Photon statistics of a free-electron laser

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A fully quantized theory of the free-electron laser in the small-signal regime is presented which allows for a calculation of the photon statistics. For an initial vacuum, we find photon antibunching if the electron momentum is below resonance. We conjecture that, in general, the free-electron laser preserves coherent states only in the absence of gain.

I. INTRODUCTION

Historically, the first explanation[1] of the gain mechanism of a free-electron laser (FEL) invoked quantum mechanics. Although Planck's constant $\hbar$ dropped out of the final expression for the gain indicating that it should be derivable from a classical approach, this was supposed to be very difficult for a long time, since the first approach[1] relied crucially on quantum recoil corrections to the frequencies of emitted photons for which there is no classical analog. There is now general agreement that all essential features of the FEL can be understood in terms of classical concepts.[2] This excludes, of course, the problem of the photon statistics of the FEL and, consequently, the very question of whether or not the FEL is a laser in the sense that it radiates a coherent state. This question albeit interesting in itself is by no means purely academic. The well-known example of multiphoton ionization of atoms[3] shows that the photon statistics of an intense monochromatic light beam can be of vital importance with respect to its interaction with matter. A general solution to this problem requires a fully quantized approach. In this paper we are far from solving the problem of the photon statistics of a free-electron *laser,* instead when speaking about a FEL we actually mean a free-electron amplifier in the small-signal cold-beam noncollective regime. No attempt has been made yet to investigate the photon statistics of a free-electron laser above threshold.

Quantum descriptions of the FEL often start from the Bambini-Renieri Hamiltonian,[4] which specifies the FEL (in the context of the Weizsäcker-Williams approximation) in a moving frame in which the frequencies of the laser and the wiggler coincide. In this frame resonance occurs when the electron is at rest, hence the electron can be treated nonrelativistically. This paper relies on a reformulation of this approach in the interaction picture in contrast to the Schrödinger or Heisenberg picture which are usually applied.[5,6]

In the interaction picture, the time-evolution operator of an electron-laser photon state is given by the time-ordered exponential of the transformed interaction Hamiltonian. If the electron momentum operator is treated as a classical c number, the problem reduces to that of a classical current interacting with a quantized radiation field. There is, however, no gain in this approximation due to the neglect of the electron quantum recoil. In an earlier approach to the same problem[7,8] this had been remedied by introducing the recoil corrections (as obtained from energy-momentum conservation) by hand into the detuning parameter, which is the only quantity to depend significantly on these very small corrections. By means of this procedure, one obtains in a very simple way all basic results of FEL theory.[8] In spite of its success, this *ad hoc* approach is not completely satisfactory. We replace it here by expanding the exact time-evolution operator up to first order in the recoil which is sufficient to describe the small-signal regime. To our knowledge, this is then the only fully quantized treatment of the FEL, which does not resort at some stage to the classical equations of motion in order to infer gain.

In Sec. II, we derive the time-evolution operator in the above-mentioned linear recoil approximation. In Sec. III, we employ it to compute gain, spread, and the photon statistics in terms of eigenstates of the photon number. If the FEL starts from the field vacuum, the resulting final state of the radiation field is bunched, antibunched, or coherent depending upon whether the electron momentum is $p>0$, $p<0$, or $p=0$, respectively. We suggest that, in general, the FEL preserves coherent states only inasmuch as gain is zero or can be neglected. This is equivalent to the startling conclusion that the FEL is a laser in the sense that it produces a coherent state only if it is not a laser in the sense that it does not amplify. In Sec. IV we compare our present results with the earlier mentioned semiphenomenological approach.[7,8] The latter turns out to be perfectly justified if the initial radiation field is either sufficiently intense or in the vacuum state. We finally relate our work to Refs. 5 and 6.

