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Photon dynamics

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A theory of the motion of photons in physical space is presented. Operators representing the photon density and the photon current density are defined in terms of a pair of vector field operators in much the same way as the energy density and the Poynting vector are defined in terms of the electric and magnetic fields. For polychromatic light, this theory predicts an ideal photon-counting rate which differs from that of the usual theory of photoelectron counting.

Many conceptual as well as practical problems in quantum optics call for a description in terms of the motion of photons in physical space. Although the electric and magnetic field operators provide a useful description of the location and flow of the field energy [through the energy density $u=(8\pi)^{-1}{:}\,E^2+B^2{:}$ and the Poynting vector $\vec{S}=(c/4\pi){:}\vec{E}\times\vec{B}{:}$], they do not provide a clear picture of the location and motion of the photons. In the theory of photoelectron counting, it is shown that the counting rate of an idealized photon detector is proportional to the product $E^{(-)}E^{(+)}$ of the positive- and negative-frequency parts of the electric field.[1] It is often inferred from this that the photon current density C (or photon flux) must also be proportional to $E^{(-)}E^{(+)}$ ($C=\alpha E^{(-)}E^{(+)}$), but if we attempt to make this idea precise, we immediately encounter difficulties. Because the photon current density is clearly independent of the detector used to measure it, the proportionality constant α must be a universal one. But the only universal constants pertaining to the photon are $\hbar$ and c, and it is not possible to construct a dimensionally correct α from these constants.[2] Therefore, the prevailing theory of photoelectron counting fails to provide a complete description of photon transport. The purpose of this paper is to present a general theory of the motion of photons in position space within the constraints imposed by photon localizability and without relying on any particular model of the photon detection process.

To describe the motion of photons in physical space, I introduce the following pair of vector field operators:

$$\vec{\psi}(\vec{x})=(2L^3)^{-1/2}\sum_{\vec{k},\lambda}\vec{\epsilon}_{\vec{k}\lambda}a_{\vec{k}\lambda}e^{i\vec{k}\cdot\vec{x}}\ , \tag{1a}$$

$$\vec{\phi}(\vec{x})=(2L^3)^{-1/2}\sum_{\vec{k},\lambda}(\vec{k}/k)\times\vec{\epsilon}_{\vec{k}\lambda}a_{\vec{k}\lambda}e^{i\vec{k}\cdot\vec{x}}\ , \tag{1b}$$

where $a_{\vec{k}\lambda}$ and $\vec{\epsilon}_{\vec{k}\lambda}$ are, respectively, the annihilation operator and polarization vector for a transverse photon of wave vector $\vec{k}$ and polarization $\lambda(=1,2)$, and L^3 is the quantization volume. I then construct from these field operators the positive-definite Hermitian operator

$$D(\vec{x})=\vec{\psi}^{\dagger}(\vec{x})\cdot\vec{\psi}(\vec{x})+\vec{\phi}^{\dagger}(\vec{x})\cdot\vec{\phi}(\vec{x})\ , \tag{2}$$

and postulate that this operator represents the density of photons in position space. It is readily verified that the integral of $D(\vec{x})$ over the quantization volume (or over all space in the limit $L\rightarrow\infty$) is the usual photon number operator,

$$\int d^3x\, D(\vec{x})=\sum_{\vec{k},\lambda}a^{\dagger}_{\vec{k}\lambda}a_{\vec{k}\lambda}\ . \tag{3}$$

To derive an equation of continuity for photons, the field equations for $\vec{\psi}$ and $\vec{\phi}$ are required. Using the usual Heisenberg equation of motion for $a_{\vec{k}\lambda}$,

$$\dot{a}_{\vec{k}\lambda}=-i\omega_k a_{\vec{k}\lambda}+i(2\pi/\hbar\omega_k)^{1/2}J_{\vec{k}\lambda}\ , \tag{4}$$

where

$$J_{\vec{k}\lambda}=\int d^3x\,\vec{u}^{*}_{\vec{k}\lambda}(\vec{x})\cdot\vec{J}(\vec{x})$$

is the projection of the electric current density operator $\vec{J}(\vec{x})$ on the transverse-mode function

$$\vec{u}_{\vec{k}\lambda}(\vec{x})=L^{-3/2}\vec{\epsilon}_{\vec{k}\lambda}\exp(i\vec{k}\cdot\vec{x})\ ,$$

one obtains from Eqs. (1) two field equations involving time derivatives of $\vec{\psi}$ and $\vec{\phi}$. Two more field equations follow from the fact that $\vec{\psi}$ and $\vec{\phi}$ are transverse. The field equations are

