Cavity Q for ergodic eigenmodes

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The Q of an overmoded (wavelength much less than cavity size) irregular cavity is calculated by making use of the ergodic theorem.

We consider here the problem of calculating the Q of a very overmoded (wavelength much less than the cavity size) irregularly shaped resonant cavity due to absorption of the electromagnetic radiation at the walls. We assume further that the volume inside the cavity can be either a vacuum, or else partially filled with anisotropic inhomogeneous dielectric or plasma. Both the scale length of the dielectric and the radius of curvature of the walls are assumed to be much larger than the radiation wavelength. However, near the cavity walls, a vacuum is assumed. Our main interest here is in applications concerning magnetically confined plasmas. For instance, consider a tokamak containing a hot plasma which radiates at the cyclotron frequency and its harmonics. An important issue is how much of this cyclotron radiation is absorbed by the walls and how much is reabsorbed by the plasma. Generally this can be calculated with a ray-tracing code where many rays are followed and the absorbtion and emission are calculated along each ray path. However, if the plasma is optically thin, as it would be at the higher harmonics, and if the reflection coefficient at the wall is near unity the rays would have to be followed for long distances before one could see how much energy is deposited in the wall.

In this paper we utilize the ergodic theorem to calculate the wall absorption for the case where the waves makes many bounces and before it is absorbed. If the local dispersion relation is

$$\omega = F(\vec{k}, \vec{r}) \ , \tag{1}$$

where the F is real, the ray equations

$$\frac{d\vec{r}}{dt} = \frac{\partial F}{\partial \vec{k}}, \quad \frac{d\vec{k}}{dt} = -\frac{\partial F}{\partial \vec{r}} \tag{2}$$

are relevant. It is evident from Eqs. (2) that the ray equations are a Hamiltonian system for which $F(\vec{k}, \vec{r})$ plays the role of the Hamiltonian, and $(\vec{k}, \vec{r})$ are momenta and coordinate variables. Thus Eqs. (2) are subject to the same phenomena of ergodicity onset and ergodic motions of other Hamiltonian systems. Here we consider the case where the solution to Eqs. (2) is ergodic[1,2] with F the only constant of the motions of the ray equations. (We note that even in plasma devices, such as tokamaks, where the plasma has a high degree of toroidal symmetry, the conducting walls surrounding the plasma commonly have corrugations and are partly composed of baffels and limiters. Thus this ergodicity assumption is probably well justified for waves which experience reflections from the walls.) In the case of ergodic ray motions it is expected that the time average over the motion of a wave packet can be obtained by a phase space average

$$\lim_{\tau \to \infty} \tau^{-1} \int_0^\tau g(\vec{k}(t), \vec{r}(t))\, dt = \int d\vec{k} \int d\vec{r}\, f(\vec{k}, \vec{r}) g(\vec{k}, \vec{r}) \tag{3}$$

where g is any function of $\vec{k}$ and $\vec{r}$ and f is the microcanonical distribution

$$f(\vec{k}, \vec{r}) = \delta(\omega - F(\vec{k}, \vec{r})) \times \left[\int\int dk\, dr\, \delta(\omega - F(\vec{k}, \vec{r})) \right]^{-1} \ . \tag{4}$$

On the basis of (4), it is expected that for an eigenmode with resonant frequency ω the wave energy density in $\vec{k}$, $\vec{r}$ is given by[3]

$$W(\vec{k}, \vec{r}) = \left[\int\int W(\vec{k}, \vec{r})\, d\vec{k}\, d\vec{r} \right] \times \frac{\delta(\omega - F(\vec{k}, \vec{r}))}{\int\int d\vec{k}\, d\vec{r}\, \delta(\omega - F(\vec{k}, \vec{r}))} \ , \tag{5}$$

if the solution of the ray equations is ergodic. This is analogous to the microcanonical ensemble in statistical mechanics where all phase space points on the energy surface [analogous to the constant $F(\vec{k}, \vec{r})$ surface] are equally likely. In writing Eqs. (4) and (5) we have in mind a cavity filled with a magnetized plasma or anisotropic dielectric. Also we consider only electromagnetic waves which propagate freely from the vacuum into the plasma and visa versa. In this case the two polarizations of plane electromagnetic waves propagating in a given direction, with given frequency are, in general, nondegenerate. That is they have different values of $|\vec{k}|$. Thus the dispersion relation $\omega = F(\vec{k}, \vec{r})$ will possess these two solutions. Hence in this case, Eq. (5) automatically gives the distribution of energy in the two polarizations. In the case of an unmagnetized plasma, isotropic dielectric or vacuum, the two independent polarizations are degenerate, that is, they have the same value of $|\vec{k}|$. In this case we supplement Eq. (5) with the information that both polarizations are equally likely. One expects that Eq. (5) holds in the limit of $k \to \infty$; however, it is not yet known how good Eq. (5) is for finite k. In this respect, recent numerical experiments are of interest[4] (typically $kL \sim 10$ in these experiments where L is a scale length). We shall be interested in applying Eq. (5) in cases where the wave experiences absorption at the conducting boundaries upon reflection. Under these circumstances, we require the following two conditions for the validity of Eq. (5). *Condition A*: The wall absorption in following a typical ray must be slow enough so that the ray wanders over a representative region of the $\omega = F(\vec{k}, \vec{r})$ surface equal to the constant surface before significant absorption takes place. *Condition B*: The dispersion relation $F(\vec{k}, \vec{r})$ must not exhibit any resonances (i.e., solutions with $k \to \infty$). Condition *B* is necessary in order that the integral in the denominator of Eq. (4) be finite.

