Stimulated emission from relativistic electrons passing through a spatially periodic longitudinal magnetic field

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Stimulated emission in the high-gain regime from a cold, relativistic beam of electrons gyrating in a combined solenoidal and longitudinally polarized periodic wiggler magnetic field is considered as a source of short-wavelength radiation. The emitted wave frequency is Doppler upshifted in proportion to the wave number of the wiggler magnetic field. Amplification is due to a ponderomotive bunching force acting on the electrons in either the transverse or axial directions. Expressions for the linear growth rate are obtained; conditions for their validity and estimates for the saturated efficiency are given.

I. INTRODUCTION

In the past several years with the advent of intense, relativistic electron beams there has been an interest in using these electron beams to generate intense, coherent electromagnetic radiation in the centimeter, millimeter, and submillimeter wavelength portions of the electromagnetic spectrum. Currently two main types of radiation mechanisms using intense, relativistic electron beams are of interest. First are the cyclotron instabilities,[1-10] gyrotron and Weibel, characterized by transverse and axial electron bunching, respectively, in which the electrons travel in a solenoidal magnetic field B_0 with the emission frequency being associated with the electron gyrofrequency or one of its harmonics. The other main type of mechanism is the free-electron laser (FEL) instability[11-16] characterized by axial electron bunching in which the electrons travel in a transversely polarized, periodic wiggler magnetic field with an emission frequency associated with the period of the wiggler magnet.

The lowbitron[17]—a longitudinal wiggler beam interaction device—is a hybrid system of the above mechanisms. A thin pencil beam of relativistic electrons with large transverse velocity $v_\perp$ acquired before entering the interaction region travels on axis in a combined uniform solenoidal magnetic field and a *longitudinally* polarized periodic wiggler magnetic field. The total imposed field on the axis is of the form

$$\vec{B}=\hat{z}[B_0+\delta B\sin(k_0z)]\ , \qquad (1)$$

where $k_0=2\pi/l$ is the wave number, l the period, and δB the amplitude of the wiggler magnetic field. The amplitudes of the solenoidal and wiggler magnetic fields can be of the same order of magnitude with $\delta B/B_0\lesssim 1$. The field given by Eq. (1) can be generated by driving current azimuthally in alternate directions through a periodic assembly of copper rings,[18] or by making a series of rings from samarium-cobalt[19] or other magnetic material, and magnetizing the rings in the axial direction as is done in systems employing periodic focusing,[20] or the field can be generated by using the technique of magnetic diffusion[21,22] in a series of copper rings. The field generated by these methods is a multiple-mirror (undulator) field which at a distance r from the axis is approximately given by

$$\vec{B}\approx\hat{z}[B_0+\delta BI_0(k_0r)\sin(k_0z)] -\hat{r}\delta BI_1(k_0r)\cos(k_0z)\ , \qquad (2)$$

with I_0 and I_1 being modified Bessel functions. Near the axis where $k_0r<1$ the field given by Eq. (2) becomes that of Eq. (1). This imposes a constraint on the radius r of the electron orbit for it to be considered as moving in a magnetic field of the form given by Eq. (1). Taking the gyromotion in the solenoidal field as dominant then, since $r\omega_c=v_\perp$, gives the constraint

$$k_0v_\perp\lesssim\omega_c\ , \qquad (3)$$

where ω_c is the relativistic electron cyclotron frequency in the solenoidal field.

The periodic magnetic field in the lowbitron is longitudinally polarized rather than transversely polarized as it is in the FEL. This has several advantages in that longitudinal modulations are more easily produced and with larger amplitudes, the periodicity of a ring system is readily changed, and an adiabatic field shaper at the electron source end is readily incorporated.[22]

In what follows we consider stimulated emission of right-hand circularly polarized radiation propagating in the same direction as the electron's travel. We only consider emission at the fundamental harmonic k_0. Amplification is due to a Lorentz $\vec{v}\times\vec{B}$ force, the ponderomotive force, which causes bunching of the electrons in both the transverse and axial directions. The bunching force travels at the phase velocity $v_{ph}=(\omega-\omega_c)/(k+k_0)$ where ω and k are the radiation frequency and wave number, respectively. When the phase velocity of the bunching force is equal to the axial electron velocity v so that

$$\omega-kv=k_0v+\omega_c\ , \tag{4}$$

the bunching force appears to be stationary with respect to the electrons, and for electrons traveling slightly faster than v_{ph} energy is given up to the electromagnetic wave. The radiation frequency is found approximately from Eq. (4) by taking $kc\approx\omega$ to give

$$\omega=\left[1+\frac{v}{c}\right]\gamma^2\left[1+\gamma^2\frac{v_\perp^2}{c^2}\right]^{-1}(k_0v+\omega_c)\ , \tag{5}$$

where

$$\gamma=\left[1-\frac{v^2}{c^2}-\frac{v_\perp^2}{c^2}\right]^{-1/2}.$$

When no wiggler magnet is present ($k_0\rightarrow 0$), Eq. (5) reduces to the frequency characteristic of the Weibel instability which indicates the lowbitron output frequency will be much larger than that of the Weibel cyclotron instability. In the limit $\omega_c\rightarrow 0$ one has the same frequency as the FEL provided $v_\perp$ in Eq. (5) is now identified as the transverse velocity imparted by the FEL wiggler magnet. For the same values of $v_\perp$, Eq. (5) indicates that the lowbitron emission frequency will be greater than that of an FEL due to the presence of ω_c.

In Sec. II we derive the dispersion relation describing the emitted radiation in the high-gain regime, i.e., $\Gamma L>1$, where Γ is the amplitude growth rate and L the interaction distance. The dispersion relation is analyzed in Sec. III for a tenuous beam of electrons all having the same transverse momentum and a cold axial momentum distribution. Section IV gives estimates for the saturated efficiency and Sec. V summarizes the results and gives several numerical examples. For convenience, we shall henceforth take the speed of light in vacuum $c=1$.

