Nonlinear constants of motion for three-level quantum systems

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We use the eight-dimensional coherence vector $\vec{S}$, recently introduced for three-level quantum systems, to derive three independent nonlinear constants of motion. Two of these were known previously, but not known to be related, and the third is new.

We have previously shown[1] the existence of a number of unexpected nonlinear constants of motion that govern the density matrix of any N-level quantum system. In this note we extend that work. We concentrate on a three-level system and show that under appropriate resonance and Cook-Shore[2] conditions the eight-dimensional coherence space can be factored exactly into three smaller subspaces, each with its own nonlinear constants of motion.

We point out that two of these constants were previously found in special contexts, while the third is new. It should be clear from our derivation that the eight-dimensional coherence vector (recently introduced in Ref. 1) plays a unifying role in relating these constants to each other.

The dynamical evolution of a three-level atomic system can be expressed in terms of its density matrix $\hat{\rho}$, which satisfies the Liouville equation

$$i\hbar\frac{\partial\hat{\rho}}{\partial t}=[\hat{H},\hat{\rho}] \; . \tag{1}$$

We have shown[1] that it can also be expressed in terms of the evolution of an eight-dimensional real coherent vector $\vec{S}$, whose components are formed in groups denoted by the symbols u, v and w and defined by

$$\begin{aligned} u_{jk}&=\rho_{jk}+\rho_{kj} \; , \\ v_{jk}&=i(\rho_{jk}-\rho_{kj}) \; , \\ w_l&=-[2/l(l+1)]^{1/2} \\ &\quad\times(\rho_{11}+\rho_{22}+\cdots+\rho_{ll}-l\rho_{l+1,l+1}) \; , \end{aligned} \tag{2}$$

where $1\le j<k\le 3$ and $1\le l\le 2$. The equations obeyed by these components are given below.

We consider an atomic system (see Fig. 1) in which nonzero dipole moments exist only between levels 1 and 2, and 2 and 3. Let there be two electromagnetic waves incident on the atom. The total electric field is given by

$$\vec{E}(z,t)=\vec{\mathscr{E}}_{12}(t)e^{i(\nu_{12}t-k_{12}z)}+\vec{\mathscr{E}}_{23}(t)e^{i(\nu_{23}t-k_{23}z)}+\text{c.c.} \tag{3}$$

We define $\alpha(t)$ and $\beta(t)$ in terms of the Rabi frequencies $\Omega_{jk}(t)$ by

$$\alpha(t)=\tfrac{1}{2}\Omega_{12}(t)=\frac{\vec{d}_{12}\cdot\vec{\mathscr{E}}_{12}(t)}{\hbar} \; , \tag{4a}$$

$$\beta(t)=\tfrac{1}{2}\Omega_{23}(t)=\frac{\vec{d}_{23}\cdot\vec{\mathscr{E}}_{23}(t)}{\hbar} \; , \tag{4b}$$

where $\vec{d}_{jk}$ is the atomic dipole moment between levels j and k. The detunings Δ_{jk} are defined as usual by

$$\Delta_{jk}=\nu_{jk}-\omega_{jk} \; , \tag{5}$$

where ω_{jk} is the frequency separation between levels j and k.

Now consider the case of exact two-photon resonance:

$$\Delta_{12}=-\Delta_{23}\equiv\Delta \; . \tag{6}$$

The equations of motion for the components of the coherent vector $\vec{S}$ [defined in Eq. (2)] are given by

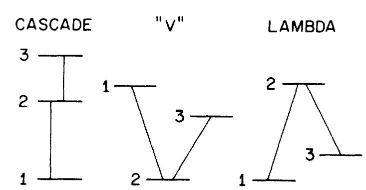

FIG. 1. The three types of three-level atom having nonzero dipole matrix elements between levels 1 and 2, and levels 2 and 3. Our analysis applies to all three types.

$$
\begin{aligned}
\dot{u}_{12} &= \Delta v_{12} + \beta v_{13} , \\
\dot{u}_{23} &= -\Delta v_{23} - \alpha v_{13} , \\
\dot{u}_{13} &= \beta v_{12} - \alpha v_{23} , \\
\dot{v}_{12} &= -\Delta u_{12} - \beta u_{13} + 2\alpha w_1 , \\
\dot{v}_{23} &= \Delta u_{23} + \alpha u_{13} \\
&\qquad - \beta w_1 + \sqrt{3}\beta w_2 , \\
\dot{v}_{13} &= -\beta u_{12} + \alpha u_{23} , \\
\dot{w}_1 &= -2\alpha v_{12} + \beta v_{23} , \\
\dot{w}_2 &= -\sqrt{3}\beta v_{23} .
\end{aligned} \tag{7}
$$

Our eight-dimensional coherence vector $\vec{S}$ is the natural generalization of the pseudo spin[3] or Bloch vector of two-level systems, and these equations are straightforward generalizations of the familiar two-level Bloch equations.[4] Those two-level equations are recovered if every symbol containing the index 3 (or the index 1) is crossed out.