II. TIME-EVOLUTION OPERATOR

We start with the one-electron nonrelativistic Hamiltonian which describes the FEL in the so-called Bambini-Renieri frame.[4] In this moving frame, the laser and wiggler frequency coincide with $\omega = ck/2$. The Hamiltonian is given by

$$H = H_0 + H_1 \,, \tag{1a}$$

$$H_0 = \frac{p^2}{2m} + \hbar\omega(a_L^\dagger a_L + a_W^\dagger a_W) \,, \tag{1b}$$

$$H_1 = i\hbar g (a_L^\dagger a_W e^{-ikz} - a_W^\dagger a_L e^{ikz}) \,. \tag{1c}$$

Here $a_L(a_L^\dagger)$ and $a_W(a_W^\dagger)$ are photon annihilation (creation) operators which represent the laser field and the wiggler field, respectively, in the Weizsäcker-Williams approximation, p and z the electron's momentum and coordinate with $[z,p]=i\hbar$, m is a renormalized electron mass, and the coupling constant g is given by

$$g = \left[\frac{4\pi c}{kV}\right] r_0 \,, \tag{2}$$

where r_0 is the classical electron radius and V is the quantization volume.

In the interaction picture, H_1 transforms to

$$H_I(t) = e^{iH_0 t/\hbar} H_1 e^{-iH_0 t/\hbar} = i\hbar g (e^{-it(\hbar k^2 + 2kp)/2m} A^\dagger a_W - \text{c.c.}) \,, \tag{3}$$

where in analogy with Ref. 5, we introduced the operator

$$A = a_L e^{ikz} \tag{4}$$

with the properties

$$[A, A^\dagger] = 1, \quad A^\dagger A = a_L^\dagger a_L \,. \tag{5}$$

In deriving Eq. (3), we used the following relations:

$$e^{i\omega t a^\dagger a} a e^{-i\omega t a^\dagger a} = a e^{-i\omega t} \quad (a = a_L, a_W) \tag{6a}$$

$$e^{ip^2 t/2m\hbar} e^{-ikz} e^{-ip^2 t/2m\hbar} = e^{-ikz} e^{it(\hbar k^2 - 2kp)/2m} \,. \tag{6b}$$

The time-evolution operator for the electron-photon state is given by

$$S(T/2, -T/2) = \mathscr{T} \exp\left[-\frac{i}{\hbar}\int_{-T/2}^{T/2} dt\, H_I(t)\right] , \tag{7}$$

where $\mathscr{T}$ is the Dyson time-ordering operator and the symmetric integration has been chosen by convenience. The interaction time $T = L/c$ is specified by the wiggler length L. Equation (7) as it stands can only be evaluated in perturbation theory. This is due to the time-ordering prescription as well as the appearance of the operator p in Eq. (3). We are now trying to get rid of both difficulties by expanding $S(T/2, -T/2)$ around some c-number average value p_0 which will be specified afterwards. Hence we write

$$S(T/2, -T/2) = S_0(T/2, -T/2) + S_1(T/2, -T/2) + \cdots \,, \tag{8a}$$

$$S_0(t_2, t_1) = \mathscr{T} \exp\left[-\frac{i}{\hbar}\int_{t_1}^{t_2} dt\, H_I(t)\right]\Bigg|_{p=p_0} \,, \tag{8b}$$

$$S_1(T/2, -T/2) = \frac{1}{\hbar}\int_{-T/2}^{T/2} dt\, S_0(T/2, t)\left[(p - p_0)\frac{\partial}{\partial p}[-iH_I(t)]\Bigg|_{p=p_0}\right] S_0(t, -T/2) \,. \tag{8c}$$

Here $S_0(T/2, -T/2)$ is the time-evolution operator in the classical recoilless approximation for the electron current. It has been shown earlier[7] that, in this approximation, the photon distribution function exhibits a Poisson distribution if initially no laser field is present, and that $S_0(T/2, -T/2)$ preserves coherent states. There is, however, no gain in this approximation since the quantum recoil of the emitted photons, which is responsible for the gain mechanism in the free-electron laser,[1] is not taken into account. The quantum recoil is accounted for up to first order by $S_1(T/2,-T/2)$. Owing to this linear approximation we are henceforth restricted to the small-signal regime. Note that in the expansion we were carefully respecting the time ordering. The square bracket in Eq. (8c) is a symbolic notation: the correct order of the operators must be inferred from Eq. (3) [see Eq. (17) below].

