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Carleman imbedding of multiplicative stochastic processes

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A class of exactly solvable nonlinear stochastic models with multiplicative Gaussian white noise is here solved by a method of linear imbedding. The models describe a continuous instability with fluctuations of the bifurcation parameter. The method of solution is a generalization of an idea, originally introduced by Carleman for the solution of deterministic rate equations. Our application of this method to the stochastic models studied here provides further insight into the applicability of the method. The solutions we find are compared with results, which have been obtained earlier by Fokker-Planck methods. Complete agreement with the Fokker-Planck results is found, contrary to recent claims by Brenig and Banai and also in disagreement with recent results of Suzuki *et al.* The method presented here also allows us, for the first time, to obtain solutions in a domain of parameter space where the Fokker-Planck equation has not been solved as yet.

I. INTRODUCTION

Instabilities in macroscopic systems driven far from thermodynamic equilibrium have become an important field of study.[1-3] It is well known, that the first instability encountered in many systems, if driven away from equilibrium, is a continuous symmetry-breaking instability. The phenomenology of such instabilities is like that of second-order phase transitions.[1-3] We will only consider here the simplest case, where the order parameter has one component and can be represented by a real variable x. Quite frequently the spatial extensions of the system under study and its correlation length near the instability are comparable, and spatial variations of the order parameter can therefore be neglected. The system is then "zero-dimensional." If there are no fluctuations, a mean-field treatment of the instability is entirely appropriate. The simplest equation of motion for x is then given by the Landau theory

$$\dot{x} = dx - bx^3 \tag{1.1}$$

with $b>0$ and d changing sign at the instability. The stable time-independent solution of (1.1) is given by

$$x = \begin{cases} 0 & \text{for } d \le 0 \\ \pm \left(\dfrac{d}{b} \right)^{1/2} & \text{for } d > 0 \, . \end{cases} \tag{1.2}$$

The relaxation of small deviations δx from these states is exponential

$$\delta x = \delta x_0 e^{-\lambda t} \tag{1.3}$$

with the rates

$$\begin{aligned} &\lambda = -d \quad \text{for } d<0 \\ &\lambda = 2d \quad \text{for } d>0 \, . \end{aligned} \tag{1.4}$$

Thus, a sharp continuous transition (bifurcation) at $d=0$ results with a critical slowing down and classical exponents given by the Landau theory. How this transition is modified is well known if thermal fluctuations are added to the picture: Because the system is zero dimensional the sharp transition disappears and is replaced by a smooth gradual change. Usually, however, this is a tiny effect if the zero-dimensional system is already macroscopic, and will be neglected henceforth. Instead, we are here interested in other fluctuations of presumably much bigger strength. We assume here that such fluctuations result from a limited control over the parameter d.

Indeed, quite often the value of d in Eq. (1.1) is itself the result of a dynamical process, which occurs on a much faster time scale than the time scale set by d itself. In such cases d is not truly constant in time, but has to be replaced by a random process

$$d \to d + F(t) \tag{1.5}$$

with a correlation time much shorter than the time scale set by d. We will assume that $F(t)$ has zero average, is δ correlated in time, and is Gaussian, i.e.,

$$\begin{aligned} &\langle F(t) \rangle = 0 \, , \\ &\langle F(t)F(0) \rangle = Q\delta(t) \, . \end{aligned} \tag{1.6}$$

These are, of course, idealizations, abstracted from functions $F(t)$ varying continuously in time. Hence, the stochastic differential equation obtained from Eq. (1.1) by (1.5),

$$\dot{x}=dx-bx^3+xF(t)\ , \tag{1.7}$$

has to be interpreted in the sense of Stratonovich or else must be transformed to the Itô representation of stochastic calculus.

One may now ask how the sharp bifurcation of Eq. (1.1) is changed by the fluctuations via Eqs. (1.5) and (1.6). Various different answers to this question have been given in the literature.

(1) By looking at the time-dependent steady-state probability distribution[5] associated with Eq. (1.5), Horsthemke *et al.*[4] have argued that Eq. (1.5) describes a sharp "noise-induced transition" at some finite positive value of d ($d=Q/2$), which is not present in the deterministic problem. The transition is *defined* to occur at that value of d, where the steady-state probability density $W_0(x)$ develops a maximum (i.e., a most probable value) at some nonvanishing value of x. This value of x was then defined to represent the order parameter. The physical shortcoming of these definitions is that they put emphasis only on the most probable value of x in a situation, where the noise is supposed to be crucial. The most probable value dominates the probability distribution only if the noise becomes very small, which is just the case where the "noise-induced transition" disappears at $d=0$. Thus the definition of a noise-induced transition along these lines, while possible logically, is physically inconsistent. Instead the order parameter of the system should be defined by a moment of the steady-state distribution if the noise is really considered as important. If this is done, no noise-induced transition is found. Instead, even in the presence of the noise, the system has a sharp transition at $d=0$.

