathscr{V}_{\vec{q}} R_z^D \mathscr{V}_{\vec{q}} \rangle_{\rm ph} R_z^D \mathscr{V}_{\vec{q}} \rangle_{\rm ph}) + \sum_{\vec{q}} \sum_{\vec{q}'(\neq \vec{q})} (\langle \mathscr{V}_{\vec{q}} R_z^D \mathscr{V}_{\vec{q}'} R_z^D \mathscr{V}_{\vec{q}} R_z^D \mathscr{V}_{\vec{q}'} \rangle_{\rm ph} + \langle \mathscr{V}_{\vec{q}} R_z^D \mathscr{V}_{\vec{q}} R_z^D \mathscr{V}_{\vec{q}'} R_z^D \mathscr{V}_{\vec{q}'} \rangle_{\rm ph}) \right] + O(\lambda^6) . \tag{2.33}$$

We have developed a prescription of the method for the practical evaluation of the dynamic conductivity tensor for an electron-phonon system. It should be stressed that the formulas obtained so far are developed independently of the single-particle representation (position, momentum, Landau, or other) and hence can be applied to a system subjected to a static magnetic field. As can be seen in (2.23) and (2.24), the $\omega$ (i.e., $z$) dependency of the conductivity tensor is entered in the self-energy superoperator and the energy denominator of $R_z^D$. Although we can evaluate the $\omega$-dependent terms arising from $\hat{X}$ [Eq. (2.24)] which may be important in the low-frequency region following the prescription outlined in this section, we shall not consider them in the present investigation. For the cyclotron-resonance problem we can safely ignore these $\omega$-dependent terms, which contain no resonance terms.[9,22] In any case, we expect the effect of these correction terms to be small and the optical-absorption line shapes, which are related to the dynamic conductivity tensor, to be mainly determined by the energy denominator $(\mathscr{H}_0+g_z^D-z)\equiv(r_z^D)^{-1}$ in (2.23).

Let us reexpress the conductivity tensor as follows:

$$\sigma_{rs}(z) = -i\hbar \lim_{\delta\to 0+} \Omega^{-1} \mathrm{tr}[(r_z^D M_s) j_r] , \tag{2.34}$$

where $M_s$ is defined by

$$M_s = \lim_{\vec{u}\to 0} \frac{\partial}{\partial u_s} \hat{X} = \lim_{\vec{u}\to 0} \frac{\partial}{\partial u_s} \langle n' \rangle_{\rm ph} . \tag{2.35}$$

It is now clear that choosing an appropriate representation for an electron state, we can evaluate the dynamic conductivity $\sigma_{rs}(\omega)$. Let us write $\mathrm{tr}[(r_z^D M_s) j_r]$ in a matrix representation:

$$\mathrm{tr}[(r_z^D M_s) j_r] = \sum_{\lambda_1,\lambda_2} \langle \lambda_1 | r_z^D M_s | \lambda_2 \rangle \langle \lambda_2 | j_r | \lambda_1 \rangle , \tag{2.36}$$

where a single-electron eigenstate is labeled by $\lambda$ and satisfies the eigenvalue equation

$$h_0 | \lambda \rangle = E_\lambda | \lambda \rangle . \tag{2.37}$$

Our problem is now reduced to the evaluation of the matrix elements for $(r_z^D M_s)$, which is a function of superoperators. Consider the quantal operator $\Psi_s(z) \equiv r_z^D M_s$. Multiplying $(r_z^D)^{-1}$ from the left, we obtain

$$(r_z^D)^{-1} \Psi_s(z) = (\mathscr{H}_0 + g_z^D - z) \Psi_s(z) = M_s . \tag{2.38}$$

We consider the $\lambda_1,\lambda_2$ element of (2.38):

$$(E_{\lambda_1} - E_{\lambda_2} - z) \langle \lambda_1 | \Psi_s(z) | \lambda_2 \rangle + \langle \lambda_1 | g_z^D \Psi_s(z) | \lambda_2 \rangle = \langle \lambda_1 | M_s | \lambda_2 \rangle . \tag{2.39}$$