From now on, we will take the semiclassical limit of the wiggler field, i.e., we will set

$$a_W^\dagger \approx a_W \approx \sqrt{N_W} \ . \tag{9}$$

This limit is reasonable because the quantum nature of the wiggler field is a mathematical device only and no quantum effects of it can have a physical meaning. With this, we obtain from Eq. (3) (for $p=p_0$),

$$[H_I(t'),H_I(t'')] = 2ig^2 N_W \sin[\beta(t'-t'')] \ , \tag{10}$$

where

$$\beta = \frac{k^2\hbar}{2m} + \frac{kp_0}{m} \ . \tag{11}$$

The commutator of the interaction Hamiltonian for $p=p_0$ at different times is therefore a purely imaginary c number. Under this condition it can be shown[9] that the time-ordering operator merely introduces a phase:

$$S_0(t_2,t_1) = e^{i\theta(t_2,t_1)} \exp\left[-\frac{i}{\hbar}\int_{t_1}^{t_2} dt'\, H_I(t') \right]\Bigg|_{p=p_0} , \tag{12a}$$

$$i\theta(t_2,t_1) = \frac{1}{2\hbar^2}\int_{t_1}^{t_2} dt' \int_{t_1}^{t'} dt''[H_I(t'),H_I(t'')] \ . \tag{12b}$$

It can easily be shown that $S_0(t_2,t_1)$ is unitary and satisfies the group property

$$S_0(t_1,t_2)S_0(t_2,t_3) = S_0(t_1,t_3) \ . \tag{13}$$

On substituting from Eq. (3) in Eq. (12a) and applying the Baker-Hausdorff formula, we get

$$S_0(t_2,t_1) = e^{i\theta(t_2,t_1)} e^{j^*(t_2,t_1)A^\dagger} e^{-j(t_2,t_1)A} \times e^{(-1/2)|j(t_2,t_1)|^2} , \tag{14}$$

where

$$j(t_2,t_1) = g\sqrt{N_W}\int_{t_1}^{t_2} dt\, e^{i\beta t} = \frac{g\sqrt{N_W}}{i\beta}(e^{i\beta t_2} - e^{i\beta t_1}) \ . \tag{15}$$

It is evident from Eq. (15) that

$$j(T/2,-T/2) \equiv j(T) = \frac{2g\sqrt{N_W}}{\beta}\sin(\beta T/2) = j^*(T) \ . \tag{16}$$

Equation (14) provides us with an explicit expression for the time-evolution operator $S_0(T/2,-T/2)$ in the classical recoilless approximation.

Next we derive an expression for the lowest order correction $S_1(T/2,-T/2)$. On substituting for $H_I(t)$ in Eq. (8c), we obtain

$$\begin{aligned}
S_1(T/2,-T/2) &= \frac{-ig\sqrt{N_W}k}{m}\int_{-T/2}^{T/2} dt\, tS_0(T/2,t)[(p-p_0)A^\dagger e^{-i\beta t} + A(p-p_0)e^{i\beta t}]S_0(t,-T/2) \\
&= -\frac{ig\sqrt{N_W}k}{m} S_0(T/2,-T/2) \\
&\quad \times \int_{-T/2}^{T/2} dt\, te^{-i\beta t}(\{p-p_0-\hbar k[\,|j(t,-T/2)|^2 + j^*(t,-T/2)A^\dagger + j(t,-T/2)A]\}[A^\dagger + \underline{j}(t,-T/2)] \\
&\qquad - [A+\underline{j}(T/2,t)]\{p-p_0-\hbar k[\,|j(T/2,t)|^2 + j^*(T/2,t)A + j(T/2,t)A^\dagger]\}) \ .
\end{aligned} \tag{17}$$

In deriving Eq. (17) we have used the following commutation relations:

$$[A,S_0(t_2,t_1)] = j^*(t_2,t_1)S_0(t_2,t_1) \ , \tag{18a}$$

$$[A^\dagger,S_0(t_2,t_1)] = j(t_2,t_1)S_0(t_2,t_1) \ , \tag{18b}$$

$$[p,A] = \hbar kA, \quad [p,A^\dagger] = -\hbar kA^\dagger \ , \tag{18c}$$

$$[p,S_0(t_2,t_1)] = -\hbar kS_0(t_2,t_1)[j^*(t_2,t_1)A^\dagger + j(t_2,t_1)A + |j(t_2,t_1)|^2] \ , \tag{18d}$$

as well as the group property [Eq. (14)].