II. DERIVATION OF THE DISPERSION RELATION

In this section we derive the dispersion relation for a right-hand circularly polarized electromagnetic wave propagating in the beam of tenuous, relativistic electrons which are traveling in the combined solenoidal and longitudinal wiggler magnetic fields. The dispersion relation is found by first determining the fluctuation in the electron distribution function induced by the propagating electromagnetic wave which in turn determines the transverse driving current. The transverse driving current is substituted in the transformed wave equation from which it is found that the amplitude of the kth mode is coupled to a sum over all other modes of the form $k-(l+s)k_0$, where l,s range from $-\infty$ to ∞ and k_0 is the wave number of the wiggler magnetic field. To uncouple the modes we treat the wiggler field as a small perturbation keeping terms to second order in it which limits l,s to the range $0,\pm 1,\pm 2$. It is furthermore assumed that terms with the resonant denominator $(k+k_0)v_3+\omega_c=\omega$, where $\omega_c=eB_0/E$ is the electron cyclotron frequency in the solenoidal field, E is the electron energy, and v_3 is the axial electron velocity, are the dominant terms. With the above assumptions the modes can be uncoupled yielding five equations with five unknown amplitudes which, upon taking their determinant, yields the desired dispersion relation.

The distribution function $f(\vec{p},z,t)$ of the tenuous beam of electrons from which the driving current is obtained satisfies the relativistic, collisionless Boltzmann equation

$$\frac{\partial f}{\partial t}+\vec{v}\cdot\vec{\nabla}f-e[\vec{v}\times\hat{z}(B_0+\delta B\sin k_0z)]\cdot\vec{\nabla}_pf=ee^{-i\omega t}\left[\vec{\mathscr{E}}(z)-\frac{i}{\omega}\vec{v}\times[\vec{\nabla}\times\vec{\mathscr{E}}(z)]\right]\cdot\vec{\nabla}_pf\ , \tag{6}$$

where B_0 and δB are the amplitudes of the solenoidal and wiggler fields, respectively. The propagating electromagnetic wave is taken to be traveling along the positive z axis, and to be right-hand circularly polarized, such that

$$\vec{\mathscr{E}}(z,t)=(\hat{x}+i\hat{y})\mathscr{E}(z)e^{-i\omega t} . \tag{7}$$

The wave is a small perturbation on the electrons' motion so that the distribution function can be expanded as the sum of a zero-order term plus a small perturbation

$$f(\vec{p},z,t)=f^{(0)}(p_3,p_\perp)+f^{(1)}(\vec{p},z,t) , \tag{8}$$

with the normalization $n=\int d^3p\, f^{(0)}$, where n is the electron number density and $p_\perp$ is the magnitude of the transverse electron momentum in cylindrical coordinates. Substituting Eq. (8) into Eq. (6) yields

$$\frac{df^{(0)}}{dt}=0 , \tag{9}$$

$$\frac{\partial f^{(1)}}{\partial t}+\vec{v}\cdot\vec{\nabla}f^{(1)}-e[\vec{v}\times\hat{z}(B_0+\delta B\sin k_0z)]\cdot\vec{\nabla}_p f^{(1)}=ee^{-i\omega t}\left[\vec{\mathscr{E}}(z)-\frac{i}{\omega}\vec{v}\times[\vec{\nabla}\times\vec{\mathscr{E}}(z)]\right]\cdot\vec{\nabla}_p f^{(0)} . \tag{10}$$

The left-hand side of Eq. (10) is the total time derivative of $f^{(1)}$. Under the assumption that the propagating electromagnetic wave is turned on adiabatically, the solution to Eq. (10) may then be expressed as a time integral over the unperturbed trajectory of the electrons

$$f^{(1)}=e\int_{-\infty}^{t}dt'e^{-i\omega t'}\left[(\hat{x}+i\hat{y})\left[\mathscr{E}'+\frac{iv_3'}{\omega}\frac{d\mathscr{E}'}{dz'}\right]-\frac{i(v_1'+iv_2')}{\omega}\frac{d\mathscr{E}'}{dz'}\hat{z}\right]\cdot\vec{\nabla}_{p'}f^{(0)} . \tag{11}$$

In Eq. (11), v_1', v_2' are the x,y components of the electron velocity, respectively, and the primed variables are the particular solutions to the unperturbed relativistic equations of motion which equal their unprimed counterparts when t', the independent variable, is equal to t.

The unperturbed equations of motion are given by

$$\frac{d\vec{p}'}{dt'}=-e\vec{v}'\times\hat{z}(B_0+\delta B\sin k_0z') , \tag{12}$$

$$\frac{dE'}{dt'}=0 , \tag{13}$$

where the electron energy $E'=E=m\gamma$ remains constant. Equation (12) is easily solved to give the components of momentum

$$p_3'=p_3 , \tag{14}$$

$$p_1'+ip_2'=(p_1+ip_2)\exp\left[i\left[\omega_c\tau-\frac{\Omega}{k_0v_3}\cos k_0(z+v_3\tau)+\frac{\Omega}{k_0v_3}\cos k_0z\right]\right] , \tag{15}$$

where $\tau=t'-t$, and $\Omega=e\delta B/E$ is the electron cyclotron frequency associated with the wiggler field. From Eqs. (14) and (15) it is obvious that the axial momentum of the electron is constant as well as the magnitude of the transverse momentum $|p_1'+ip_2'|=|p_1+ip_2|=p_\perp$. However, the x and y components of momentum separately are of course not constant and are obtained from Eq. (15) by taking the real and imaginary parts, respectively. The electron coordinates [obtained from Eqs. (14) and (15)] are

$$z'=z+v_3\tau , \tag{16}$$

$$r'-r=V\exp\left[i\frac{\Omega}{k_0v_3}\cos k_0z\right]\sum_{q=-\infty}^{\infty}(-i)^qJ_q\left[\frac{\Omega}{k_0v_3}\right]e^{iqk_0z}\frac{(e^{i(\omega_c+qk_0v_3)\tau}-1)}{i(\omega_c+qk_0v_3)} , \tag{17}$$

where $r=x+iy$, $V=v_1+iv_2$, and J_q is an ordinary Bessel function of order q. For $\Omega=0$ it is seen from Eq. (17) that the radius of the orbit remains constant corresponding to simple helical motion in the solenoidal field. In the absence of the solenoidal field, $\omega_c=0$, the $q=0$ term in Eq. (17) grows linearly in τ with the radius of the orbit becoming unbounded unless Ω/k_0v_3 is close to a zero of J_0 in which case the radius of the orbit remains bounded. Also, in the presence of both B_0 and δB the radius of the orbit grows linearly in τ for $\omega_c=-qk_0v_3$. For our purposes we will be interested in parameters such that typically $k_0v_3>\omega_c$ and $\Omega/k_0v_3<1$. In this range of parameters the radius of the orbit remains bounded. Since the argument of the Bessel functions appearing in Eq. (17) is small, the $q=0,\pm1$ terms are dominant with the Bessel functions being expanded in terms of their arguments. The radius of the orbit is then essentially a circle with "wiggles" on it.