We now assume that $\alpha(t)$ and $\beta(t)$ have the same time dependence but possibly different amplitudes:

$$\alpha(t) = a\Omega_0(t) , \tag{8a}$$

$$\beta(t) = b\Omega_0(t) . \tag{8b}$$

Here a and b are arbitrary constants. We shall refer to pulses of the form (8) as generalized Cook-Shore pulses.[2] Note that a subcategory of considerable interest (monochromatic plane-wave laser fields) is obtained by taking $\Omega_0 = \text{const}$.

We now show that when conditions (6) and (8) hold true, the time evolution of the eight-dimensional coherence vector $\vec{S}$ can be analyzed in terms of the time evolution of three independent vectors of dimensions three, four, and one, rotating in three disjoint subspaces of those dimensions. We also derive three new corresponding constants of the motion which represent the squares of the lengths of these three vectors. Their sum is the square of the length of the coherence vector $\vec{S}$, which is known to be a constant of the motion.

We first make the following linear coordinate transformation:

$$
\begin{aligned}
U &= \frac{1}{\sqrt{\alpha^2+\beta^2}} (\alpha u_{12} + \beta u_{23}) , \\
V &= \frac{1}{\sqrt{\alpha^2+\beta^2}} (-\alpha v_{12} + \beta v_{23}) , \\
W &= \frac{1}{2(\alpha^2+\beta^2)} [-(2\alpha^2+\beta^2) w_1 \\
&\qquad + \sqrt{3}\beta^2 w_2 + 2\alpha\beta u_{13}] , \\
v_{13} &= v_{13} , \\
\mathcal{U} &= \frac{1}{\sqrt{\alpha^2+\beta^2}} (\beta u_{12} - \alpha u_{23}) , \\
\mathcal{V} &= \frac{1}{\sqrt{\alpha^2+\beta^2}} (\beta v_{12} + \alpha v_{23}) , \\
\mathcal{W} &= \frac{1}{\alpha^2+\beta^2} [-\alpha\beta w_1 - \sqrt{3}\alpha\beta w_2 - (\alpha^2-\beta^2) u_{13}] \\
\mathfrak{W} &= \frac{1}{2(\alpha^2+\beta^2)} [-\sqrt{3}\beta^2 w_1 + (2\alpha^2-\beta^2) w_2 \\
&\qquad - 2\sqrt{3}\alpha\beta u_{13}] .
\end{aligned} \tag{9}
$$

This is an orthogonal transformation. Note that it can be expressed as

$$
\begin{bmatrix} v_{13} \\ \mathcal{U} \\ U \\ \mathcal{V} \\ V \\ \mathcal{W} \\ W \\ \mathfrak{W} \end{bmatrix}
=
\left[\begin{array}{c|cc|cc|ccc}
M_1 & & & & & & & \\ \hline
 & \multicolumn{2}{c|}{M_2} & & & & 0 & \\
 & & & & & & & \\ \hline
 & & & \multicolumn{2}{c|}{\tilde{M}_2} & & & \\
 & & & & & & & \\ \hline
 & & 0 & & & & & \\
 & & & & & & M_3 & \\
 & & & & & & &
\end{array}\right]
\begin{bmatrix} v_{13} \\ u_{12} \\ u_{23} \\ v_{12} \\ v_{23} \\ u_{13} \\ w_1 \\ w_2 \end{bmatrix} \tag{10}
$$

where

$$
M_1 = 1 , \quad M_2 = \frac{1}{\sqrt{\alpha^2+\beta^2}} \begin{bmatrix} \beta & -\alpha \\ \alpha & \beta \end{bmatrix} , \quad \tilde{M}_2 = \frac{1}{\sqrt{\alpha^2+\beta^2}} \begin{bmatrix} \beta & \alpha \\ -\alpha & \beta \end{bmatrix} ,
$$

$$
M_3 = \frac{1}{\alpha^2+\beta^2} \begin{bmatrix} -(\alpha^2-\beta^2) & -\alpha\beta & -\sqrt{3}\alpha\beta \\ \alpha\beta & -\frac{1}{2}(2\alpha^2+\beta^2) & \frac{\sqrt{3}}{2}\beta^2 \\ -\sqrt{3}\alpha\beta & -\frac{\sqrt{3}}{2}\beta^2 & \frac{1}{2}(2\alpha^2-\beta^2) \end{bmatrix} , \tag{11}
$$

and where all of the M's are orthogonal matrices. Thus the transformation (10) preserves the length of the vector $\vec{S}$. That is, if we denote by $\vec{T}$ the transformed vector and denote by M the transformation matrix on the right-hand side of Eq. (10), we have from