$$\vec{\nabla}\cdot\vec{\psi}=0\ , \tag{5a}$$

$$\vec{\nabla}\cdot\vec{\phi}=0\ , \tag{5b}$$

$$\vec{\nabla}\times\vec{\psi}+\frac{1}{c}\frac{\partial\vec{\phi}}{\partial t}=\frac{1}{c}\vec{S}_1\ , \tag{5c}$$

$$\vec{\nabla}\times\vec{\phi}-\frac{1}{c}\frac{\partial\vec{\psi}}{\partial t}=\frac{i}{c}\vec{S}_2\ , \tag{5d}$$

where

$$\vec{S}_1(\vec{x})=\frac{i\pi^{1/2}}{(2\pi)^3}\int d^3x'\int\frac{d^3k}{(\hbar\omega_k)^{1/2}}\frac{\vec{k}\times\vec{J}(\vec{x}')}{k}\times e^{i(\vec{x}-\vec{x}')\cdot\vec{k}}\ , \tag{6a}$$

$$\vec{S}_2(\vec{x})=\frac{\pi^{1/2}}{(2\pi)^3}\int d^3x'\int\frac{d^3k}{(\hbar\omega_k)^{1/2}}\frac{\vec{k}\times[\vec{k}\times\vec{J}(\vec{x}')]}{k^2}\times e^{i(\vec{x}-\vec{x}')\cdot\vec{k}}\ , \tag{6b}$$

are Hermitian source terms, here expressed in the mode continuum limit ($L\rightarrow\infty$). Now the time derivative of the photon density, Eq. (2), is evaluated by substituting for the time derivatives of $\vec{\psi}$, $\vec{\phi}$, $\vec{\psi}^{\dagger}$, and $\vec{\phi}^{\dagger}$ the expressions provided by Eqs. (5) and their Hermitian conjugates. It is found that D satisfies the equation of continuity

$$\frac{\partial D}{\partial t}+\vec{\nabla}\cdot\vec{C}=Q\ , \tag{7}$$

where

$$\vec{C}=c(\vec{\psi}^{\dagger}\times\vec{\phi}-\vec{\phi}^{\dagger}\times\vec{\psi}) \tag{8}$$

is the photon current density, and

$$Q=\vec{S}_1\cdot\vec{\phi}+\vec{\phi}^{\dagger}\cdot\vec{S}_1+i(\vec{S}_2\cdot\vec{\psi}-\vec{\psi}^{\dagger}\cdot\vec{S}_2) \tag{9}$$

is the operator representing the number of photons created per unit volume and per unit time by the radiating current $\vec{J}$.

The interpretation of $\vec{C}$ as the photon current density was inferred from the position it holds in the equation of continuity (7). Additional support for this interpretation of $\vec{C}$ is provided by an analysis of the response of an idealized photon detector. For a detector consisting of N noninteracting two-level atoms, it is well known that, within the electric-dipole and rotating-wave approximations, the operator

$$M=\sum_{\vec{k},\lambda}a^{\dagger}_{\vec{k}\lambda}a_{\vec{k}\lambda}+\sum_{i=1}^{N}\sigma_i^{\dagger}\sigma_i \tag{10}$$

is a constant of the motion for the field-detector system, where $\sigma_i^{\dagger}=|2\rangle_{ii}\langle 1|$ and $\sigma_i=|1\rangle_{ii}\langle 2|$ are the excitation and deexcitation operators of the ith detector atom. The constancy of M means that for each excitation of a detector atom, i.e., for each count registered by the detector, one photon disappears from the field. Using Eq. (3), the time derivative of (10) may be written as

$$R=-\frac{d}{dt}\int d^3x\,D\ , \tag{11}$$

where $R=d(\sum_i\sigma_i^{\dagger}\sigma_i)/dt$ is the counting rate of the detector. Now suppose the detector atoms occupy a small volume V_d with surface S. Then, if V' denotes the volume of space outside of the detector, Eq. (11) reads

$$R=-\frac{d}{dt}\int_{V_d}d^3x\,D-\int_{V'}d^3x\frac{\partial D}{\partial t}\ . \tag{12}$$

The integral over V_d represents the number of photons that have entered the detector volume, but have not been absorbed by the detector. In most cases encountered in practice, this number of photons is entirely negligible. Therefore, I discard the first term in (12) and convert the second term to the form

$$R=\oint_S da\,\vec{n}\cdot\vec{C} \tag{13}$$

by means of the continuity equation in $V'(\partial D/\partial t=-\vec{\nabla}\cdot\vec{C})$ and the divergence theorem, $\vec{n}$ being the inward normal to the surface S. Thus the counting rate of the detector equals the integral over the detector's surface of the inward normal component of $\vec{C}$. This argument is easily modified to treat a detector whose atoms each have a continuum of excited states rather than a single excited state. Such a model more closely describes the photoelectric detection process. Equation (13) is again obtained, but now R represents the rate of excitation to the continuum, i.e., the rate of photoelectric counts.