It is clearly seen that if $g_z^D = 0$ (no interaction), absorption spectra have a sharp peak at $E_{\lambda_1} - E_{\lambda_2} - z = 0$. However, due to the interaction incorporated in $g_z^D$ such peaks are lifetime-broadened. We would like to emphasize that our results depend only on the assumptions that phonons are uncorrelated and that the statistical operator is factorizable. The former assumption may be relaxed in our formalism. The latter is likely to be valid when one considers the system of low-electron densities such as in a semiconductor. We did not appeal to the weak-interaction assumption. Hence, the formula could be applied to the strong interaction case. Solving this equation for $\langle \lambda_1 | \Psi_s(z) | \lambda_2 \rangle$, one can, in principle, evaluate the conductivity tensor. Unfortunately, it is not feasible since $g_z^D$ is generally not diagonal. However, we evaluate it by making an appropriate approximation to the matrix elements of $g_z^D$. In the next section we shall illustrate the approximation procedure and demonstrate a method to evaluate the self-energy superoperator.

## III. APPROXIMATION PROCEDURE

An exact solution of transport equation for $\langle\lambda_1|\Psi_s(z)|\lambda_2\rangle$, Eq. (2.39), is not feasible because the self-energy operator $g_z^D$ is, in general, nondiagonal. In order to proceed further we therefore need an approximation scheme for the evaluation of $\langle\lambda_1|g_z^D\Psi_s(z)|\lambda_2\rangle$. The approximation scheme has two distinct stages. Firstly, we consider only the lowest-order contribution to $g_z^D$ from its series expansion, Eq. (2.30) or Eq. (2.33). That is, we assume

$$g_z^D \approx -b_z = -\sum_{\vec{q}} \langle \mathscr{V}_{\vec{q}} R_z^D \mathscr{V}_{\vec{q}} \rangle_{\rm ph} , \tag{3.1}$$

where $R_z^D$ also contains $b_z$:

$$R_z^D = (\mathscr{H}_0 + \mathscr{H}_{\rm ph} - b_z - z)^{-1} . \tag{3.2}$$

We note that this approximation to the self-energy operator contains higher-order effects of the interaction through the appearance of $b_z$ in $R_z$ and should be valid even for quite strong interactions. The second stage of approximation involves a scheme for evaluating $\langle\lambda_1|b_z\Psi_s(z)|\lambda_2\rangle$ which will be outlined below.

### A. Second-order non-self-consistent approximation (weak-interaction case)

In order to see the structure of the collision term, let us consider the crude approximation which is applicable only if the interaction between electron and phonon is weak. Then Eq. (3.1) can be further approximated:

$$g_z^D \approx -b_z^0 = -\sum_{\vec{q}} \langle \mathscr{V}_{\vec{q}} R_z^0 \mathscr{V}_{\vec{q}} \rangle_{\rm ph} , \tag{3.3}$$

where

$$R_z^0 = (\mathscr{H}_0 + \mathscr{H}_{\rm ph} - z)^{-1} . \tag{3.4}$$

Since the collision operator is a function of superoperators, to evaluate the matrix element of such a function is a complicated matter. The practical calculation involving superoperators can be facilitated by using a general technique due to Resibois.[25] In this method a four-leg structure (which should be used for calculating the matrix elements of superoperators) is reduced to two legs by introducing a new representation of such operators so that the tetradic algebra[19,20] reduces to the familiar algebra. This general technique exhibits a power for evaluating the self-consistent collision term to be discussed in the next section.

Let us specify the $\lambda_1,\lambda_2$ matrix element of an arbitrary operator $A$ by a new matrix representation:

$$\langle\lambda_1|A|\lambda_2\rangle = A_{\lambda_1-\lambda_2}\left[\frac{\lambda_1+\lambda_2}{2}\right] . \tag{3.5}$$

If we introduce a pair of variables $(\nu,\mu)$ replacing the pair $(\lambda_1,\lambda_2)$ such that

$$\nu \equiv \frac{\lambda_1+\lambda_2}{2}, \quad \mu \equiv \lambda_1-\lambda_2 , \tag{3.6}$$

then (3.5) is expressed by the new notation

$$\langle\lambda_1|A|\lambda_2\rangle = \langle \nu+\tfrac{1}{2}\mu|A|\nu-\tfrac{1}{2}\mu\rangle \equiv A_\mu(\nu) . \tag{3.7}$$