$$\vec{T}=\underline{M}\vec{S} \tag{12a}$$

and

$$\tilde{\underline{M}}=\underline{M}^{-1} \tag{12b}$$

the result

$$|\vec{S}|^2=|\vec{T}|^2 . \tag{13}$$

A remarkable consequence of this transformation follows from the observation that if the components of $\vec{T}$ are ordered according to

$$\vec{T}(U,V,W;v_{13},\mathscr{U},\mathscr{V},\mathscr{W};\mathfrak{W}) \tag{14}$$

then the equations of motion for $\vec{T}$ become

$$\frac{d\vec{T}}{dt}=\underline{\Delta}\vec{T} , \tag{15}$$

where $\underline{\Delta}$ is a block-diagonal matrix of the form

$$\underline{\Delta}=\begin{bmatrix}\underline{\Delta}_3 & & \\ & \underline{\Delta}_4 & \\ & & \underline{\Delta}_1\end{bmatrix} , \tag{16}$$

where each of the $\underline{\Delta}$'s is antisymmetric. That is, we have

$$\underline{\Delta}_3=\begin{bmatrix}0 & -\Delta & 0\\ \Delta & 0 & 2\epsilon\\ 0 & -2\epsilon & 0\end{bmatrix} , \tag{17a}$$

where

$$\epsilon\equiv(\alpha^2+\beta^2)^{1/2} ,$$

$$\underline{\Delta}_4=\begin{bmatrix}0 & -\epsilon & 0 & 0\\ \epsilon & 0 & \Delta & 0\\ 0 & -\Delta & 0 & -\epsilon\\ 0 & 0 & \epsilon & 0\end{bmatrix} , \tag{17b}$$

$$\underline{\Delta}_1=[0] . \tag{17c}$$

Clearly we can now analyze the time evolution of the eight-dimensional coherence vector $\vec{T}$ in terms of the time evolution of three separate vectors

$$\vec{T}_3(t)=\begin{bmatrix}U(t)\\ V(t)\\ W(t)\end{bmatrix}, \ \vec{T}_4(t)=\begin{bmatrix}v_{13}(t)\\ \mathscr{U}(t)\\ \mathscr{V}(t)\\ \mathscr{W}(t)\end{bmatrix} , \tag{18}$$

$$\vec{T}_1(t)=[\mathfrak{W}(t)] .$$

The central point is that, not only is the length of $\vec{T}$ conserved, but because of the antisymmetric nature of the $\underline{\Delta}$ matrices, the lengths of $\vec{T}_1$, $\vec{T}_3$, and $\vec{T}_4$ are also separately conserved. That is, in two-photon resonance conditions, one has three exact nonlinear conservation laws:

$$U(t)^2+V(t)^2+W(t)^2=\text{const} , \tag{19a}$$

$$v_{13}(t)^2+\mathscr{U}(t)^2+\mathscr{V}(t)^2+\mathscr{W}(t)^2=\text{const} , \tag{19b}$$

$$\mathfrak{W}(t)^2=\text{const} . \tag{19c}$$

The reduced equation of motion $d\vec{T}_3/dt=\underline{\Delta}_3\vec{T}_3$, and thus (19a), was first obtained by Brewer and Hahn,[5] and Eq. (19c) was first stated in a different form (for the pure case) by Gray, Whitley, and Stroud.[6] The four-dimensional conservation law (19b) is presented here for the first time.

Furthermore, we note that under exact one-photon as well as two-photon resonance conditions ($\Delta=0$), $\underline{\Delta}_4$ also becomes block-diagonal, and each block is antisymmetric. Thus $\vec{T}_4$ can be further decomposed into two two-component vectors $\vec{T}_4=\vec{T}_{2a}\oplus\vec{T}_{2b}$, each of which also has constant length:

$$|\vec{T}_{2a}|^2=v_{13}(t)^2+\mathscr{U}(t)^2=\text{const} , \tag{20a}$$

$$|\vec{T}_{2b}|^2=\mathscr{V}(t)^2+\mathscr{W}(t)^2=\text{const} . \tag{20b}$$

In summary, we have shown that the familiar equations for two-photon and one-photon transitions in three-level quantum systems (7) can be regrouped with unexpected results. In the presence of two monochromatic lasers (or Cook-Shore pulses, more generally), and under two-photon-resonant conditions, the eight-dimensional space can be decomposed into three independent spaces. The quantum evolution of the three-level system is thus characterized by three independent coherence vectors, each with its own nonlinear conservation law. The overall coherence vector $\vec{S}$ used in our discussion allows one to unify the two of these conservation laws that were known before, with each other and with the third (new) conservation law.

In addition, the block-factored form of $\underline{A}$ can be shown to provide a unifying approach to the theory of simultons.[7] It also suggests that analogies to quark-model results in particle physics may be of interest in quantum optics. We will devote separate papers to these additional topics.

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