We take the unperturbed distribution function $f^{(0)}$ to be a function only of the zeroth-order constants of the motion $p_\perp$ and p_3. $f^{(0)}$ then automatically satisfies Eq. (9) and using the facts that $p'_\perp=p_\perp, p'_3=p_3$ it is easy to show that

$$\vec{\nabla}_{p'}f^{(0)}=\left[\hat{x}\frac{p'_1}{p_\perp}+\hat{y}\frac{p'_2}{p_\perp}\right]\frac{\partial f^{(0)}}{\partial p_\perp}+\hat{z}\frac{\partial f^{(0)}}{\partial p_3}\ . \tag{18}$$

Substituting Eqs. (14) and (15) along with the Fourier transform of the field amplitude,

$$\mathscr{E}(z')=\frac{1}{\sqrt{2\pi}}\int dq\, e^{iqz'}\mathscr{E}(q)\ ,$$

into Eq. (11) and performing the time integration yields

$$f^{(1)}=ep_\perp e^{i\varphi-i\omega t}\sum_{l=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}(-1)^l J_l\left[\frac{\Omega}{k_0v_3}\right]J_s\left[\frac{\Omega}{k_0v_3}\right]\exp i(l+s)\left[\frac{\pi}{2}+k_0z\right]$$
$$\times\int\frac{dq}{\sqrt{2\pi}}\frac{e^{iqz}\mathscr{E}(q)G(q)}{i[-\omega+\omega_c+(lk_0+q)v_3]}\ , \tag{19}$$

where

$$G(q)\equiv\frac{1}{p_\perp}\left[1-\frac{qv_3}{\omega}\right]\frac{\partial f^{(0)}}{\partial p_\perp}+\frac{q}{\omega E}\frac{\partial f^{(0)}}{\partial p_3}$$

and the angle φ is given by $p_2=p_1\tan\varphi$. The induced transverse driving current is defined by

$$\vec{j}=-e\int d^3p\,\vec{V}f^{(1)}\ . \tag{20}$$

Substituting for $f^{(1)}$ and taking the spatial Fourier transform of the current,

$$\vec{j}(k)=\frac{1}{\sqrt{2\pi}}\int dz\,\vec{j}(z)e^{-ikz}\ ,$$

yields upon performing the φ integration

$$\vec{j}(k)=i\pi e^2e^{-i\omega t}(\hat{x}+i\hat{y})\int_0^\infty dp_\perp p_\perp^2\int_{-\infty}^{\infty}dp_3v_\perp\sum_{l=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}(-1)^lJ_l\left[\frac{\Omega}{k_0v_3}\right]J_s\left[\frac{\Omega}{k_0v_3}\right]$$
$$\times\frac{e^{i(l+s)\pi/2}\mathscr{E}(k-(l+s)k_0)G(k-(l+s)k_0)}{-\omega+\omega_c+(k-sk_0)v_3}\ , \tag{21}$$

where the magnitude of the transverse velocity is given by $v_\perp E=p_\perp$. Combining Maxwell's equations and spatially Fourier transforming the resulting wave equation gives, upon using Eq. (21) for the current,

$$(\omega^2-k^2)\mathscr{E}(k)=4\pi^2F\omega e^2\int_0^\infty dp_\perp p_\perp^2\int_{-\infty}^{\infty}dp_3v_\perp\sum_{l=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}(-1)^lJ_l\left[\frac{\Omega}{k_0v_3}\right]J_s\left[\frac{\Omega}{k_0v_3}\right]$$
$$\times\frac{e^{i(l+s)\pi/2}\mathscr{E}(k-(l+s)k_0)G(k-(l+s)k_0)}{-\omega+\omega_c+(k-sk_0)v_3}\ . \tag{22}$$

In Eq. (22) we have appended a phenomenological filling factor F which describes the coupling of the electron beam to the electromagnetic wave. For a uniform plane wave and infinitely wide electron beam F is unity, and for a finite beam cross section F is close to unity when the electron beam radius exceeds that of the electromagnetic beam. For the case when the radius of the electron beam is less than that of the electromagnetic beam F is approximately given by the ratio of the electron beam area to the electromagnetic beam area.

From Eq. (22) we see that the amplitude $\mathscr{E}(k)$ on the left-hand side is coupled to an infinite sum of amplitudes involving the harmonics $\mathscr{E}(k-(l+s)k_0)$. To uncouple the amplitudes, in order to find the dispersion relation, we note that for experimentally accessible parameters the argument of the Bessel functions is typically small, $\Omega/k_0v_3<1$. The Bessel functions can then be expanded in terms of their arguments with the dominant terms in the summations occurring for $l,s=0,\pm1,\pm2$. Keeping only the terms up to second order in the small parameter Ω/k_0v_3 we find that the approximate expression for Eq. (22) is given by

$$\begin{aligned}(\omega^2-k^2)\mathscr{E}(k)=4\pi^2e^2\omega F\int_0^\infty dp_\perp p_\perp^2\int_{-\infty}^{\infty}dp_3v_\perp\\ \times\Bigg[\mathscr{E}(k)G(k)\left[\frac{Q^2}{4}H_1+\frac{Q^2}{4}H_{-1}+H_0\right]+\frac{iQ}{2}\mathscr{E}(k+k_0)G(k+k_0)(H_1-H_0)\\ +\frac{iQ}{2}\mathscr{E}(k-k_0)G(k-k_0)(H_{-1}-H_0)\\ +\frac{Q^2}{4}\mathscr{E}(k+2k_0)G(k+2k_0)(H_1-\tfrac{1}{2}H_0-\tfrac{1}{2}H_2)\\ +\frac{Q^2}{4}\mathscr{E}(k-2k_0)G(k-2k_0)(H_{-1}-\tfrac{1}{2}H_{-2}-\tfrac{1}{2}H_0)\Bigg]\ ,\end{aligned}\tag{23}$$

where the small parameter Q is defined as $Q\equiv\Omega/k_0v_3$, and $H_j=[-\omega+\omega_c+(k+jk_0)v_3]^{-1}$, $j=0,\pm1,\pm2.\ldots$ For $Q=0$ the above result reduces to the dispersion relation for the electron cyclotron instability. Since we are interested in the resonance H_1, we take terms proportional to H_1 in Eq. (23) to be dominant. Note that the coefficients of H_1 are proportional to Q (or Q^2) and thus there is a lower bound on Q in order that the terms with resonances H_1 be largest.