(2) Schenzle and Brand[6] have solved exactly the Fokker-Planck equation associated with Eq. (1.7). They found that the stochastic system still exhibits a critical slowing down at $d=0$, which is modified, however, into the rate

$$\lambda=d^2/2Q \tag{1.8}$$

for $d>0$. For $d<0$, they also found $\lambda=d^2/2Q$. However, the eigenfunctions of the Fokker-Planck operator in that domain were not normalizable, and thus no safe conclusions about the relaxation rates could be drawn. For $d>0$, no additional points of critical slowing down were found by the Fokker-Planck analysis. Therefore, if transition points are defined either by the appearance of an order parameter, defined as a moment of the steady-state distribution, or as points of critical slowing down, the noisy system only makes a transition at $d=0$, and a noise-induced transition does not exist in this model.

(3) The results of Schenzle and Brand[6] have recently been criticized by Suzuki *et al.*[7] and by Brenig and Banai[8] on different grounds. The criticism of Suzuki[7] was based on the boundary conditions employed by Schenzle and Brand[6] when solving the Fokker-Planck equation. Proposing a weaker boundary condition Suzuki argued that new solutions of the Fokker-Planck equation would be allowed displaying a critical slowing down at certain values of $d>0$, and a noise-induced transition would reappear.

Up to now Suzuki's criticism could be refuted on two grounds: (i) The weaker boundary conditions he proposes (L_1 integrability of all solutions) are not sufficient to impose a Hilbert space structure on the eigenvalue problem associated with the Fokker-Planck equation, and to formulate a completeness relation for its eigenfunctions. Hence, they give only an incomplete characterization of its solutions.

(ii) It has been shown by direct calculation,[9] that the eigenfunctions found by Schenzle and Brand[6] completely exhaust the sum rule which follows from the condition

$$\langle x^2(t)x^2(0)\rangle\,|_{t=0}=\langle x^4\rangle\ . \tag{1.9}$$

Since each eigenfunction contributes a positive term to the sum rule, this finding proves that the spectrum found by Schenzle and Brand is indeed complete.

However, it would certainly be nice to have a different exact method of solution, which would proceed without explicit use of the disputed boundary condition. Various approximate solutions of Eq. (1.7) have been presented recently.[10,11] The purpose of this paper is to give just such an independent exact solution. The results obtained can be compared with the Fokker-Planck results for $d>0$, and complete agreement is obtained. This result presents a third, and perhaps the strongest, argument against Suzuki's criticism[7] and the noise-induced transition[4] for $d>0$. As a nice by-product, for $d<0$ the new method leads to new and somewhat surprising results. A solution of Eq. (1.7) without use of the Fokker-Planck method was already attempted by Brenig and Banai.[8] Indeed, the method of solution we want to present here is based on the same idea as theirs, the linear imbedding of Eq. (1.7). The difficulties inherent in the linear imbedding of Eq. (1.7) will be discussed in detail below and in the bulk of this paper. These difficulties have not been surmounted in the paper of Brenig and Banai,[8] and consequently, they led to incorrect conclusions. In particular, the discrepancies with the Fokker-Planck results which they reported, are entirely due to an incorrect evaluation of the linear imbedding.

In the bulk of this paper we proceed in the following way. In Sec. II we introduce and illustrate Carleman's method of linear imbedding for solving the deterministic Eq. (1.1).[12,13] [This method has also recently been applied to the Lorenz model of thermal convection, however, with uncertain conclusions.[14] For a very recent formulation of this method for deterministic systems (cf. also Ref. 15).] The typical difficulties associated with this method here already make their appearance: These difficulties arise from the fact that the solution of the problem is formally obtained as an infinite series, whose radius of convergence covers only a very small part of the physically important parameter space. In general, extensive resummations and reorderings of the series have to be carried out, before a meaningful representation of the solution in the entire parameter space is obtained. For the deterministic problem the solution is obtained in Sec. II in the form of a geometrical series, which can be summed inside its radius of convergence, and then be analytically continued to obtain the solution in closed form.