Let us define an operator $(\mu|\tilde{h}(\nu)|\mu')$ in $\nu$ space associated with superoperator $\mathscr{h}$ such that

$$(\mu|\tilde{h}(\nu)|\mu') \equiv \eta^{\mu'} h_{\mu-\mu'}(\nu)\eta^{-\mu} - \eta^{-\mu'} h_{\mu-\mu'}(\nu)\eta^{\mu} , \tag{3.8}$$

where we have introduced the shift operators $\eta^{\pm\mu}$ which replace a function of $\nu$ by the same function of $\nu\pm\frac{1}{2}\mu$:

$$\eta^{\pm\mu} f(\nu) = f(\nu\pm\tfrac{1}{2}\mu) . \tag{3.9}$$

It is noted that an operator $\tilde{h}(\nu)$ is just another way of writing the superoperator $\mathscr{h}$ and a matrix operator $(\mu|\tilde{h}(\nu)|\mu')$ is a matrix with respect to one set of quantum numbers and an operator with respect to the other. It is then easy to verify that the matrix elements of a commutator can be expressed in terms of $(\mu|\tilde{h}(\nu)|\mu')$:

$$\begin{aligned}\langle\lambda_1|\mathscr{h}A|\lambda_2\rangle &= (\mathscr{h}A)_\mu(\nu) \\ &\equiv \langle \nu+\tfrac{1}{2}\mu|\mathscr{h}A|\nu-\tfrac{1}{2}\mu\rangle \\ &= \sum_{\mu'} (\mu|\tilde{h}(\nu)|\mu')A_{\mu'}(\nu) ,\end{aligned} \tag{3.10}$$

where the explicit form of the unperturbed and the perturbed parts of $\mathscr{h}$ are, respectively, given by making use of (2.2), (2.37), (3.7), and (3.9):

$$\begin{aligned}(\mu|\tilde{h}_0(\nu)|\mu') &\equiv (E_{\nu+\mu/2}-E_{\nu-\mu/2})\delta_{\mu',\mu} \\ &\equiv h_0(\nu,\mu)\delta_{\mu',\mu} ,\end{aligned} \tag{3.11}$$

$$\begin{aligned}(\mu|\tilde{V}(\nu)|\mu') \equiv &\langle \nu+\tfrac{1}{2}\mu|V|\nu-\tfrac{1}{2}\mu+\mu'\rangle\eta^{\mu'-\mu} \\ &-\langle \nu+\tfrac{1}{2}\mu-\mu'|V|\nu-\tfrac{1}{2}\mu\rangle\eta^{-\mu'+\mu} .\end{aligned} \tag{3.12}$$

It is noted that the unperturbed superoperator $\mathscr{h}_0$ is diagonal in the $\mu$ variable.

Since we know the mathematical device outlined above, the evaluation of the collision term is straightforward. Applying it to the present problem, one obtains

$$\langle\lambda_1 \mid b^0\Psi \mid \lambda_2\rangle \equiv (b^0\Psi)_\mu(\nu) \equiv \sum_{\mu'} (\mu \mid \tilde{b}^0(\nu) \mid \mu')\Psi_{\mu'}(\nu)$$

$$\equiv \sum_{\vec{q}} \sum_{\mu'} \sum_{\mu''} \langle (\mu \mid \tilde{V}_{\vec{q}}(\nu) \mid \mu')[h_0(\nu,\mu') + \mathscr{H}_{\rm ph} - z]^{-1} (\mu' \mid \tilde{V}_{\vec{q}}(\nu) \mid \mu'')\rangle_{\rm ph} \Psi_{\mu''}(\nu)$$

$$= \sum_{\vec{q}} \sum_{\mu'} \sum_{\mu''} \{(N_{\vec{q}}+1)[\langle \nu+\tfrac{1}{2}\mu \mid \gamma_{\vec{q}} \mid \nu-\tfrac{1}{2}\mu+\mu'\rangle (E_{\nu-\mu/2+\mu'} - E_{\nu-\mu/2} + \hbar\omega_{\vec{q}} - z)^{-1}$$