Since $|j|^2 << 1$,[10] Eq. (17) can be somewhat simplified. Applying the square bracket in Eq. (17) to a state $|\bar{p},N\rangle$, the resulting state is a superposition of states $|\bar{p},N\rangle$, $|\bar{p}\mp\hbar k,N\pm1\rangle$ and $|\bar{p}\mp2\hbar k,N\pm2\rangle$. With the choice of p_0 specified in Sec. III, the eigenvalue $\bar{p}-p_0$ never vanishes. We can then safely neglect $|j(t,-T/2)|^2$ and $|j(T/2,t)|^2$ in Eq. (17). Moreover, it turns out that the underlined j's never contribute significantly except when multiplied with $p-p_0$. The resulting expression for $S(T/2,-T/2)$ is then

$$S(T/2,-T/2)=S_0(T/2,-T/2)\left[1-\frac{ig\sqrt{N_W}k}{m}\int_{-T/2}^{T/2}dt\, te^{-i\beta t}\times(\{p-p_0-\hbar k[j^*(t,-T/2)A^\dagger+j(t,-T/2)A]\}A^\dagger - A\{p-p_0-\hbar k[j^*(T/2,t)A+j(T/2,t)A^\dagger]\} + (p-p_0)[j(t,-T/2)-j(T/2,t)])\right]. \quad (19)$$

The last term proportional to $p-p_0$ is negligible for $N>>1$. It can easily be shown that $S(T/2,-T/2)$ as given by Eq. (19) is unitary up to the order of k/m.

$S(T/2,-T/2)$ apparently depends on the choice of p_0. We are now going to show that up to the order of k/m it is actually independent of p_0. According to Eqs. (15) and (11) we have

$$\frac{\partial j(T/2,-T/2)}{\partial p_0}=\frac{k}{m}\frac{\partial j(T/2,-T/2)}{\partial\beta}$$

and hence using Eq. (18b)

$$\frac{\partial S_0(T/2,-T/2)}{\partial p_0}=\frac{k}{m}\frac{\partial j(T/2,-T/2)}{\partial\beta}\times S_0(T/2,-T/2)(A^\dagger-A)\,, \quad (20)$$

where we have neglected the derivative of the phase $i\theta(T/2,-T/2)$ since it contributes only to higher orders. Calculating then the derivative $\partial S(T/2,-T/2)/\partial p_0$ from Eq. (19), $\partial S_0(T/2,-T/2)/\partial p_0$ cancels against the derivative of the integrand thus leaving us with

$$\frac{\partial S(T/2,-T/2)}{\partial p_0}=O\left[\frac{k}{m}\right]^2. \quad (21)$$

III. PHOTON STATISTICS

We shall first consider an initial number state $|\bar{p},N\rangle$ which satisfies

$$p\,|\bar{p},N\rangle=\bar{p}\,|\bar{p},N\rangle\,, \quad (22a)$$

$$A\,|\bar{p},N\rangle=\sqrt{N}\,|\bar{p}+\hbar k,N-1\rangle\,, \quad (22b)$$

$$A^\dagger|\bar{p},N\rangle=\sqrt{N+1}\,|\bar{p}-\hbar k,N+1\rangle\,. \quad (22c)$$

Exploiting the arbitrariness of the expansion parameter p_0, we fix it by

$$p_0=\bar{p}-\tfrac{1}{2}\hbar k\,. \quad (23)$$

This will provide us with the most symmetric explicit results. Moreover, the parameter β then reads

$$\beta=k\bar{p}/m\,, \quad (24)$$

so that resonance at $\bar{p}=0$ becomes explicitly obvious. It then follows that

$$\langle\bar{\bar{p}},n\,|S(T/2,-T/2)|\bar{p},N\rangle = \langle n\,|S_0|N\rangle+\frac{ig\sqrt{N_W}\hbar k^2}{m}\int_{-T/2}^{T/2}dt\,te^{-i\beta t}\{\tfrac{1}{2}\sqrt{N+1}\langle n\,|S_0|N+1\rangle+\tfrac{1}{2}\sqrt{N}\langle n\,|S_0|N-1\rangle + (N+\tfrac{1}{2})[j(t,-T/2)-j(T/2,t)]\langle n\,|S_0|N\rangle + j^*(t,-T/2)\sqrt{(N+1)(N+2)}\langle n\,|S_0|N+2\rangle - j^*(T/2,t)\sqrt{N(N-1)}\langle n\,|S_0|N-2\rangle\}\,. \quad (25)$$

Here we used the abbreviations $S_0=S_0(T/2,-T/2)$, $\bar{\bar{p}}=\bar{p}+(N-n)\hbar k$, and $|l\rangle=|\bar{p}-(l-N)\hbar k,l\rangle$.