In Sec. III, the stochastic problem of Eq. (1.7) is reformulated by linear imbedding and formally solved. The time-dependent moments and correlation functions are represented again by infinite series, whose radius of convergence decreases in time and goes to zero for $t \to \infty$. In Sec. IV the formal series for the two-time correlation functions in the steady state is summed up and represented by a contour integral. In reordered form the representation of the correlation functions obtained here coincides exactly with their representation in terms of the eigensolutions of the Fokker-Planck operator. The asymptotic form of the correlation functions for $t \to \infty$ is also evaluated in Sec. IV. In Sec. V the analysis of Sec. IV is repeated for the time-dependent moments. Exact expressions are obtained for $d \geq 0$ and $d < 0$. For $d \geq 0$ the results are compared with the Fokker-Planck results and complete agreement is obtained. For $d < 0$ Fokker-Planck results for the moments are not yet available. We show here that the moments for $d < 0$ in general decay in time by a sum of discrete exponentials and an integral over a continuum of exponentials. This result is surprising, since the Fokker-Planck spectrum in that domain only consists of a pure continuum. We show, however, that no contradiction arises from this result. Section VI contains a discussion of our results and the final conclusions. How the exponential decay of moments is reconciled with a continuous spectrum of the Fokker-Planck operator is shown in Appendix A by looking at the linear problem associated with Eq. (1.7) for $d < 0$. Appendix B contains a brief summary of the Fokker-Planck results necessary for a comparison with the results obtained here.

Before proceeding further we choose in Eq. (1.7) the special values $b = 1$ and $Q = 2$ without restriction of generality. Indeed, the equations

$$\begin{aligned} \dot{x} &= dx - x^3 + xF(t) , \\ \langle F(t)F(0) \rangle &= 2\delta(t) , \end{aligned} \tag{1.10}$$

which we will solve is the scaled form of the whole class of stochastic equations

$$\begin{aligned} \dot{x} &= dx - bx^{1+\gamma} + xF(t), \\ \langle F(t)F(0) \rangle &= Q\delta(t) . \end{aligned} \tag{1.11}$$

The results for the class of models (1.11) can be obtained from our results by rescaling in our results

$$\begin{aligned} t &\to \tfrac{1}{8}\gamma^2 Q t , \\ x &\to \left[\frac{4b}{\gamma Q} x^{\gamma} \right]^{1/2} , \\ d &\to \frac{4}{\gamma Q} d . \end{aligned} \tag{1.12}$$

II. LINEAR IMBEDDING OF THE DETERMINISTIC PROBLEM

In order to introduce the method of Carleman we first consider Eq. (1.1) for $F = 0$. Its solution by direct integration is

$$x(t) = \left[\frac{d}{\left[\frac{d}{x_0^2} - 1 \right] e^{-2dt} + 1} \right]^{1/2} . \tag{2.1}$$

The same solution can be derived[13] introducing

$$y_1 = x, y_2 = x^2, \ldots, y_n = x^n, \ldots , \tag{2.2}$$

and rewriting Eq. (1.1), with $F=0$, in the form

$$\frac{dy_n}{dt}=ndy_n-ny_{n+2},\quad n=1,2,3,\ldots, \tag{2.3}$$

with the initial condition $y_n(0)=x_0^n$. The nonlinear problem has thereby been replaced by a linear problem with infinite number of dimensions. We restrict attention to the domain $x>0$, and also choose $x_0>0$.

Equation (2.3) is easily solved by the Laplace transformation

$$\tilde{y}_n(\lambda)=\int_0^\infty e^{-\lambda t}y_n(t)dt\ . \tag{2.4}$$

We obtain

$$(\lambda-nd)\tilde{y}_n+n\tilde{y}_{n+2}=x_0^n\ , \tag{2.5}$$

or in matrix form,

$$\begin{bmatrix}\lambda-ld & l & 0 & 0 & \cdots\\ 0 & \lambda-(l+2)d & l+2 & 0 & \cdots\\ 0 & 0 & \lambda-(l+4)d & (l+4) & \cdots\\ \vdots & \vdots & \vdots & \vdots\end{bmatrix}\begin{bmatrix}\tilde{y}_l\\ \tilde{y}_{l+2}\\ \tilde{y}_{l+4}\\ \vdots\end{bmatrix}=\begin{bmatrix}x_0^l\\ x_0^{l+2}\\ x_0^{l+4}\\ \vdots\end{bmatrix} \tag{2.6}$$

with $l=1,2,\ldots,$.

By direct matrix inversion we obtain

$$\begin{bmatrix}\tilde{y}_l\\ \tilde{y}_{l+2}\\ \tilde{y}_{l+4}\\ \vdots\end{bmatrix}=\begin{bmatrix}\frac{1}{\lambda-ld} & \frac{-l}{(\lambda-ld)[\lambda-(l+2)d]} & \frac{l(l+2)}{(\lambda-ld)[\lambda-(l+2)d][\lambda-(l+4)d]} & \cdots\\ 0 & \frac{1}{\lambda-(l+2)d} & \frac{-(l+2)}{[\lambda-(l+2)d][\lambda-(l+4)d]} & \cdots\\ 0 & 0 & \frac{1}{\lambda-(l+4)d} & \cdots\\ \vdots & \vdots & \vdots\end{bmatrix}\begin{bmatrix}x_0^l\\ x_0^{l+2}\\ x_0^{l+4}\\ \vdots\end{bmatrix}, \tag{2.7}$$

i.e.,

$$\tilde{y}_{l+2m}=-\frac{x_0^{l+2m}}{2d\Gamma\left[\frac{l}{2}+m\right]}\sum_{n=0}^{\infty}\left[\frac{x_0^2}{d}\right]^n\frac{\Gamma\left[\frac{l}{2}+m+n\right]\Gamma\left[-\frac{\lambda}{2d}+\frac{l}{2}+m\right]}{\Gamma\left[-\frac{\lambda}{2d}+\frac{l}{2}+m+n+1\right]}\ . \tag{2.8}$$