To proceed with the uncoupling of the amplitudes, we keep only the terms in Eq. (23) proportional to H_1 which yields an equation involving $\mathscr{E}(k)$, $\mathscr{E}(k+k_0)$, and $\mathscr{E}(k+2k_0)$. Next, we replace k appearing in Eq. (23) by $k+k_0$ and keep only the terms proportional to the resonance $[-\omega+\omega_c+(k+k_0)v_3]^{-1}$ yielding an equation involving $\mathscr{E}(k)$, $\mathscr{E}(k+k_0)$, $\mathscr{E}(k+2k_0)$, $\mathscr{E}(k-k_0)$, and $\mathscr{E}(k+3k_0)$. Successively replacing k by $k-k_0$, $k+2k_0$, and $k+3k_0$ in Eq. (23), and keeping only the terms with the resonance $[-\omega+\omega_c+(k+k_0)v_3]^{-1}$ then yields five equations involving the amplitudes $\mathscr{E}(k)$, $\mathscr{E}(k+k_0)$, $\mathscr{E}(k+2k_0)$, $\mathscr{E}(k-k_0)$, and $\mathscr{E}(k+3k_0)$. They are

$$(k^2-\omega^2)\mathscr{E}(k)=\chi_0\mathscr{E}(k)-i\chi_1\mathscr{E}(k+k_0)+\chi_2\mathscr{E}(k+2k_0)\ ,\tag{24a}$$

$$[(k+k_0)^2-\omega^2]\mathscr{E}(k+k_0)=-\chi_4\mathscr{E}(k+k_0)-i\chi_5\mathscr{E}(k)-i\chi_6\mathscr{E}(k+2k_0)+\chi_7\mathscr{E}(k-k_0)+\chi_8\mathscr{E}(k+3k_0)\ ,\tag{24b}$$

$$[(k-k_0)^2-\omega^2]\mathscr{E}(k-k_0)=\chi_3\mathscr{E}(k+k_0)\ ,\tag{24c}$$

$$[(k+2k_0)^2-\omega^2]\mathscr{E}(k+2k_0)=-i\chi_1\mathscr{E}(k+k_0)+\chi_0\mathscr{E}(k)+\chi_2\mathscr{E}(k+2k_0)\ ,\tag{24d}$$

$$[(k+3k_0)^2-\omega^2]\mathscr{E}(k+3k_0)=\chi_3\mathscr{E}(k+k_0)\ ,\tag{24e}$$

where the effective susceptibilities χ are

$$\chi_0=-\tfrac{1}{4}I[Q^2G(k)]\ ,\tag{25a}$$

$$\chi_1=\tfrac{1}{2}I[QG(k+k_0)]\ ,\tag{25b}$$

$$\chi_2=-\tfrac{1}{4}I[Q^2G(k+2k_0)]\ ,\tag{25c}$$

$$\chi_3=\tfrac{1}{8}I[Q^2G(k+k_0)]\ ,\tag{25d}$$

$$\chi_4=I[G(k)]\ ,\tag{25e}$$

$$\chi_5 = -\tfrac{1}{2} I[QG(k)] , \tag{25f}$$

$$\chi_6 = -\tfrac{1}{2} I[QG(k+2k_0)] , \tag{25g}$$

$$\chi_7 = \tfrac{1}{8} I[Q^2 G(k-k_0)] , \tag{25h}$$

$$\chi_8 = \tfrac{1}{8} I[Q^2 G(k+3k_0)] . \tag{25i}$$

Here

$$Q = \frac{\Omega}{k_0 v_3} ,$$

$$G(k) = \frac{1}{p_\perp} \left[1 - \frac{k v_3}{\omega} \right] \frac{\partial f^{(0)}}{\partial p_\perp} + \frac{k}{\omega E} \frac{\partial f^{(0)}}{\partial p_3} ,$$

and the integral operator I is given by

$$I = 4\pi^2 e^2 \omega F \times \int_0^\infty dp_\perp p_\perp^2 \int_{-\infty}^\infty dp_3 \frac{v_\perp}{-\omega + \omega_c + (k+k_0) v_3} .$$

Taking the determinant of Eqs. (24a)–(24e) yields the dispersion relation

$$1 + \frac{\chi_4}{D_1} - \frac{\chi_0}{D_0} - \frac{\chi_2}{D_2} + \frac{\chi_1\chi_6 - \chi_2\chi_4}{D_1 D_2} + \frac{\chi_1\chi_5 - \chi_0\chi_4}{D_0 D_1} = 0 , \tag{26}$$

where $D_n = (k + nk_0)^2 - \omega^2$ and where we have only kept terms up to second order in the small parameter Q. The above dispersion relation shows that the three transversely polarized modes of amplitude $\mathscr{E}(k)$, $\mathscr{E}(k+k_0)$, $\mathscr{E}(k+2k_0)$ are coupled together by the wiggler magnetic field. In the region where $D_1 \approx 0$ the first two terms in Eq. (26) are dominant and describe the electron cyclotron instability at the effective wave number $k + k_0$. For our purposes we will be interested in the region where $D_0 \approx 0$ and $(k+k_0)v_3 + \omega_c \approx \omega$ simultaneously. In this region the dominant terms in Eq. (26) give

$$D_0 \left[1 + \frac{\chi_4}{D_1} \right] = \chi_0 + \frac{\chi_0\chi_4 - \chi_1\chi_5}{D_1} . \tag{27}$$

Since χ_0, $\chi_0\chi_4$, $\chi_1\chi_5$ are proportional to Q^2, Eq. (27) indicates that a freely propagating electromagnetic mode D_0 is coupled to the beam cyclotron mode $1 + (\chi_4/D_1)$, by the wiggler magnetic field. For a sufficiently low density of electrons the terms proportional to $\chi_0\chi_4$, $\chi_1\chi_5$ are negligible in Eq. (27) compared to χ_0 since they are of order ω_p^4 ($\omega_p^2 = 4\pi n e^2/m$ the nonrelativistic plasma frequency squared and m is the rest mass) and χ_0 is of order ω_p^2.