$$\times \langle \nu-\tfrac{1}{2}\mu+\mu' \mid \gamma^\dagger_{\vec{q}} \mid \nu-\tfrac{1}{2}\mu+\mu''\rangle \Psi_{\mu''}(\nu-\tfrac{1}{2}\mu+\tfrac{1}{2}\mu'')$$

$$+ \langle \nu+\tfrac{1}{2}\mu-\mu'' \mid \gamma_{\vec{q}} \mid \nu+\tfrac{1}{2}\mu-\mu'\rangle (E_{\nu+\mu/2} - E_{\nu+\mu/2-\mu'} - \hbar\omega_{\vec{q}} - z)^{-1}$$

$$\times \langle \nu+\tfrac{1}{2}\mu-\mu' \mid \gamma^\dagger_{\vec{q}} \mid \nu-\tfrac{1}{2}\mu\rangle \Psi_{\mu''}(\nu+\tfrac{1}{2}\mu-\tfrac{1}{2}\mu'')]$$

$$+ N_{\vec{q}}[\langle \nu+\tfrac{1}{2}\mu \mid \gamma^\dagger_{\vec{q}} \mid \nu-\tfrac{1}{2}\mu+\mu'\rangle (E_{\nu-\mu/2+\mu'} - E_{\nu-\mu/2} - \hbar\omega_{\vec{q}} - z)^{-1}$$

$$\times \langle \nu-\tfrac{1}{2}\mu+\mu' \mid \gamma_{\vec{q}} \mid \nu-\tfrac{1}{2}\mu+\mu''\rangle \Psi_{\mu''}(\nu-\tfrac{1}{2}\mu+\tfrac{1}{2}\mu'')$$

$$+ \langle \nu+\tfrac{1}{2}\mu-\mu'' \mid \gamma^\dagger_{\vec{q}} \mid \nu+\tfrac{1}{2}\mu-\mu'\rangle (E_{\nu+\mu/2} - E_{\nu+\mu/2-\mu'} + \hbar\omega_{\vec{q}} - z)^{-1}$$

$$\times \langle \nu+\tfrac{1}{2}\mu-\mu' \mid \gamma_{\vec{q}} \mid \nu-\tfrac{1}{2}\mu\rangle \Psi_{\mu''}(\nu+\tfrac{1}{2}\mu-\tfrac{1}{2}\mu'')]\}$$

$$- \sum_{\vec{q}} \sum_{\mu'} \sum_{\mu''} \{(N_{\vec{q}}+1)[\langle \nu+\tfrac{1}{2}\mu \mid \gamma_{\vec{q}} \mid \nu-\tfrac{1}{2}\mu+\mu'\rangle (E_{\nu-\mu/2+\mu'} - E_{\nu-\mu/2} + \hbar\omega_{\vec{q}} - z)^{-1}$$

$$\times \langle \nu-\tfrac{1}{2}\mu+\mu'-\mu'' \mid \gamma^\dagger_{\vec{q}} \mid \nu-\tfrac{1}{2}\mu\rangle \Psi_{\mu''}(\nu-\tfrac{1}{2}\mu+\mu'-\tfrac{1}{2}\mu'')$$

$$+ \langle \nu+\tfrac{1}{2}\mu \mid \gamma_{\vec{q}} \mid \nu+\tfrac{1}{2}\mu-\mu'+\mu''\rangle (E_{\nu+\mu/2} - E_{\nu+\mu/2-\mu'} - \hbar\omega_{\vec{q}} - z)^{-1}$$

$$\times \langle \nu+\tfrac{1}{2}\mu-\mu' \mid \gamma^\dagger_{\vec{q}} \mid \nu-\tfrac{1}{2}\mu\rangle \Psi_{\mu''}(\nu+\tfrac{1}{2}\mu-\mu'+\tfrac{1}{2}\mu'')]$$