Inverting the Laplace transform by

$$y_{l+2m}(t)=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}d\lambda\, e^{\lambda t}\tilde{y}_{l+2m}(\lambda)\ , \tag{2.9}$$

we obtain a formal solution as the double sum

$$y_{l+2m}(t)=x_0^{l+2m}\sum_{k=0}^{\infty}\exp(l+2m+2k)dt\sum_{n=k}^{\infty}\left[\frac{x_0^2}{d}\right]^n\frac{(-1)^k\Gamma\left[\frac{l}{2}+m+n\right]}{\Gamma\left[\frac{l}{2}+m\right]\Gamma(k+1)\Gamma(n-k+1)}\ . \tag{2.10}$$

It should be noted that each term of the infinite sum over k diverges for $t\to\infty$ if $d>0$. Therefore, the formal solution has to be summed up before letting t become large (if $d>0$), in order to obtain a meaningful expression. In particular, for $l=2$ and $m=0$ we obtain

$$y_2(t)=x_0^2e^{2dt}\sum_{k=0}^{\infty}(-1)^k e^{2k\,dt}\sum_{n=k}^{\infty}\begin{bmatrix}n\\ k\end{bmatrix}\left[\frac{x_0^2}{d}\right]^n\ . \tag{2.11}$$

Exchanging the order of the two sums we obtain

$$y_2(t)=x_0^2e^{2dt}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k e^{2k\,dt}\begin{bmatrix} n \\ k \end{bmatrix}\left[\frac{x_0^2}{d}\right]^n . \tag{2.12}$$

The finite sum over k is easily carried out with the result

$$y_2(t)=x_0^2e^{2dt}\sum_{n=0}^{\infty}\left[\frac{x_0^2}{d}\right]^n(1-e^{2dt})^n . \tag{2.13}$$

The domain of convergence of this series is

$$\left|\frac{x_0^2}{d}(1-e^{2dt})\right|<1 .$$

Thus, for $d>0$, the radius of convergence in the complex x_0 plane vanishes as $t\rightarrow\infty$.

Inside the domain of convergence, the infinite sum over n may be carried out with the result

$$y_2(t)=\frac{d}{(dx_0^{-2}-1)e^{-2dt}+1} . \tag{2.14}$$

From Eq. (2.14) the result (2.1) for arbitrary x_0 and d follows by analytical continuation:

$$x(t)=[y_2(t)]^{1/2} . \tag{2.15}$$

For the deterministic problem the solution by linear imbedding is, of course, rather awkward and much more complicated than the direct integration of the nonlinear equation, and the method is therefore not particularly useful. However, the method may have some advantages for the multidimensional case.[11] The method proves also to be useful for the stochastic problem, which will be considered in this paper.

The most important point, which can be learned from the preceding solution of the deterministic problem is the following: The solution obtained by linear imbedding in the form of an infinite series (2.11) is, in general, purely formal. Each term of this series may diverge for $t\rightarrow\infty$ in a domain of parameter space where the solution it represents is completely analytical. One has to realize, therefore, that the time dependence of each single term in the series (2.11) has nothing whatsoever to do with the time dependence of the solution represented by the entire sum. Rather, it is necessary to formally sum up the series in some way, before any meaningful conclusions can be drawn.

The reason for the "wrong" time dependence of each term in the series (2.10) is a direct consequence of the linear imbedding. Indeed, the eigenvalues of the triangular matrix in Eq. (2.6), which determine the time dependence of the individual terms of the sum, are simply given by the matrix elements on the diagonal; these are, due to Eq. (2.3), just positive integer multiples of the eigenvalues of the original problem (1.1), linearized for small amplitudes. For $d>0$, such a linearization introduces an exponentially diverging time dependence, which in Eq. (1.1), is only stabilized by the nonlinear term. However, since the eigenvalues are independent of the nonlinear terms, each single term in the infinite sum remains unstabilized. Stabilization can occur only by mutual cancellation of the individual terms in the sum. Thereby the diverging exponential time dependence [e.g., in Eq. (2.11)] can be changed into a finite converging time dependence [e.g., in Eq. (2.14)]. These features are typical for Carleman imbedding and will reappear also in the treatment of the stochastic problem.