The photon-distribution function for the radiation field is then given by

$$P(n)=|\langle\bar{\bar{p}},n\,|S(T/2,-T/2)|\,\bar{p},N\rangle|^2$$
$$=|\langle n\,|S_0|\,N\rangle|^2-\frac{\hbar k^2}{m}\frac{\partial j(T)}{\partial\beta}[\sqrt{N+1}\langle n\,|S_0|\,N+1\rangle+\sqrt{N}\langle n\,|S_0|\,N-1\rangle$$
$$-j(T)\sqrt{N(N-1)}\langle n\,|S_0|\,N-2\rangle+j(T)\sqrt{(N+1)(N+2)}\langle n\,|S_0|\,N+2\rangle]\langle N\,|S_0^\dagger|\,n\rangle\ . \tag{26}$$

It can be shown[7,8] that

$$\langle n\,|S_0|\,N\rangle=\left[\frac{N!}{n!}\right]^{1/2}e^{i\theta(T/2,-T/2)}e^{(-1/2)j^2(T)}j^{n-N}(T)L_N^{n-N}[j^2(T)]\ , \tag{27}$$

where the L_N^{n-N} are Laguerre polynomials. In view of Eq. (27), $\langle\alpha\,|S_0|\,\beta\rangle\langle\gamma\,|S_0^\dagger|\,\delta\rangle$ is real, which has been used in deriving Eq. (26). Owing to the unitarity of $S(T/2,-T/2)$, $P(n)$ should be properly normalized at least up to the order of k/m. Actually we find as a consequence of $S_0S_0^\dagger=1$,

$$\sum_{n=0}^{\infty}P(n)=1\ .$$

The first term in Eq. (26) corresponds to the photon distribution in the absence of quantum recoil.[7,8] For $N=0$,[11] it yields the earlier mentioned Poisson statistics. The second and third term are responsible for gain, as will be shown below. They destroy Poisson statistics even for $N=0$. One can also easily convince oneself that the $P(n)$ for $N=0$ are *not* the first-order expansion of a Poisson distribution with a different mean value: for $N=0$, Eqs. (26) and (27) yield

$$P(n)=\frac{1}{n!}e^{-j^2(T)}j^{2n}(T)\left[1-\frac{\hbar k^2}{mj(T)}\frac{\partial j(T)}{\partial\beta}[n^2-(2n+1)j^2(T)+j^4(T)]\right]\ , \tag{28a}$$

whereas the shifted Poisson distribution is

$$\bar{P}(n)=\frac{1}{n!}e^{-[j^2(T)+\epsilon]}[j^2(T)+\epsilon]^{2n}$$
$$=\frac{1}{n!}e^{-j^2(T)}j^{2n}(T)\left[1-\epsilon+\frac{2n\epsilon}{j^2(T)}+\cdots\right]\ . \tag{28b}$$

Here ϵ might be specified by Eq. (29) below, according to

$$\epsilon=-\frac{\hbar k^2}{m}j(T)\frac{\partial j(T)}{\partial\beta}\ .$$

Obviously, the discrepancy between Eqs. (28a) and (28b) is considerable. This leads us to conjecture that the FEL radiates or preserves a coherent state only inasmuch as gain can be neglected.

A further interesting observation can be made when comparing Eqs. (25) and (26). The expression in curly braces in Eq. (25) is, for $N>>1$, proportional to N, whereas the term in square brackets in Eq. (26) is only proportional to $\sqrt{N}$ since the last two terms almost cancel for $N>>1$. This indicates that the phase of $\langle\bar{\bar{p}},n\,|S(T/2,-T/2)\,|\bar{p},N\rangle$ reacts much earlier to increasing laser-field strengths than its modulus, i.e., the applicability of the first-order recoil approximation depends upon the quantity to be calculated. Gain and spread (and all higher moments) can be calculated from $P(n)$; the expectation value of the field as well as two-time field-correlation functions, however, would incorporate the phase.