$$+ N_{\vec{q}}[\langle \nu+\tfrac{1}{2}\mu \mid \gamma^\dagger_{\vec{q}} \mid \nu-\tfrac{1}{2}\mu+\mu'\rangle (E_{\nu-\mu/2+\mu'} - E_{\nu-\mu/2} - \hbar\omega_{\vec{q}} - z)^{-1}$$

$$\times \langle \nu-\tfrac{1}{2}\mu+\mu'-\mu'' \mid \gamma_{\vec{q}} \mid \nu-\tfrac{1}{2}\mu\rangle \Psi_{\mu''}(\nu-\tfrac{1}{2}\mu+\mu'-\tfrac{1}{2}\mu'')$$

$$\times \langle \nu+\tfrac{1}{2}\mu \mid \gamma^\dagger_{\vec{q}} \mid \nu+\tfrac{1}{2}\mu-\mu'+\mu''\rangle (E_{\nu+\mu/2} - E_{\nu+\mu/2-\mu'} + \hbar\omega_{\vec{q}} - z)^{-1}$$

$$\times \langle \nu+\tfrac{1}{2}\mu-\mu' \mid \gamma_{\vec{q}} \mid \nu-\tfrac{1}{2}\mu\rangle \Psi_{\mu''}(\nu+\tfrac{1}{2}\mu-\mu'+\tfrac{1}{2}\mu'')]\}\ , \tag{3.13}$$

where $N_{\vec{q}}$ is the Planck distribution for phonons and is given by

$$N_{\vec{q}} = \mathrm{Tr}^{(\rm ph)}(\rho_{\rm ph} b^\dagger_{\vec{q}} b_{\vec{q}}) = (e^{\beta\hbar\omega_{\vec{q}}} - 1)^{-1}\ . \tag{3.14}$$

It is noted that expression (3.13) is exact. This general expression looks very complicated. To see the physical picture of (3.13), let us consider as an example the simplest situation of an isotropic translationally invariant system. Let us assume an electron-phonon interaction (operator) of the form

$$\gamma_{\vec{q}} = C_{\vec{q}} \exp(i\vec{q}\cdot\vec{r}/\hbar)\ , \tag{3.15}$$

where $C_{\vec{q}}$ is a constant, which depends on the type of phonon studied, and that the one-electron states are characterized by the electron momentum. For such a case, one can easily find from (3.13) that

$$\left\langle \vec{P}_1 \left| \sum_{\vec{q}} \langle \mathscr{V}_{\vec{q}} R_z^0 \mathscr{V}_{\vec{q}} \rangle_{\rm ph} \Psi \right| \vec{P}_2 \right\rangle$$

$$= \sum_{\vec{q}} |C_{\vec{q}}|^2 \{ (N_{\vec{q}}+1)[(E_{1-\vec{q}} - E_2 + \hbar\omega_{\vec{q}} - z)^{-1} + (E_1 - E_{2-\vec{q}} - \hbar\omega_{\vec{q}} - z)^{-1}]$$
$$+ N_{\vec{q}}[(E_{1+\vec{q}} - E_2 - \hbar\omega_{\vec{q}} - z)^{-1} + (E_1 - E_{2+\vec{q}} + \hbar\omega_{\vec{q}} - z)^{-1}] \} \langle \vec{P}_1 | \Psi | \vec{P}_2 \rangle$$
$$- \sum_{\vec{q}} |C_{\vec{q}}|^2 \{ (N_{\vec{q}}+1)[(E_{1-\vec{q}} - E_2 + \hbar\omega_{\vec{q}} - z)^{-1} + (E_1 - E_{1-\vec{q}} - \hbar\omega_{\vec{q}} - z)^{-1}] \langle \vec{P}_1 - \vec{q} | \Psi | \vec{P}_2 - \vec{q} \rangle$$
$$+ N_{\vec{q}}[(E_{1+\vec{q}} - E_2 - \hbar\omega_{\vec{q}} - z)^{-1} + (E_1 - E_{2+\vec{q}} + \hbar\omega_{\vec{q}} - z)^{-1}] \langle \vec{P}_1 + \vec{q} | \Psi | \vec{P}_2 + \vec{q} \rangle \} , \quad (3.16)$$

where 1 and 2 refer to the electron momenta $\vec{P}_1$, $\vec{P}_2$ and $E_1$ is an eigenvalue of the eigenvalue equation:

$$h_0 | \vec{P}_1 \rangle = E_1 | \vec{P}_1 \rangle = (\vec{P}_1^2/2m^*) | \vec{P}_1 \rangle . \quad (3.17)$$