III. LINEAR IMBEDDING OF THE STOCHASTIC PROBLEM

We now turn to the stochastic problem defined by Eq. (1.10). A direct generalization of the linear imbedding described in Sec. II is obtained by introducing the moments

$$y_n(t)=\langle x^n(t)\rangle . \tag{3.1}$$

Equations (1.10) are then equivalent to the linear infinite hierarchy

$$\dot{y}_n(t)=n(d+n)y_n-ny_{n+2},\quad n=1,2,\ldots, . \tag{3.2}$$

A comparison of Eq. (3.1) with Eq. (2.3) shows that the triangular structure of the linear problem has not been changed by the introduction of the fluctuating force. This circumstance still allows for an exact formal solution of the infinite hierarchy to be obtained, as was first noted by Brenig and Banai.[8] It is easy to see that choosing additive, rather than multiplicative, noise in Eq. (2.1) destroys the triangular structure of the hierarchy (3.2); the present methods of solution are therefore not applicable to that case.

Equation (3.2) is now solved by the same steps which led from Eq. (2.3) to Eq. (2.10). First, however, it is necessary to specify a suitable initial condition for all moments $y_n(t)$ at $t=0$. In principle, the moments of any desired initial probability distribution $w(x)$ may be used. However, in order to eliminate the necessity of solving Eqs. (3.2) anew for every given initial distribution, it is best to choose a δ function $w(x)=\delta(x-x_0)$ at $t=0$, i.e.,

$y_n(0)=x_0^n$. The results may then later be specialized to any desired initial distribution $w(x)$ by computing

$$\int dx_0 y_n(0) w(x_0) .$$

Clearly, what we are calculating by using the initial condition $y_n(0)=x_0^n$ are just the moments of the conditional probability density of the stochastic process under study.

By Laplace transformation and direct matrix inversion we obtain

$$\begin{bmatrix} \tilde{y}_l \\ \tilde{y}_{l+2} \\ \tilde{y}_{l+4} \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda-\lambda_{l0}} & \frac{-l}{(\lambda-\lambda_{l0})(\lambda-\lambda_{l1})} & \frac{l(l+2)}{(\lambda-\lambda_{l0})(\lambda-\lambda_{l1})(\lambda-\lambda_{l2})} & \cdots \\ 0 & \frac{1}{\lambda-\lambda_{l1}} & \frac{-(l+2)}{(\lambda-\lambda_{l1})(\lambda-\lambda_{l2})} & \cdots \\ 0 & 0 & \frac{1}{\lambda-\lambda_{l2}} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} x_0^l \\ x_0^{l+2} \\ x_0^{l+4} \\ \vdots \end{bmatrix} , \qquad (3.3)$$

where

$$\lambda_{ln}=(l+2n)(d+l+2n) . \qquad (3.4)$$

The components of Eq. (3.3) are

$$\tilde{y}_{l+2m}=x_0^{l+2m}\sum_{n=0}^{\infty}(-2x_0^2)^n \times \frac{\Gamma\left(\frac{l}{2}+m+n\right)}{\Gamma\left(\frac{l}{2}+m\right)} \prod_{k=m}^{n+m}\frac{1}{\lambda-\lambda_{lk}} . \qquad (3.5)$$

Inverting the Laplace transform we obtain

$$y_{l+2m}(t)=x_0^{l+2m}\sum_{n=0}^{\infty}\frac{e^{\lambda_{ln+m}t}}{\Delta_{n0}\Delta_{n1}\cdots\Delta_{n\,n-1}} \times \sum_{k=n}^{\infty}\frac{\Gamma\left(\frac{l}{2}+m+k\right)}{\Gamma\left(\frac{l}{2}+m\right)} \times \frac{(-2x_0^2)^k}{\Delta_{n\,n+1}\Delta_{n\,n+2}\cdots\Delta_{nk}} , \qquad (3.6)$$

where

$$\Delta_{pq}=\lambda_{lm+p}-\lambda_{lm+q} = -4(q-p)\left(q+p+l+2m+\frac{d}{2}\right) . \qquad (3.7)$$

Equation (3.6) may be simplified formally by carrying out the sum over k and obtaining

$$y_p(t)=x_0^p\sum_{n=0}^{\infty}\frac{e^{\lambda_{pn}t}}{n!}\left(-\frac{x_0^2}{2}\right)^n \times \frac{\Gamma\left(\frac{p}{2}+n\right)\Gamma\left(n+p+\frac{d}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(2n+p+\frac{d}{2}\right)} \times {}_1F_1\left(\frac{p}{2}+n;p+\frac{d}{2}+2n+1;\frac{x_0^2}{2}\right) . \qquad (3.8)$$

Throughout we use the standard notation for the hypergeometric functions ${}_nF_m$ (cf. Ref. 16).