The photon-distribution function $P(n)$ allows for the calculation of all the moments,

$$\langle n^r\rangle=\sum_{n=0}^{\infty}n^rP(n)\ .$$

When investigating the (anti-) bunching properties of the emitted radiation, however, we will find that extensive cancellations erase all leading terms. Hence Eq. (16), which is based on the already approximated Eq. (19), is insufficient and we have to return to Eq. (17). We then find using the commutation relations (18a)—(18d):

$$\langle n\rangle=\langle\bar{p},N\,|\,S(T/2,-T/2)^{\dagger}A^{\dagger}AS(T/2,-T/2)\,|\,\bar{p},N\rangle$$
$$=N+j^2(T)-\frac{\hbar k^2}{m}j(T)\frac{\partial j(T)}{\partial\beta}(2N+1)+\delta\ , \tag{29}$$

$$\langle n^2\rangle=\langle\bar{p},N\,|\,S(T/2,-T/2)^{\dagger}(A^{\dagger}A)^2S(T/2,-T/2)\,|\,\bar{p},N\rangle$$
$$=[N+j^2(T)]^2+j^2(T)(2N+1)-\frac{\hbar k^2}{m}j(T)\frac{\partial j(T)}{\partial\beta}[4N^2+2N+1+4j^2(T)(2N+1)]$$
$$+[4N+2j^2(T)+1]\delta\ , \tag{30a}$$

$$\Delta n^2=\langle(n-\langle n\rangle)^2\rangle=j^2(T)(2N+1)-\frac{\hbar k^2}{m}j(T)\frac{\partial j(T)}{\partial\beta}[1+2j^2(T)(2N+1)]+(2N+1)\delta\ , \tag{30b}$$

where

$$\delta=ig\sqrt{N_W}\hbar k^2j(T)/m\int_{-T/2}^{T/2}dt\,te^{-i\beta t}[2\,|\,j(T/2,t)\,|^2+2\,|\,j(t,-T/2)\,|^2-j^2(T/2,t)-j^2(t,-T/2)]\ .$$

The second term in Eq. (29) represents spontaneous emission. For $N>>1$ it is negligible with respect to the third term, which is the usual gain expression. Via Madey's theorem[12] this is related to the first term in the spread (30b). Inasmuch as $N\sim\hbar^{-1}$, all terms in Eq. (29), except the one in the factor $2N+1$, contribute as classical terms to the quantity $\hbar\omega\langle n\rangle$ [notice that $j(T)\sim\hbar^{-1/2}$ in view of Eq. (15)]. This includes the last term which we would not have obtained from the approximate Eqs. (19) or (26). It gives corrections to spontaneous emission and is negligible for all N. This quantity δ is also negligible in Eq. (30a). In the spread (30b), however, due to extensive cancellations the second and the third term, which involves δ, are of comparable magnitude. Notice that the spread is increased for positive and decreased for negative gain.

From Eqs. (29) and (30), we have for $N=0$

$$\Delta n^2-\langle n\rangle=-\frac{2\hbar k^2}{m}j^3(T)\frac{\partial j(T)}{\partial\beta}\ , \tag{31}$$

where δ has cancelled. Hence the radiation field which evolves by spontaneous emission is bunched for $\beta>0$, i.e., if the electron momentum is above resonance ($\bar{p}>0$), antibunched for $\beta<0$ ($\bar{p}<0$), and in a coherent state for $\beta=0$ ($\bar{p}=0$). This is a genuine quantum effect which cannot be obtained by any classical analysis. Intuitively we can understand the phenomenon of photon antibunching in a FEL by noting first that the classical current leads to a coherent state of the field, i.e., a Poisson distribution function. The effect of recoil for $\beta<0$ is to remove "bunches" of photons from the coherent state, thus leading to a narrower distribution function. The situation here is therefore similar to the multiphoton absorption process in atoms,[13] where photon antibunching has also been predicted.

The present analysis is based on an initial vacuum state. We conjecture that similar results concerning photon antibunching would be obtained for an arbitrary initial coherent state. A careful analysis of this problem within the framework of a many-electron theory remains to be carried out.