Equation (27) determines the relation between the electromagnetic wave frequency ω and wave number k. Since we are interested in an amplifier, ω is taken to be a real specifiable parameter and we look for situations where k has a negative imaginary part. The analysis of the dispersion relation is carried out in the next section.

III. ANALYSIS OF THE DISPERSION RELATION

In this section we analyze the dispersion relation given by Eq. (27) for various limiting cases. First, we determine the relevant effective susceptibilities for a beam of electrons having all the same transverse momentum and a "cold" distribution in axial momentum. With the resulting expressions the dispersion relation is examined first for a low beam density and large wiggler amplitude, then for a small wiggler amplitude and large beam density, and finally for a large wiggler amplitude and large beam density.

We take all the electrons to have the same transverse momentum with the distribution in axial momentum being cold so that the unperturbed distribution $f^{(0)}$ is given by

$$f^{(0)} = \frac{\delta(p_\perp - p_{\perp 0})\delta(p_3 - p)}{2\pi p_\perp} , \tag{28}$$

where $p = mv\gamma$ is now the axial electron momentum and v the corresponding axial electron velocity. Using the above expression for $f^{(0)}$ in χ_0, χ_1, χ_4, χ_5 and integrating by parts once over momentum yields

$$\chi_0 = \frac{Q^2 w_p^2 F}{8\gamma} \left[\frac{(w^2 - k^2 - kk_0) v_{10}^2}{[(k+k_0)v + w_c - w]^2} + \frac{2(w - kv - kv_{10}^2/v)}{(k+k_0)v + w_c - w} \right] , \tag{29}$$

$$\chi_4 = \frac{-w_p^2 F}{2\gamma} \left[\frac{2[w - (k+k_0)v]}{(k+k_0)v + w_c - w} - \frac{D_1 v_{10}^2}{[(k+k_0)v + w_c - w]^2} \right] , \tag{30}$$

$$\chi_1\chi_5=\frac{-Q^2w_p^4F}{16\gamma^2}\left[\frac{2[w-(k+k_0)v]-(k+k_0)p_{10}^2/pE}{(k+k_0)v+w_c-w}-\frac{D_1v_{10}^2}{[(k+k_0)v+w_c-w]^2}\right]$$

$$\times\left[\frac{2[w-(k+k_0)v]-kp_{10}^2/pE}{(k+k_0)v+w_c-w}+\frac{w^2-k^2-kk_0}{[(k+k_0)v+w_c-w]^2}\right]. \tag{31}$$

In evaluating the momentum integrals we have assumed that the distribution function $f^{(0)}$ is more rapidly varying than the resonant denominator $[(k+k_0)v_3+w_c-w]^{-1}$, i.e., the width of the distribution is less than the width of the resonant denominator which requires that the following inequalities be satisfied:

$$\frac{\Gamma}{w}>\frac{(1+v)k_0v+w_c}{wv}\frac{\Delta p}{p} \quad\text{and}\quad \frac{\Gamma}{w}>\left[\frac{v_{10}}{v}\right]^2\frac{\Delta p_\perp}{p_{10}}, \tag{32}$$

where $\Gamma=-\mathrm{Im}k=$ amplitude gain per unit length, and Δp, $\Delta p_\perp$ are the small axial and transverse spreads in electron momenta, respectively.

Maximum gain is obtained when the velocity matching wave number $k_m\equiv(w-w_c-k_0v)/v$ is precisely equal to w which occurs when $w=w_m\equiv(w_c+k_0v)/(1-v)$. With this assumption in Eqs. (29)–(31) the dispersion relation given by Eq. (27) becomes, after some tedious algebra,

$$x^2\{[x-a/4+(a^2/16-b)^{1/2}][x-a/4-(a^2/16-b)^{1/2}]\}=-\frac{k_0bQ^2}{8w_pF^{1/2}}x-\frac{Q^2b^2}{16}, \tag{33}$$

where $x\equiv(k-w)/w_pF^{1/2}$, $a=(F^{1/2}w_p/w)\times(w_c/vk_0)$, $b=v_{10}^2/2\gamma v^2$, and $Q=e\delta B/k_0p<1$. In Eq. (33) we have retained only the largest coefficients of each power of x keeping in mind the assumptions

$$k\approx w>>k_0,w_p,w_c;k_0>>|k-w|;vk_0\gtrsim w_c;$$
$$w_c,k_0>w_p.$$

The first term on the left-hand side of Eq. (33) is due to the freely propagating electromagnetic mode, while the curly bracketed term is due to the beam cyclotron mode with the coupling of these modes given by the terms on the right-hand side of the equation. From the above inequalities we see that the quantity a is very small and in what follows we will be assuming that $b>>a^2/16$. In order to evaluate Eq. (33) we will look at various limiting cases and determine the real and imaginary parts of the mismatch parameter x.

Case 1. In this limit we take $|x|>>b^{1/2}>>a/4$ so that the beam cyclotron mode is but weakly excited. Furthermore, we will assume that the beam density is sufficiently low so that $|x|^4>>Q^2b^2/16$. The dispersion relation then becomes a simple cubic equation given by

$$x^3+\frac{k_0bQ^2}{8w_pF^{1/2}}=0. \tag{34}$$

The solution to the above equation giving amplification is

$$k-w=\tfrac{1}{4}(Fw_p^2k_0bQ^2)^{1/3}-i\frac{\sqrt{3}}{4}(Fw_p^2k_0bQ^2)^{1/3}, \tag{35}$$

which gives for the growth rate

$$\Gamma_1=\frac{\sqrt{3}}{4}(Fw_p^2k_0bQ^2)^{1/3}. \tag{36}$$

The above result has the same functional dependence upon Q, w_p, k_0, and γ as does the cold beam, Compton effect, free electron laser growth rate.[14] Because of the dependence of Γ_1 on $(v_{10}/v)^{2/3}$ which is typically less than one, the growth rate given by Eq. (36) will be somewhat less than that for the free electron laser with the same values of Q, w_p, k_0, γ, unless $v_{10}/v\sim1$. The expression given by Eq. (36) is valid only when Eq. (32) is satisfied along with

$$\frac{\sqrt{243}}{64}\left[\frac{k_0}{F^{1/2}w_p}\right]^{5/3}\frac{Q^{4/3}}{b^{1/3}}>>\frac{\sqrt{3}}{4}\left[\frac{k_0}{F^{1/2}w_p}\right]^{1/3}b^{1/3}Q^{2/3}>>b^{1/2}>>a/4, \tag{37}$$

which is obeyed for sufficiently large wiggler magnetic field amplitudes and low beam densities. The amplification in this parameter range is due to axial electron bunching.