Equation (3.8) is in agreement with an expression derived by Brenig and Banai[8] along somewhat different lines. Let us note, however, that Eq. (3.8) is still a purely formal expression and does not give a true representation of $\langle x^{l+2m}(t)\rangle$ for $t>0$ by a convergent series. Indeed, all λ_{ln+m} for sufficiently large n are positive due to Eq. (3.4), and hence the corresponding terms in the sum (3.8) diverge individually for large t. The reason for this behavior has already been discussed for the deterministic solution. The same arguments apply to the stochastic problem *a fortiori*, since the sign of λ_{nm} for sufficiently large n is *always* positive regardless of the sign of d. The "eigennvalues" λ_{ln} appearing after linear imbedding are entirely determined by the linear part of the stochastic equation

(1.10). They are therefore only mathematical constructs without any physical significance. In particular, these eigenvalues are different from and not simply related to the eigenvalues appearing in the Fokker-Planck solution[6] after separation of the time variable. Only the latter eigenvalues (which are listed in Appendix B) are directly related to the physical time dependence. We will show below how the physical spectrum is obtained from the purely formal solution (3.8).

Before doing so, it is useful to generate from Eq. (3.8) a similarly formal representation of the stationary correlation functions, by

$$\langle x^p(t)x^q(0)\rangle = \int dx_0 w_0(x_0) x_0^q y_p(t) \;. \tag{3.9}$$

Here $w_0(x)$ is the steady-state distribution of the process. $y_p(t)$ is given by Eq. (3.8) and depends explicitly on the initial amplitude x_0. The steady-state distribution is given by

$$w_0(x)=\delta(x) \tag{3.10}$$

for $d \le 0$, and

$$w_0(x)=Nx^{d-1}\exp-\tfrac{1}{2}x^2 \;, \tag{3.11}$$

with

$$N=2^{1-d/2}\Gamma^{-1}(d/2) \tag{3.12}$$

for $d \ge 0$ (cf. Appendix B).

Using Eq. (3.10) we obtain, for $d \le 0$, $p>0$, and $q>0$,

$$\langle x^p(t)x^q(0)\rangle = 0 \;. \tag{3.13}$$

This simple result for the steady-state correlation functions in the domain $d \le 0$ must, of course, be expected, if one looks at the stochastic differential equation (1.10). For $d<0$ the point $x=0$ is attractive. The only opposing forces are the fluctuations $xF(t)$ which become weak for small x and vanish for $x=0$. Thus, the system is captured at $x=0$ for long times [i.e., $w_0(x)=\delta(x)$] and remains there forever. All correlation functions in the steady state for $d \le 0$ must therefore vanish, and the only domain of interest, as far as the stationary correlation functions are concerned, is the domain $d>0$. A similar conclusion does not hold for the transient moments $y_n(t)$, which show nontrivial behavior both for $d>0$ and $d \le 0$.

For $d \ge 0$ the following formal representation of the stationary correlation functions is obtained after using Eqs. (3.8), (3.11), and (3.12) in Eq. (3.9):

$$\langle x^p(t)x^q(0)\rangle = \frac{2^{(p+q)/2}\Gamma(1-q/2)}{\Gamma(d/2)\Gamma(p/2)} \times \sum_{n=0}^{\infty} \frac{c_n}{n!}(-1)^n e^{\lambda_{pn}t} \tag{3.14}$$

with

$$c_n = \frac{\Gamma\left[\frac{p}{2}+n\right]\Gamma\left[p+\frac{d}{2}+n\right]\Gamma\left[\frac{p+q}{2}+\frac{d}{2}+n\right]\Gamma\left[n+\frac{p}{2}+\frac{d}{4}\right]}{\Gamma\left[\frac{p+d}{2}+n+1\right]\Gamma\left[\frac{p-q}{2}+n+1\right]} \;. \tag{3.15}$$

Again, the purely formal nature of this representation of the correlation function must be stressed. It is signaled by the divergence of the exponentials in Eq. (3.14) for $t \to \infty$, where the correlation functions should relax to simple constants given by $\langle x^p\rangle\langle x^q\rangle$. Another signal is the apparent divergence of $\Gamma(1-q/2)$ in Eq. (3.14) for the $q=2n$, $n \gtrsim 1$ integer, where the correlation functions are not expected to display any anomalous behavior. It is therefore clear, that the infinite series in (3.14) has to be resummed and reordered, before a true representation of the correlation functions is obtained. This will be done in the following section.