If we try to obtain corresponding results for a coherent state we run into the same difficulties which are already inherent in Ref. 5. Let us first take an initial electron field coherent state[5] $||\alpha\rangle$ with $A||\alpha\rangle=\alpha||\alpha\rangle$. The lowest-order contribution to the gain,

$$\langle\alpha||S(T/2,-T/2)^{\dagger}A^{\dagger}AS(T/2,-T/2)||\alpha\rangle$$
$$=|\,\alpha+j\,|^2+\cdots\ ,$$

is strongly phase dependent, and the same occurs to all higher orders. This is not surprising since in contrast to a state $|\,\bar{p},N\rangle=|\,\bar{p}\rangle\,|\,N\rangle$, in which both the field and the electron are uniformly distributed in space, an electron field coherent state contains inbuilt correlations which reflect the classical initial conditions. Before reasonable results for an ensemble of electrons can be obtained, the phase of the coherent state must be averaged over, analogously to the averaging over classical initial conditions.[14] Generally, such an averaging procedure will not preserve a coherent state. The necessity of averaging does not occur in case of a state $|\,\bar{p}N\rangle$.

Alternatively we might consider the amplification of a field-coherent state, i.e., $|\,in\rangle=|\,p\rangle\,|\,v\rangle$ with $a_L\,|\,v\rangle=v\,|\,v\rangle$. Since

$$A\,|p\rangle\,|v\rangle = v\,|p+\hbar k\rangle\,|v\rangle\ ,$$

in view of the orthogonality of electron states with different momenta, only terms with equal numbers of A's and $A^\dagger$'s survive. Hence we are essentially back to the results for photon number states. This conclusion, however, depends crucially on the orthogonality relation $\langle p\,|p+\hbar k\rangle=0$, which involves the quantum recoil. To use it in the zeroth-order term where recoil has been neglected otherwise, does not seem to be consistent. Moreover, making use of this orthogonality requires an extremely monochromatic electron beam.

We showed that starting from the field vacuum $N=0$, due to the presence of gain, the FEL does not radiate a coherent state. We believe this suggests that the FEL also conserves coherent states (be it field-coherent states or some averaged electron field coherent states) only inasmuch as gain is neglected.

IV. DISCUSSION

If the quantum recoil is neglected, we are left with the simple model of a classical current interacting with a quantized radiation field which is exactly solvable. This leads to the photon statistics as established in Eq. (27). The most essential features of the FEL, however, gain and electron bunching, have dropped out of this approximation. In an earlier approach[7,8] this had been remedied by reintroducing the recoil by hand into the detuning parameter. This procedure yielded correct results for gain, spread, and all other basic properties of FEL's. We are now going to compare our present exact first-order calculation of the photon statistics with the former semiphenomenological approach.

With the just-mentioned procedure[7,8] we have, instead of Eq. (26),

$$\tilde{P}(n)=\left[1+(n-N)\Delta z\frac{\partial}{\partial z}\right]|\langle n\,|S_0\,|N\rangle|^2\ , \tag{32}$$

where we proceeded as indicated in Eqs. (23) and (24) of Ref. 7 or Eqs. (2.16) and (2.18) of Ref. 8. In our present notation

$$z=j(T)^2,\quad \Delta z=-\frac{\hbar k^2}{m}j(T)\frac{\partial j(T)}{\partial\beta}\ . \tag{33}$$

Using the zeroth-order matrix element (27), which is common to both approaches, doing the derivative indicated in Eq. (32), and re-expressing $\tilde{P}(n)$ in terms of matrix elements of S_0 we obtain

$$\tilde{P}(n)=|\langle n\,|S_0\,|N\rangle|^2+\Delta z(n-N)\left[\left(\frac{n-N}{z}-1\right)\langle n\,|S_0\,|N\rangle-2\langle n\,|S_0\,|N-1\rangle\right]\langle N\,|S_0^\dagger\,|n\rangle\ . \tag{34}$$

To compare with Eq. (26) we evaluate the matrix element $\langle\bar{p},n\,|[p,S_0]\,|\bar{p},N\rangle$ [Eq. (18d)] which yields the relation

$$(n-N)\langle n\,|S_0\,|N\rangle=j(T)(\langle n\,|S_0\,|N-1\rangle\sqrt{N}+\langle n\,|S_0\,|N+1\rangle\sqrt{N+1})+j^2(T)\langle n\,|S_0\,|N\rangle\ . \tag{35}$$