Case 2. We take the opposite limit of Case 1, $b^{1/2} >> |x| >> a/4$, with the beam cyclotron mode again weakly excited. In this limit the dispersion relation becomes a simple quadratic given by

$$x^2 + \frac{k_0 Q^2}{8F^{1/2} w_p} x + \frac{Q^2 b}{16} = 0 \; . \quad (38)$$

The solution to the above equation giving amplification is

$$k - w = -\frac{k_0 Q^2}{16} - i\frac{F^{1/2} w_p}{2} \left[\frac{Q^2 b}{4} - \left[\frac{k_0 Q^2}{8F^{1/2} w_p} \right]^2 \right]^{1/2} . \quad (39)$$

Amplification occurs when the first term under the square root is dominant, which occurs for sufficiently large beam densities and incident transverse beam velocity $v_{\perp 0}$. In the limit where the first term under the square root is much larger than the other term the growth rate is given by

$$\Gamma_2 = \frac{F^{1/2} w_p Q b^{1/2}}{4} \; . \quad (40)$$

The expression for Γ_2 is valid when the conditions on allowable momentum spread given by Eq. (32) are satisfied as well as the inequalities

$$b^{1/2} >> \frac{Q b^{1/2}}{4} >> \frac{k_0}{F^{1/2} w_p} \frac{Q^2}{8} >> \frac{a}{4} \; . \quad (41)$$

In the present case, the growth rate Γ_2 depends linearly upon the quantity $w_p Q$ whereas Γ_1 depended upon $(w_p Q)^{1/3}$. When conditions of Case 2 apply the amplification Γ_2 is due to transverse bunching of the electrons whereas Γ_1 was due to axial bunching. Note from Eq. (35) that the wave phase velocity when axial bunching occurs is less than the speed of light in vacuum, whereas Eq. (39) shows that the wave phase velocity is greater than the speed of light in vacuum when transverse bunching occurs.

Case 3. For this case we take $b^{1/2} >> a/4$ and assume that the beam cyclotron mode is strongly excited with $x \approx -ib^{1/2}$. With these assumptions Eq. (33) becomes

$$x^2(x + ib^{1/2})(-i2b^{1/2}) = -\frac{k_0 b Q^2}{8F^{1/2} w_p} x - \frac{Q^2 b^2}{16} \; . \quad (42)$$

Substituting $x \approx -ib^{1/2}$ for x^2 appearing on the left-hand side and x on the right-hand side of Eq. (42) and keeping the dominant terms results in

$$k - w = \frac{k_0 Q^2}{16} - iF^{1/2} w_p b^{1/2} \; . \quad (43)$$

The growth rate is then given by

$$\Gamma_3 = F^{1/2} w_p b^{1/2} \; . \quad (44)$$

This expression for the growth rate is similar to that of the Weibel cyclotron instability,[9] and is independent of the wiggler amplitude δB. Γ_3 is valid as long as Eq. (32) is satisfied and

$$\left[\frac{b}{3} \right]^{3/2} >> \frac{k_0}{F^{1/2} w_p} \frac{Q^2 b}{16} >> \frac{b^{5/2}}{2} \left[\frac{w_p F^{1/2}}{k_0} \right]^2 , \quad (45)$$

which requires a sufficiently large incident transverse velocity $v_{\perp 0}$ and low beam densities. The lower bound on Q appearing in Eq. (45) has been estimated from the requirement $|\chi_0(k+k_0)| >> |\chi_4(k-k_0)|$ namely that the terms in Eq. (23) proportional to the resonant denominator $[(k+k_0)v_3 + w_c - w]^{-1}$ be dominant. For the present case the amplification is due to axial electron bunching. Also, in this case the wave phase velocity is seen to be less than the speed of light in vacuum as it was for axial bunching Case 1. Next, we will obtain estimates for the efficiency of radiative energy extraction at saturation.

IV. EFFICIENCY ESTIMATES

In this section we derive estimates for the efficiency of radiative energy extraction at saturation. It is assumed that the saturation mechansim is trapping of the electrons in the periodic potential wells of the bunching wave. The trapped electrons will all be moving at the phase velocity of the bunching force $(w - w_c)/(k + k_0) = v$. The difference in energy between the initial electron energy and trapped electron energy is then determined by the above expression to be

$$|\Delta E| = \frac{pv\,|\Delta k|}{(1+v)k_0 + w_c} \; , \quad (46)$$

where $\Delta k = \mathrm{Re}(k - w)$. Assuming that all of the energy loss from the electrons is converted to radiation, the efficiency is then the ratio of ΔE to initial electron kinetic energy

TABLE I. Summary of operating regimes.

Parameter ordering	Peak amplitude gain	Wave phase velocity and bunching mechanism	Saturated efficiency
$\frac{\Gamma_1}{w} > \frac{(1+v)vk_0+w_c}{wv}\frac{\Delta p}{p};\left[\frac{v_{\perp 0}}{v}\right]^2\frac{\Delta p_\perp}{p_{\perp 0}}$		$\frac{w}{k} \approx 1-\frac{(Fw_p^2k_0bQ^2)^{1/3}}{4w} < 1$	
$0.243\left[\frac{k_0}{F^{1/2}w_p}\right]^{5/3}\frac{Q^{4/3}}{b^{1/3}} >> 0.433\left[\frac{k_0}{F^{1/2}w_p}\right]^{1/3} b^{1/3}Q^{2/3} >> b^{1/2} >> \frac{a}{4}$	$\Gamma_1=\frac{\sqrt{3}}{4}(Fw_p^2k_0bQ^2)^{1/3}$	axial bunching	$\eta=\frac{v^2\gamma(Fw_p^2k_0bQ^2)^{1/3}}{4(\gamma-1)[(1+v)k_0+w_c]}$
$\frac{\Gamma_2}{w} > \frac{(1+v)vk_0+w_c}{wv}\frac{\Delta p}{p};\left[\frac{v_{\perp 0}}{v}\right]^2\frac{\Delta p_\perp}{p_{\perp 0}}$		$\frac{w}{k} \approx 1+\frac{k_0Q^2}{16w} > 1$	$\eta=\frac{v^2\gamma k_0Q^2}{16(\gamma-1)[(1+v)k_0+w_c]}$
$b^{1/2} >> \frac{Qb^{1/2}}{4} >> \frac{k_0}{F^{1/2}w_p}\frac{Q^2}{8} >> \frac{a}{4}$	$\Gamma_2=\frac{F^{1/2}w_pQb^{1/2}}{4}$	transverse bunching	
$\frac{\Gamma_3}{w} > \frac{(1+v)vk_0+w_c}{wv}\frac{\Delta p}{p};\left[\frac{v_{\perp 0}}{v}\right]^2\frac{\Delta p_\perp}{p_{\perp 0}}$		$\frac{w}{k} \approx 1-\frac{k_0Q^2}{16w} < 1$	
$\left[\frac{b}{3}\right]^{3/2} >> \frac{k_0}{F^{1/2}w_p}\frac{Q^2b}{16} >> \frac{b^{5/2}}{2}\left[\frac{w_pF^{1/2}}{k_0}\right]^2$ $b^{1/2} >> \frac{a}{4}$	$\Gamma_3=F^{1/2}w_pb^{1/2}$	axial bunching	$\eta=\frac{v^2\gamma k_0Q^2}{16(\gamma-1)[(1+v)k_0+w_c]}$