IV. TWO-TIME CORRELATION FUNCTIONS IN THE STEADY STATE

A. General expressions

After Gaussian linearization of the exponent of $\exp\lambda_{pn}t$ with respect to n by

$$e^{\lambda_{pn}t} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} d\xi \exp[-\xi^2+2\xi(p+2n)\sqrt{t}+(p+2n)d\cdot t] \;, \tag{4.1}$$

the formal series (3.14) can be summed up under the ξ integral. We obtain for $d>0$ and $q\neq 2$,

$$\langle x^p(t)x^q(0)\rangle=\frac{\Gamma\left[1-\frac{q}{2}\right]\Gamma\left[p+\frac{d}{2}+1\right]\Gamma\left[\frac{p+q}{2}+\frac{d}{2}\right](2)^{(p+q)/2}}{\Gamma\left[\frac{d}{2}\right]\Gamma\left[\frac{p+d}{2}+1\right]\Gamma\left[\frac{p-q}{2}+1\right]}\tilde{F}(t) \tag{4.2}$$

with

$$\tilde{F}(t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}d\xi\, e^{-\xi^2+2\xi p\sqrt{t}+p\,dt}$$
$$\times {}_4F_3\left[\frac{p}{2},p+\frac{d}{2},\frac{p+q+d}{2},\frac{p}{2}+\frac{d}{4}+1;\frac{p}{2}+\frac{d}{4},\frac{p+d}{2}+1,\frac{p-q}{2}+1;-\exp(4\xi\sqrt{t}+2dt)\right]. \tag{4.3}$$

We now represent the hypergeometric function ${}_4F_3$ by a Mellin-Barnes integral,[16] whereupon the integral over ξ can be done. We obtain

$$\langle x^p(t)x^q(0)\rangle=\frac{2^{(p+q)/2+1}\Gamma\left[1-\frac{q}{2}\right]}{\Gamma\left[\frac{p}{2}\right]\Gamma\left[\frac{d}{2}\right]}\int_{-i\infty}^{+i\infty}\frac{ds}{2\pi i}e^{-(d^2/4)t+4t(s+d/4+p/2)^2}\Gamma(-s)\left[s+\frac{d}{4}+\frac{p}{2}\right]$$
$$\times\frac{\Gamma\left[\frac{p}{2}+s\right]\Gamma\left[p+\frac{d}{2}+s\right]\Gamma\left[\frac{p+q}{2}+\frac{d}{2}+s\right]}{\Gamma\left[\frac{p}{2}+\frac{d}{2}+s+1\right]\Gamma\left[\frac{p-q}{2}+s+1\right]}, \tag{4.4}$$

where the contour of integration is chosen such, that all poles of $\Gamma(-s)$ (i.e., $s=0,1,2,\ldots,$) are to its right and all other poles of the integrand are to its left (cf. Fig. 1). Equation (4.4) is equivalent to Eqs. (3.14) and (3.15). Indeed, all that has been done is the replacement of the sum over all positive integers in Eq. (3.14) by a contour integral over s from $-i\infty$ to $+i\infty$ in Eq. (4.4), which circles the poles of $\Gamma(-s)$. The direct correspondence between the integrand and the sum is obvious.

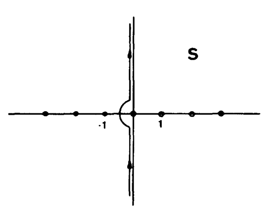

FIG. 1. Contour of the integral in Eq. (4.4) in the complex s plane for $p=2$, $q=1$, $d=6$. Open circles, poles of $\Gamma(-s)$; full circles, poles of $\Gamma[(p/2)+s]$, $\Gamma[p+(d/2)+s]$, $\Gamma[(p+q)/2+(d/2)+s]$.

However, the integral representation can now also be used in domains of parameter space, where the original sum did not converge. In order to simplify the integrand we introduce the integration variable κ by

$$s=i\kappa-\frac{d}{4}-\frac{p}{2}\,. \tag{4.5}$$

All poles of $\Gamma(-i\kappa+d/4+p/2)$ then have to be to the right of the contour of integration, while all other poles have to be to its left. Let us now deform the integration contour for κ in such a way that the contour is along the real κ axis. We discuss the cases $d>0$, $p>0$, and $q>0$. By the indicated deformation of the contour of integration, a finite number of poles of

$$\Gamma\left[\frac{p}{2}+s\right]=\Gamma\left[i\kappa-\frac{d}{4}\right]$$

with

$$s+\frac{p}{2}=i\kappa-\frac{d}{4}=-n,\quad 0\le n\le n_0 \tag{4.6}$$

where n_0 is the largest integer smaller than $d/4$, wander from the left-hand side of the contour to the right-hand side (cf. Fig. 2). After the deformation of the contour, the residua of the integrand at these poles, multiplied by $2\pi i$, have to be added separately to the integral. This finite sum takes the form $\sum_{n=0}^{n_0} c_n\,(p,q)e^{-\lambda_n t}$, where