In this representation, the single-electron current operator $j_r$ is also necessarily diagonal, hence in the expression for the conductivity we need $\langle \vec{P}_1 | \sum \langle \mathscr{V}_{\vec{q}} R_z^0 \mathscr{V}_{\vec{q}} \rangle_{\rm ph} \Psi | \vec{P}_1 \rangle$. Changing $2 \rightarrow 1$ in (3.16) and taking the limit $\delta \rightarrow 0^+$, this expression shows the gain-loss structure appearing in the quantum Boltzmann collision integral for an electron-phonon system.[3,4,20] For the static case ($\omega \rightarrow 0$), the resultant expression agrees to (A2.11) of Barker.[20] It is noted that in (3.13) the first four and the last four terms correspond to the gain and the loss structure of terms, respectively. It should be emphasized that expression (3.13) is general and does not depend on any representation for an electron state. Therefore, it can be applied to a system subjected to a static magnetic field.

### B. Second-order self-consistent approximation (strong-interaction case)

When the interaction between electron and phonon is strong, one has to take account of higher-order effects of the interaction. In practice the operator $g_z^D$ may be approximated by $b_z$ given in (3.1). The replacement of $R_z^D$ by $R_z^D(b_z)$ in $b_z^0$ is, however, an important correction to the scattering vertex since $b_z$ in the energy denominator of $R_z^D$ is responsible for the higher-order effects. Therefore, even the second-order approximation for $g_z^D$ is beyond the Born approximation customarily employed in the evaluation of the scattering rate.

Let us consider the collision term:

$$\langle \lambda_1 | b\Psi | \lambda_2 \rangle = \sum_{\mu'} (\mu | \tilde{b}(\nu) | \mu') \Psi_{\mu'}(\nu) . \quad (3.18)$$

Since the energy denominator in $R_z^D$ contains $b_z$ which is, in general, not diagonal, it is not feasible to evaluate (3.18). Hence we need an approximation. As seen in the previous section, for relaxive (diagonal $\vec{j}$) transport, the operator $b_z^0$ is diagonal in $\mu$. Following this remarkable property, we assume here that $b_z$ is also diagonal in $\mu$:

$$(\mu | \tilde{b}(\nu) | \mu') \simeq [b(\nu)]_\mu \delta_{\mu',\mu} . \quad (3.19)$$

Substituting Eq. (3.19) into Eq. (3.18) and after a straightforward calculation, we obtain the following equation for the self-energy:

$$[b(\nu)]_\mu = \sum_{\vec{q}} \sum_{\mu'} [(N_{\vec{q}}+1)(\langle \nu + \tfrac{1}{2}\mu | \gamma_{\vec{q}} | \nu - \tfrac{1}{2}\mu + \mu' \rangle \{ E_{\nu-\mu/2+\mu'} - E_{\nu-\mu/2} + \hbar\omega_{\vec{q}} - [b(\nu - \tfrac{1}{2}\mu + \tfrac{1}{2}\mu')]_{\mu'} - z \}^{-1}$$
$$\times \langle \nu - \tfrac{1}{2}\mu + \mu' | \gamma_{\vec{q}}^\dagger | \nu + \tfrac{1}{2}\mu \rangle + \langle \nu + \tfrac{1}{2}\mu - \mu' | \gamma_{\vec{q}}^\dagger | \nu - \tfrac{1}{2}\mu \rangle$$
$$\times \{ E_{\nu+\mu/2} - E_{\nu+\mu/2-\mu'} - \hbar\omega_{\vec{q}} - [b(\nu + \tfrac{1}{2}\mu - \tfrac{1}{2}\mu')]_{\mu'} - z \}^{-1} \langle \nu - \tfrac{1}{2}\mu | \gamma_{\vec{q}} | \nu + \tfrac{1}{2}\mu - \mu' \rangle)$$
$$+ N_{\vec{q}} (\langle \nu + \tfrac{1}{2}\mu | \gamma_{\vec{q}}^\dagger | \nu - \tfrac{1}{2}\mu + \mu' \rangle \{ E_{\nu-\mu/2+\mu'} - E_{\nu-\mu/2} - \hbar\omega_{\vec{q}} - [b(\nu - \tfrac{1}{2}\mu + \tfrac{1}{2}\mu')]_{\mu'} - z \}^{-1}$$