$$\eta = \left| \frac{\Delta E}{m(\gamma-1)} \right| = \left| \frac{v^2 \gamma \Delta k}{(\gamma-1)[(1+v)k_0+w_c]} \right| . \quad (47)$$

From Eq. (35) we see that the efficiency when Case 1 conditions apply is

$$\eta_1 = \frac{v^2\gamma(F w_p^2 k_0 b Q^2)^{1/3}}{4(\gamma-1)[(1+v)k_0+w_c]} , \quad (48)$$

while from Eqs. (39) and (43) when Case 2 and Case 3 conditions apply the efficiency is given by

$$\eta_{2,3} = \frac{v^2 \gamma k_0 Q^2}{16(\gamma-1)[(1+v)k_0+w_c]} . \quad (49)$$

It is noteworthy that although in Case 3 the gain Γ_3 is independent of Q (and is therefore independent of the wiggler amplitude), the efficiency η_3 is strongly dependent on Q.

When $\gamma >> 1$, $k_0 >> w_c$, $v \sim 1$ it is seen from Eq. (49) that the efficiency for Case 2 and Case 3 is $\eta_{2,3} = Q^2/32$. Since $Q < 1$, $\eta_{2,3}$ are limited to values less than 3.1%. For Case 1 the efficiency is less than or equal to 0.29 Γ_1/k_0 and since $\Gamma_1/k_0 < 1$ the efficiency for this case will be limited to values less than 29%. Typically for experimentally accessible parameters, the efficiency is expected to be less than a few percent. To increase the efficiency it may be possible to use the various efficiency enhancement techniques suggested to improve the efficiency of the free-electron laser such as tapering the wiggler magnet amplitude and period[23] or use a depressed collector[24] as in conventional traveling wave tubes.

V. SUMMARY AND NUMERICAL EXAMPLES

We have given a relativistic, classical derivation of the gain coefficients in the high gain regime ($\Gamma L > 1$ where L is the interaction distance) for stimulated emission from a cold, pencil beam of electrons traveling in a combined solenoidal and longitudinally polarized wiggler magnetic field. The amplification is due to the Lorentz $\vec{v} \times \vec{B}$ or ponderomotive force inducing transverse and axial bunching of the electrons with the wiggler magnet coupling a freely propagating electromagnetic mode to the beam cyclotron mode.

In Table I we summarize the analytic expressions for the peak amplitude gain with conditions on the various parameters for the applicability of the gain expressions and the saturated efficiency estimates. In Table I, $b = (v_{\perp 0}/v)^2/2\gamma$, $Q = e\delta B/k_0 p$, $a = (w_p F^{1/2}/w)(w_c/k_0 v)$, $w_c = eB_0/\gamma m$, c = speed of light = 1, F = filling factor, Δp, $\Delta p_\perp$ are the small axial and transverse spreads in momentum, and the emission frequency is at the fundamental harmonic k_0, i.e., $w = (1+v)\gamma^2[vk_0+w_c]/(1+\gamma^2 v_{\perp 0}^2)$. From Table I it is seen that for a given k_0 and b, the peak amplitude gain Γ_1 occurs for a large wiggler amplitude and low beam density, Γ_2 occurs for small wiggler amplitude and large beam density, and Γ_3 occurs for large wiggler amplitude and large beam density. Note, however, that the beam density must remain small enough to allow us to neglect self-electrostatic fields. Also, from Table I for the case when Γ_1 applies the saturated efficiency in the extreme relativistic limit is limited to values much less than 29% and for the other two cases it is limited to values less than 3%.

In Table II, we give numerical examples for the

TABLE II. Radiation characteristics of the lowbitron, FEL, and Weibel instabilties (for beam parameters in Sec. V).

		Radiation frequency (GHz)	Peak power gain 2Γ (cm^{-1})	Maximum momentum spread $\Delta p/p$	$\Delta p_\perp/p_{\perp 0}$	Saturated efficiency (%)	Saturated power (kW)
lowbitron	(1)	260	0.031	10^{-3}	10^{-3}	0.1	1.0
	(2)	260	0.026	10^{-3}	10^{-3}	0.03	428
	(3)	260	0.7	10^{-2}	10^{-2}	0.3	4283
FEL	(1)	296	0.1	10^{-4}		2	20
	(2)	423	0.39	10^{-4}		3	42 830
	(3)	296	0.95	10^{-4}		3	42 830
Weibel	(1)	72	0.018	10^{-3}	10^{-3}	0.001	0.01
	(2)	72	0.7	10^{-2}	10^{-2}	1.2	17 132
	(3)	72	0.7	10^{-2}	10^{-2}	1.2	17 132

three cases considered in Table I and compare with the FEL and Weibel instabilities. In making the comparisons, we calculated the growth rates for the FEL instability from formulas given in Ref. 14 and 15 and for the Weibel instability from formulas given in Ref. 9. We also note that whereas the FEL instabilities have been observed experimentally, there are as yet no convincing experiments that demonstrate radiation from the Weibel instability. The allowable momentum spreads for the lowbitron and Weibel instabilities were estimated from Eq. (32); the momentum spread for the FEL instability was obtained from Refs. 14 and 15. In all three cases we take $F=1$, $\gamma=3$, $v_{\perp 0}=0.37$ (c=speed of light $=1$), $k_0=6\ \mathrm{cm}^{-1}$, $B_0=10$ kG. We also assume a beam radius of 1.89 mm which equals one Larmor radius. For the FEL the initial transverse velocity $v_{\perp 0}=0$ and the transverse velocity is imparted by the transverse wiggler magnet. In Case 1 the beam current is $I=0.38$ A with $\delta B=8.33$ kG, case 2 has a beam current $I=567$ A with $\delta B=2.78$ kG, and Case 3 has a beam current $I=567$ A with $\delta B=8.33$ kG.