It is consistent with our earlier approximations to drop the last term on the right-hand side of Eq. (35), which then can also be used to simplify Eqs. (25) and (26). Introducing Eq. (35) in Eq. (34) we obtain

$$\tilde{P}(n)=|\langle n\,|S_0\,|N\rangle|^2+\frac{\Delta z}{j(T)}\langle N\,|S_0^\dagger\,|n\rangle[\sqrt{N}\langle n\,|S_0\,|N-1\rangle+j(T)\sqrt{(N+1)(N+2)}\langle n\,|S_0\,|N+2\rangle$$
$$-j(T)\sqrt{N(N-1)}\langle n\,|S_0\,|N+2\rangle+j(T)\langle n\,|S_0\,|N\rangle]\ . \tag{36}$$

This differs from Eq. (26) only by the presence of the last term. It is this term which destroys unitarity so that $\sum_{n=0}^{\infty}\tilde{P}(n)\neq1$. The last term can be safely neglected for $N>>1$. For small N inspection of the explicit form [Eq. (27)] of $\langle n\,|S_0\,|N\rangle$ shows that it only contributes significantly for $n=N$. Hence all moments calculated by means of $\tilde{P}(n)$ instead of $P(n)$ are reliable for $N>>1$ as well as $N=0$. This justifies the semiphenomenological approach[7,8] for all cases of interest. It also shows that the latter cannot be trusted whenever recoil related modifications of $\langle n\,|S_1(T/2,-T/2)\,|N\rangle$ become important, since the process of introducing the recoil by hand fails to reproduce Eq. (25).

Our work differs from Ref. 5 mainly by using the interaction instead of the Schrödinger picture. In Ref. 5 quantum fluctuations of the momentum operator are neglected by approximating

$$p^2=[\langle p\rangle+(p-\langle p\rangle)]^2\approx 2p\langle p\rangle-\langle p\rangle^2 \,. \quad (37)$$

If $\langle p\rangle$ is assumed to be constant, the resulting Hamiltonian no longer allows for gain. This is easily demonstrated by calculating $\langle N \mid \exp(iHt)A^{\dagger}A\exp(-iHt)\mid N\rangle$ with the Hamiltonian approximated according to Eq. (37). Hence in Ref. 5, $\langle p\rangle$ is assumed to be time dependent and to be given by a classical trajectory. One is then left with an explicitly time-dependent Hamiltonian, which is, moreover, ambiguous since the classical trajectories behave completely different depending on the classical initial conditions.[14] Since this procedure cannot be considered to be a consistent quantum-mechanical approach, conclusions drawn from it regarding genuine quantum-mechanical entities such as the evolving photon statistics do not seem to be reliable.

Our linear recoil approximation is similar to Eq. (37); we apply it, however, to the interaction picture-time evolution operator and not to the complete Hamiltonian. Up to that final expansion, the exact H_0 has been used. Speaking in terms of quantum-mechanical perturbation theory we have approximated the vertices, but not the propagators. The importance of retaining "quantum fluctuations" in the momentum is also evident from a semiclassical treatment; see Eq. (10) of Ref. 15 or Eq. (3.7) of Ref. 16. If within the mentioned equations the second-order terms are dropped, the gain is lost.

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[10]In view of Eqs. (15) and (2),

$$|j(T)|^2\le g^2N_WT^2=4\pi(r_0L/k)^2(N_W/V)(1/V)\,,$$

which is proportional to the energy density N_W/V of the wiggler field as well as the inverse quantization volume. The latter must be chosen larger than the actual volume of the system, which is Ld^2 with $d\sim 1$ cm. With the numerical parameters of the Stanford experiments we then have $|j(T)|^2\lesssim 10^{-4}$. For arbitrary t_1 and t_2, $|j(t_2,t_1)|^2$ is of the same order of magnitude. Note that, although $N_W>>1$ is required, the limit $N_W\to\infty$ corresponding to an infinite wiggler field is excluded due to our first-order expansion.

[11]In principle, the Bambini-Renieri frame which equates the laser and the wiggler frequency, restricts spontaneous emission to just one mode. If we want to consider different modes, we have to introduce a different frame in each case, which is then mainly reflected in different values of $\bar{p}$. Hence, by varying $\bar{p}$, we can still cover all modes of spontaneous emission (in axial direction).

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