$$\times\langle\nu-\tfrac{1}{2}\mu+\mu'\,|\,\gamma_{\vec{q}}\,|\,\nu+\tfrac{1}{2}\mu\rangle+\langle\nu+\tfrac{1}{2}\mu-\mu'\,|\,\gamma_{\vec{q}}\,|\,\nu-\tfrac{1}{2}\mu\rangle$$

$$\times\{E_{\nu+\mu/2}-E_{\nu+\mu/2-\mu'}+\hbar\omega_{\vec{q}}-[b(\nu+\tfrac{1}{2}\mu-\tfrac{1}{2}\mu')]_{\mu'}-z\}^{-1}\langle\nu-\tfrac{1}{2}\mu\,|\,\gamma^{\dagger}_{\vec{q}}\,|\,\nu+\tfrac{1}{2}\mu-\mu'\rangle)]$$

$$-\sum_{\vec{q}}\sum_{\mu'}[(N_{\vec{q}}+1)(\langle\nu+\tfrac{1}{2}\mu\,|\,\gamma_{\vec{q}}\,|\,\nu-\tfrac{1}{2}\mu+\mu'\rangle\{E_{\nu-\mu/2+\mu'}-E_{\nu-\mu/2}+\hbar\omega_{\vec{q}}-[b(\nu-\tfrac{1}{2}\mu+\tfrac{1}{2}\mu')]_{\mu'}-z\}^{-1}$$

$$\times\langle\nu-\tfrac{3}{2}\mu+\mu'\,|\,\gamma^{\dagger}_{\vec{q}}\,|\,\nu-\tfrac{1}{2}\mu\rangle\eta^{2(\mu'-\mu)}+\langle\nu+\tfrac{1}{2}\mu\,|\,\gamma_{\vec{q}}\,|\,\nu+\tfrac{3}{2}\mu-\mu'\rangle$$

$$\times\{E_{\nu+\mu/2}-E_{\nu+\mu/2-\mu'}-\hbar\omega_{\vec{q}}-[b(\nu+\tfrac{1}{2}\mu-\tfrac{1}{2}\mu')]_{\mu'}-Z\}^{-1}$$

$$\times\langle\nu+\tfrac{1}{2}\mu-\mu'\,|\,\gamma^{\dagger}_{\vec{q}}\,|\,\nu-\tfrac{1}{2}\mu\rangle\eta^{2(\mu-\mu')})$$

$$+N_{\vec{q}}(\langle\nu+\tfrac{1}{2}\mu\,|\,\gamma^{\dagger}_{\vec{q}}\,|\,\nu-\tfrac{1}{2}\mu+\mu'\rangle\{E_{\nu-\mu/2+\mu'}-E_{\nu-\mu/2}-\hbar\omega_{\vec{q}}-[b(\nu-\tfrac{1}{2}\mu+\tfrac{1}{2}\mu')]_{\mu'}-z\}^{-1}$$

$$\times\langle\nu-\tfrac{3}{2}\mu+\mu'\,|\,\gamma_{\vec{q}}\,|\,\nu-\tfrac{1}{2}\mu\rangle\eta^{2(\mu-\mu')}+\langle\nu+\tfrac{1}{2}\mu\,|\,\gamma^{\dagger}_{\vec{q}}\,|\,\nu+\tfrac{3}{2}\mu-\mu'\rangle$$

$$\times\{E_{\nu+\mu/2}-E_{\nu+\mu/2-\mu'}+\hbar\omega_{\vec{q}}-[b(\nu+\tfrac{1}{2}\mu-\tfrac{1}{2}\mu')]_{\mu'}-z\}^{-1}$$

$$\times\langle\nu+\tfrac{1}{2}\mu-\mu'\,|\,\gamma_{\vec{q}}\,|\,\nu-\tfrac{1}{2}\mu\rangle\eta^{2(\mu-\mu')})]\,. \tag{3.20}$$