We now calculate the damping of an ergodic eigenmode due to finite wall conductivity. The Q of the cavity is defined as

$$Q = \frac{\omega \int\int W(\vec{k}, \vec{r}) d\vec{k}\, d\vec{r}}{\oint \eta_s |\vec{J}_s|^2 d^2 r} , \tag{6}$$

where η_s is the surface skin resistivity $\eta_s = (\mu_0 \omega / 2\sigma)^{1/2}$, σ is the wall conductivity which may be a function of position on the wall, $\vec{J}_s$ is the rms surface current density vector, and the integral, $\oint \eta_s d^2 r$, is taken over the boundary of the cavity. Here we wish to calculate $|\vec{J}_s|^2$ (and hence Q).

We recall that, by assumption, near the cavity walls the plasma density is zero. In this region $F(\vec{k}, \vec{r}) = |\vec{k}| c$, where c is the speed of light, and from Eq. (5) $W(\vec{k}, \vec{r}) \sim \delta(\omega - |\vec{k}| c)$. Thus, near the walls, $W(\vec{k}, \vec{r})$ is independent of $\vec{r}$ and the direction of $\vec{k}$. To treat this case we recall that $\lambda \ll L$ and consider a plane wave in vacuum which is incident on a plane conductor. We also assume, as mentioned previously, that in the vacuum region near the wall $W(\vec{k}, \vec{r})$ is the same for each of the two independent polarizations. Let θ be the angle of incidence, and ϕ the angle that the magnetic field vector $\vec{H}$ makes with the plane of incidence. Then, from Maxwell's equations and the condition that $\vec{E}$ tangential vanish, the current $\vec{J}_s$ created by this plane wave is

$$\vec{J}_s = 2H \cos\phi \hat{\alpha} + 2H \sin\phi \cos\theta \hat{\beta} , \tag{7}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are unit vectors in the surface of the conductor, respectively, perpendicular to and in the plane of incidence. Averaging over ϕ and the solid angle of the incident waves we have that

$$|\vec{J}_s|^2 = \int_0^{2\pi} \frac{d\phi}{2\pi} \int \frac{d\Omega}{2\pi} [4H^2(\cos^2\phi + \sin^2\phi \cos^3\theta)] , \tag{8}$$

where $d\Omega = 2\pi \sin\theta\, d\theta$, the integral over $d\Omega$ is from $\theta = 0$ to $\pi/2$ (i.e., only over incident waves), and H^2 may be related to the average energy density of the eigenmode in the vacuum near the wall

$$\tfrac{1}{2}\mu_0 H^2 = \int\int W(\vec{k}, \vec{r}) d\vec{k}\, d\vec{r} \times \frac{V \int \delta(\omega - |\vec{k}| c) dk}{\int\int \delta(\omega - F(\vec{k}, \vec{r})) d\vec{k}\, d\vec{r}} , \tag{9}$$

where V denotes the volume enclosed by the cavity. From Eq. (8), $|\vec{J}_s|^2 = \frac{4}{3} H^2$. Thus

$$Q = \frac{3\mu_0 c^3}{16\pi\omega \oint \eta_s d^2 r V} \int\int \delta(\omega - F(\vec{k}, \vec{r})) d\vec{k}\, d\vec{r} . \tag{10}$$

For vacuum, $F(\vec{k}, \vec{r}) = |\vec{k}| c$ and Eq. (10) yields

$$Q = (3\mu_0\omega) \left(4 \oint \eta_s d^2 r\right)^{-1} ,$$

where the two independent polarizations have been taken into account.

Similar arguments to those above can be applied to Helmholtz equation in two dimensions (x and y), $\nabla^2\psi + k^2\psi = 0$ with the eigenfunctions zero on the boundary with the result that

$$\left\langle \left(\frac{\partial\psi}{\partial\eta}\right)^2 \right\rangle = k^2 \int \psi^2 dx\, dy \Big/ \int dx\, dy , \tag{11}$$

where $\partial/\partial\eta$ denotes the normal derivative, and $\langle \cdots \rangle$ denotes the average over the boundary. This expression has been checked in numerical experiments by McDonald and Manheimer[5] using the model of Ref. 6. Reasonable agreement was obtained, thus giving an indication of the correctness of our approach. To obtain (11) consider a plane wave of amplitude $\tilde{\psi}$ incident on a plane boundary where $\psi = 0$. This leads to a normal derivative of ψ at the boundary of amplitude $2\tilde{\psi} k \cos\theta$, where θ is the angle of incidence. Averaging $(2\tilde{\psi} k \cos\theta)^2$ over θ for incident waves $(0 \le \theta \le \pi/2)$ then yields Eq. (11).

If the wave has nonzero damping rate $\gamma(\vec{k}, \vec{r})$ in the plasma then the cavity Q is $(Q_w^{-1} + Q_p^{-1})^{-1}$ where Q_p is the Q resulting from absorption in the plasma and Q_w is given by Eq. (10). Thus the fraction of energy absorbed by the wall is $Q_p(Q_p + Q_w)^{-1}$. The Q

resulting from the plasma losses is

$$Q_p = \omega / 2\bar{\gamma} \ , \qquad (12)$$

where[2]

$$\bar{\gamma} = \frac{\int\int d\vec{k}\, d\vec{r}\, \gamma(\vec{k},\vec{r})\delta(\omega - F(\vec{k},\vec{r}))}{\int\int d\vec{k}\, d\vec{r}\, \delta(\omega - F(\vec{k},\vec{r}))} \ . \qquad (13)$$

This may be useful in calculating the amount of energy emitted by cyclotron radiation which is absorbed by the wall in a thermonuclear fusion device.

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