Table II shows that for the given parameters the lowbitron and FEL operate at a much higher emission frequency than the Weibel instability. In all three cases, the lowbitron and FEL have comparable peak output power growth rates. In all three cases the momentum spread requirements for the lowbitron and Weibel instabilities are somewhat less stringent than for the FEL. For the low density Case 1 the lowbitron and FEL have a much better efficiency than the Weibel instability, while for the high density Cases 2 and 3 the FEL and Weibel instabilities have better efficiency than the lowbitron.

In conclusion, we have described the basic properties of a novel source of coherent radiation capable of generating or amplifying electromagnetic radiation in the submillimeter wavelength range. It uses a longitudinal, periodic wiggler magnetic field which interacts with an electron beam having initial transverse energy. This results in a frequency upshift given by Eq. (5) of the right circularly polarized wave propagating along the guiding magnetic field. The process can be viewed as a three-wave parametric coupling between a freely propagating electromagnetic wave, a beam cyclotron mode supported by the gyrating electrons, and the periodic wiggler fields. It is noteworthy that this resonant coupling requires that the wiggler magnetic field be longitudinal. In the case of a purely transverse periodic magnetic field and a uniform longitudinal guide field, there are no resonances whose frequency is given by Eq. (5). In this latter situation one finds[25] solutions corresponding to gyrotron and Weibel modes on the one hand, and FEL modes on the other hand, and they are uncoupled except when $k_0 v=\omega_c$.

Our calculations are performed for thin, solid paraxial electron beams whose transverse dimensions satisfy the inequality $k_0 r<1$. This gives assurance that the periodic magnetic field be primarily longitudinal and of the form given by Eq. (1). When $k_0 r>1$, the longitudinal and transverse periodic perturbations become comparable in magnitude, and a solution to this problem becomes intractable. Of course, in experiments that use thick beams or annular beams, lowbitron-type modes satisfying Eq. (5) may well exist due to the presence of the longitudinal wiggler field component.

The periodic, multiple-mirror wiggler which generates the longitudinal field modulation has several advantages over the circularly polarized bifilar wigglers used in conventional FEL's. It is easier to construct, it gives larger amplitudes, its periodicity is easily changed, and an input adiabatic field shaper is readily incorporated.[17,22] It has also considerable advantage over transverse, linearly polarized wigglers. It is known that in traversing a linearly polarized wiggler, longitudinal oscillations are induced in the electron's motion which can cause particle untrapping[26] from the potential "buckets" when the excursions become comparable with the radiation wavelength. In the lowbitron configuration, the axial electron momentum is constant and this difficulty does not arise.

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[1]R. Q. Twiss, Aust. J. Phys. 11, 564 (1958).
[2]J. Schneider, Phys. Rev. Lett. 2, 504 (1959).
[3]A. V. Gaponov, Izv. Vyssh. Uchebn. Zaved. Radiofiz. 2, 837 (1959).
[4]J. L. Hirshfield and J. M. Wachtel, Phys. Rev. Lett. 12, 533 (1964).
[5]J. L. Hirshfield and V. L. Granatstein, IEEE Trans. Microwave Theory Tech. MTT-25, 522 (1977).
[6]P. Sprangle and A. T. Drobot, IEEE Trans. Microwave Theory Tech. MTT-25, 528 (1977).
[7]H. Uhm, R. C. Davidson, and K. R. Chu, Phys. Fluids 21, 1877 (1978).
[8]K. R. Chu and J. L. Hirshfield, Phys. Fluids 21, 461 (1978).
[9]J. L. Hirshfield, K. R. Chu, and S. Kainer, Appl. Phys. Lett. 33, 847 (1978).
[10]S. Ahn, V. Granatstein, and J. L. Hirshfield, editors, Collection of Papers on the Electron Cyclotron Maser (Gyrotron), Naval Research Laboratory Report, Washington D. C., NRL Memorandum No. 3937 (unpublished).
[11]V. P. Sukhatme and P. A. Wolff, J. Appl. Phys. 44, 2331 (1973).
[12]W. B. Colson, Phys. Lett. 59A, 187 (1976).
[13]T. Kwan, J. M. Dawson and A. T. Lin, Phys. Fluids, 20, 581 (1977).
[14]N. M. Kroll and W. A. McMullin, Phys. Rev. A 17, 300 (1978).
[15]P. Sprangle and R. A. Smith, Phys. Rev. A 21, 293 (1980).
[16]R. C. Davidson and H. Uhm, Phys. Fluids 23, 2076 (1980).
[17]W. A. McMullin and G. Bekefi, Appl. Phys. Lett. 39, 845 (1981).
[18]D. B. McDermott, T. C. Marshall, S. P. Schlesinger, R. K. Parker, and V. L. Granststein, Phys. Rev. Lett. 41, 1368 (1978).
[19]K. Halbach, LBL Univ. of Calif., Accel. and Fusion Res. Div. Report No. LBL 11393 (unpublished).
[20]G. R. Brewer, in *Focusing of Charged Particles,* edited by A. Septier (Academic, New York, 1967), p. 73.
[21]K. D. Jacobs, R. E. Shefer, and G. Bekefi, Appl. Phys. Lett. 37, 583 (1980).
[22]K. D. Jacobs, G. Bekefi, and J. R. Freeman, J. Appl. Phys. 52, 4977 (1981).
[23]N. M. Kroll, P. L. Morton, and M. N. Rosenbluth, IEEE J. Quant. Electron. QE-17, 1436 (1981).
[24]L. R. Elias, Phys. Rev. Lett. 42, 977 (1979).
[25]W. A. McMullin and R. C. Davidson (unpublished).
[26]H. Boehmer, M. Z. Caponi, J. Munch, G. Neil, and N. Schoen, Proceedings Erice, Sicily Workshop, August 1980 (in press).