Multiplying $\Psi_{\mu}(\nu)$ from the right on both sides of Eq. (3.20), one can easily obtain the collision term (3.18) of the transport equation (2.39). Prasad has also obtained expressions for the collision term but by means of a diagram technique. The physical meaning of (3.20) is clearly understood by considering the corresponding terms for the weakly interacting case: The first two terms of (3.20) correspond to the gain part and the last two to the loss part of the collision term. As can be seen from Eq. (3.20), $[b(\nu)]_{\mu}$ has to be determined self-consistently since it appears in the energy denominators on the right-hand side. For strong interactions this causes a significant correction. It should be mentioned that expression (3.20) is valid for arbitrary frequencies of the probing electric field and that the approximation of the diagonality of $b_z$ (or more generally that of $g_z^D$) is strictly valid for the translational invariance of the system (relaxive transport). The further evaluation of (3.20) for a particular electron-phonon system and the application to resonance problems will be reported in a separate paper.

One of the authors (A.S.) wishes to thank Professor S. Fujita, State University of New York at Buffalo, for enlightening discussions at an early stage of the present work.

## APPENDIX: DERIVATION OF EQ. (2.23)

Let us consider the equation

$$\langle R_z^D\hat{X}\rangle_{\rm ph}=\langle(\mathcal{L}_0+\mathcal{H}_{\rm ph}+g_z^D-z)^{-1}\hat{X}\rangle_{\rm ph}\,, \tag{A1}$$

where $\hat{X}$ is any quantum operator. Let the phonon states be labeled $M,N,P,\ldots$. Noting that $\rho_{\rm ph}$ and $H_{\rm ph}$ are diagonal in $|M\rangle$ and that $g_z^D$ does not depend on phonon coordinates, we have (A1) as

$$\langle R_z^D\hat{X}\rangle_{\rm ph}=\langle(\mathcal{L}_0+\mathcal{H}_{\rm ph}+g_z^D-z)^{-1}\hat{X}\rangle_{\rm ph}$$

$$=\mathrm{tr}^{(\rm ph)}[\rho_{\rm ph}(\mathcal{L}_0+\mathcal{H}_{\rm ph}+g_z^D-z)^{-1}\hat{X}]$$

$$=\sum_{M,N}\langle M\,|\,\rho_{\rm ph}\,|\,N\rangle\langle N\,|\,(\mathcal{L}_0+\mathcal{H}_{\rm ph}+g_z^D-z)^{-1}\hat{X}\,|\,M\rangle$$

$$
\begin{aligned}
&= \sum \langle M \mid \rho_{\mathrm{ph}} \mid N \rangle [\mathcal{A}_0 + (N-M)\hbar\omega_q + g_z^D - z]^{-1} \langle N \mid \hat{X} \mid M \rangle \\
&= \sum \langle M \mid \rho_{\mathrm{ph}} \mid M \rangle \delta_{M,N} [\mathcal{A}_0 + (N-M)\hbar\omega_q + g_z^D - z]^{-1} \langle N \mid \hat{X} \mid M \rangle \\
&= \sum \langle M \mid \rho_{ph} \mid M \rangle (\mathcal{A}_0 + g_z^D - z)^{-1} \langle M \mid \hat{X} \mid M \rangle \\
&= (\mathcal{A}_0 + g_z^D - z)^{-1} \sum_M \langle M \mid \rho_{\mathrm{ph}} \mid M \rangle \langle M \mid \hat{X} \mid M \rangle \\
&= (\mathcal{A}_0 + g_z^D - z)^{-1} \langle \hat{X} \rangle_{\mathrm{ph}} \, . \qquad \text{(A2)}
\end{aligned}
$$

Equation (A2) is valid for any quantum operator $\hat{X}$. Therefore we can write $n_z'$ as in (2.23).


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