It must be emphasized that the present theory is not equivalent to the usual theory of photoelectron counting. The latter theory employs the formula $R'=\xi E^{(-)}E^{(+)}$ for the counting rate, where ξ is a constant describing the sensitivity of the detector. For an x-polarized polychromatic plane wave traveling in the z direction and a detector at $z=0$ with a flat sensitive surface of area A normal to the z axis, Eq. (13) yields the ideal counting rate

$$R = cA(\psi_x^\dagger \phi_y + \phi_y^\dagger \psi_x) = cAL^{-3} \sum_{k,k'} a_k^\dagger a_{k'} , \quad (14)$$

while the theory of photoelectron counting predicts the rate

$$R' = \xi E_x^{(-)} E_x^{(+)} = 2\pi\xi\hbar L^{-3} \sum_{k,k'} (\omega_k \omega_{k'})^{1/2} a_k^\dagger a_{k'} , \quad (15)$$

where the sums in (14) and (15) are restricted to x-polarized field modes propagating in the z direction. The two expressions are essentially equivalent for quasimonochromatic light, for in this case $(\omega_k \omega_{k'})^{1/2}$ may be replaced by the center frequency ω of the radiation and may be taken outside of the sum in (15) to give $R' = R$, provided $\xi = cA/2\pi\hbar\omega$, which is the condition for unit quantum efficiency. But if the radiation is, say, a superposition of two monochromatic coherent states with amplitudes $\alpha_1 \exp(i\omega_1 t)$ and $\alpha_2 \exp(i\omega_2 t)$ and widely separated frequencies, e.g., two laser beams, then the time-averaged expectation value of R, namely,

$$\langle R \rangle = cA(|\alpha_1|^2 + |\alpha_2|^2)/L^3 ,$$

is the correct photon impact rate on the area A (since $|\alpha_1|^2/L^3$ and $|\alpha_2|^2/L^3$ are the photon number densities in the two beams and the photons move with speed c), while the time-averaged expectation value of R',

$$\langle R' \rangle = 2\pi\xi\hbar(\omega_1 |\alpha_1|^2 + \omega_2 |\alpha_2|^2)/L^3 ,$$

is not proportional to the photon flux for any value of ξ. In fact, for a fixed value of the photon impact rate $\langle R \rangle$, $\langle R' \rangle$ can have any value from $2\pi\xi\hbar\omega_1 \langle R \rangle / cA$ to $2\pi\xi\hbar\omega_2 \langle R \rangle / cA$ depending on the ratio of the amplitudes of the two beams. Therefore, Eq. (15) is an inappropriate expression for the photon-counting rate when the radiation is polychromatic, while Eq. (14) appears to be satisfactory.[3]

If one accepts the above interpretations of D and $\vec{C}$, then the operator n_V representing the number of photons in a volume V must be given by the integral of D over that volume,

$$n_V = \int_V d^3x \, D(\vec{x}) , \quad (16)$$

and the operator n_T representing the number of photons that cross a surface S in the time interval $[t, t+T]$ must take the form

$$n_T = \int_t^{t+T} dt' \int_S da \, \vec{n} \cdot \vec{C}(\vec{x}, t') , \quad (17)$$

where $\vec{n}$ is the unit normal to the surface S in the direction of interest. The principle shortcoming of the present theory is that n_V and n_T do not possess all of the properties of number operators for arbitrarily small V, S, and T. This reflects the well-known difficulty of localizing photons in position space. According to the original criteria of Newton and Wigner for the localizability of elementary systems,[4] the photon is not localizable. This means that, strictly speaking, there does not exist a probability density for the position of the photon, nor does there exist a photon number operator for the volume V.[5] This last conclusion was challenged by Jauch and Piron who, by modifying the criteria for localizability, were able to show that an operator representing the number of photons in a given volume can be rigorously defined,[6] but not as the integral of a photon density operator.[7] These results demand that D and $\vec{C}$ be reinterpreted as coarse-grained representatives of the photon density and current density, respectively; i.e., as operators that yield correct expressions for n_V and n_T so long as V, S, and T are sufficiently large. This interpretation is supported by a calculation which shows that, when the linear dimensions of V and S are large compared to the photon wavelengths λ and T is long compared to the periods c/λ of the photons, n_V and n_T acquire all of the usual properties of number operators. The calculation, which for the sake of brevity will not be given here, is essentially equivalent to that presented by Mandel in his analysis of the photon number operator for the volume V.[8] Given that n_V and n_T are number operators, one can, by standard arguments, derive formulas for the probability $p_V(m)$ that V contains m photons and the probability $p_T(m)$ that m photons cross the surface S in the time interval $[t, t+T]$. The results are

$$p_V(m) = \mathrm{Tr}[\rho :n_V^m \exp(-n_V):]/m! , \quad (18)$$

$$p_T(m) = \mathrm{Tr}[\rho :n_T^m \exp(-n_T):]/m! , \quad (19)$$

where ρ is the density operator of the radiation field. Moreover, when S is the sensitive surface of an ideal photon detector, $p_T(m)$ is the photon-count distribution registered by the detector.[9] The utility of these results derives from the fact that the required conditions on V, S, and T are met in most, if not all practical applications. Note that the photon-count distribution $p_T(m)$ can differ appreciably from that of the prevailing theory when the radiation is polychromatic.

In calculating the fields $\vec{\psi}$ and $\vec{\phi}$ radiated by the electric current $\vec{J}$, the following results are often quite useful. The non-Hermitian fields $\vec{\psi}$, $\vec{\phi}$ are derivable from a pair of transverse Hermitian vector potentials:

$$\vec{\psi}=\vec{\nabla}\times\vec{a}_1-\frac{i}{c}\frac{\partial\vec{a}_2}{\partial t}\ , \tag{20}$$

$$\vec{\phi}=\frac{1}{c}\frac{\partial\vec{a}_1}{\partial t}+i\vec{\nabla}\times\vec{a}_2\ . \tag{21}$$

In terms of the potentials, the field equations (5) read

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{a}_1=-\frac{\vec{S}_1}{c}\ , \tag{22}$$

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{a}_2=-\frac{\vec{S}_2}{c}\ . \tag{23}$$

Finally, Eqs. (6) can be simplified somewhat by evaluating the integrals over $\vec{k}$. The results,

$$S_n^i(\vec{x})=\sum_{j=1}^{3}\int d^3x' K_n^{ij}(\vec{x}-\vec{x}')J^j(\vec{x}')\ , \tag{24}$$

where

$$K_1^{ij}(\vec{x})=3(2^5\pi^2\hbar c)^{-1/2}\sum_{L=1}^{3}\epsilon^{ijL}x^L/x^{7/2}\ , \tag{25}$$

$$K_2^{ij}(\vec{x})=(2^5\pi^2\hbar c)^{-1/2}(x^2\delta^{ij}-5x^ix^j)/x^{9/2}\ , \tag{26}$$

provide a clearer view of the relation between the photon source terms $\vec{S}_1$ and $\vec{S}_2$ and the electric current density $\vec{J}$.

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[1]R. J. Glauber, in *Quantum Optics and Electronics,* edited by C. DeWitt *et al.* (Gordon and Breach, New York, 1965), p. 65.

[2]For the special case of quasimonochromatic light, a photon current density proportional to $E^{(-)}E^{(+)}$ can be defined by using the characteristic frequency ω of the radiation together with $\hbar$ and c in the construction of α. But such a special case can hardly be made the basis of a general description of photon transport.

[3]The formula $R'=\xi E^{(-)}E^{(+)}$ is derived on the assumption that the detector has a frequency-independent transition probability (Ref. 1). The present theory challenges the view that such a detector measures the photon flux when the radiation is polychromatic.

[4]T. D. Newton and E. P. Wigner, Rev. Mod. Phys. 21, 400 (1949).

[5]A. S. Wightman, Rev. Mod. Phys. 34, 845 (1962).

[6]J. M. Jauch and C. Piron, Helv. Phys. Acta, 40, 559 (1967).

[7]W. O. Amrein, Helv. Phys. Acta, 42, 149 (1969).

[8]Mandel defines a photon number operator for the volume V as the integral of $D_M=2\vec{\psi}^{\dagger}\cdot\vec{\psi}$ over that volume. [Phys. Rev. 144, 1071 (1966)]. In the coarse-grained limit under consideration, Mandel's expression for n_V is equivalent to ours, and both agree with the rigorous result of Jauch and Piron (Ref. 7). But we have rejected D_M as a photon density because it is not possible to define a photon current density which together with D_M satisfies an equation of continuity.

[9]For a detector of quantum efficiency η, n_V is replaced by ηn_V in Eq. (19) to obtain the correct photon-count distribution.