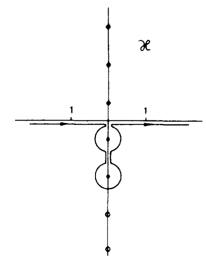

FIG. 2. Contour in the complex κ plane before deformation into the real κ axis, for $p=2$, $q=1$, $d=6$. The poles are related to the poles in Fig. 1 by Eq. (4.5). In this example $n_0=1$.

$$c_n(p,q)=\frac{2^{(p+q)/2+1}\left[\frac{d}{4}-n\right]\Gamma\left[-n+\frac{p+d}{2}\right]\Gamma\left[-n+\frac{q+d}{2}\right]\Gamma\left[n+\frac{p}{2}\right]\Gamma\left[n+\frac{q}{2}\right]}{\Gamma\left[\frac{d}{2}\right]\Gamma\left[\frac{p}{2}\right]\Gamma\left[\frac{q}{2}\right]n!\,\Gamma\left[-n+1+\frac{d}{2}\right]} \tag{4.7}$$

and

$$\lambda_n=2n(d-2n)\ \text{ for }\ 0\le n\le n_0\ . \tag{4.8}$$

The complete symmetry of $c_n(p,q)$ with respect to p,q should be noted. In order to obtain this symmetrical form we have used the transformation

$$\frac{\Gamma\left[1-\frac{q}{2}\right]}{\Gamma\left[1-\frac{q}{2}-n\right]}=(-1)^n\frac{\Gamma\left[\frac{q}{2}+n\right]}{\Gamma\left[\frac{q}{2}\right]}\ . \tag{4.9}$$

Even more important is the fact, that all exponentials in the finite sum now decay in time. Thus, each term of the sum remains meaningful even for $t\to\infty$. Note also the cancellation in Eq. (4.9), of the divergence at $q=2n$, mentioned at the end of Sec. III.

The remaining part of the correlation function now consists of a contour integral along the real κ axis, whose integrand is given by the integrand of Eq. (4.4), transformed by Eq. (4.5). Symmetry of the integrand with respect to p,q can again be achieved by the identity

$$\frac{i\kappa\Gamma\left[1-\frac{q}{2}\right]}{\Gamma\left[i\kappa-\frac{q}{2}-\frac{d}{4}+1\right]\Gamma\left[i\kappa+\frac{d}{4}+1\right]}=f(i\kappa)\frac{\Gamma\left[-i\kappa-\frac{d}{4}\right]\Gamma\left[-i\kappa+\frac{q}{2}+\frac{d}{4}\right]}{\Gamma\left[\frac{q}{2}\right]}\ , \tag{4.10}$$

where $f(i\kappa)$ is given by

$$f(i\kappa)=-i\kappa\frac{\sin\pi\left[-i\kappa+\frac{q}{2}+\frac{d}{4}\right]\sin\pi\left[i\kappa+\frac{d}{4}\right]}{\pi\sin\pi\frac{q}{2}}\equiv f_+ + f_-\ , \tag{4.11}$$

$$f_\pm(i\kappa)=\pm f_\pm(-i\kappa)\ .$$

Inserting these transformations into Eq. (4.4), it becomes clear upon inspection that only the part $f_+(i\kappa)$ of

$f(i\kappa)$, which is even under $i\kappa \to -i\kappa$, contributes to the integral. This part is given by

$$f_+(i\kappa) = \frac{\kappa}{2\pi}\sinh 2\pi\kappa \; . \tag{4.12}$$

Note, that the denominator $\sin\pi q/2$ present in Eq. (4.11), cancels from the symmetrical part $f_+(i\kappa)$ which contributes to the integral over κ. Again, we can see here, explicitly, the disappearance of the divergence at $q=2n$, which seemed to plague the formal result (3.14). Using Eq. (4.12) with (4.10) to rewrite the integrand, we obtain

$$\langle x^p(t)x^q(0)\rangle = \sum_{n=0}^{n_0} c_n(p,q)e^{-\lambda_n t} + \int_0^\infty d\kappa^2 c(\kappa^2,p,q)e^{-\lambda(\kappa^2)t} \; , \tag{4.13}$$

where n_0, $c_n(p,q)$, and λ_n are given by Eqs. (4.6), (4.7), and (4.8),

$$\lambda(\kappa^2) = \frac{d^2}{4} + 4\kappa^2 \tag{4.14}$$

and

$$c(\kappa^2,p,q) = 2^{(p+q)/2-1}\frac{\sinh 2\pi\kappa}{\pi^2}\frac{\left|\Gamma\left[i\kappa-\frac{d}{4}\right]\Gamma\left[i\kappa+\frac{p}{2}+\frac{d}{4}\right]\Gamma\left[i\kappa+\frac{q}{2}+\frac{d}{4}\right]\right|^2}{\Gamma\left[\frac{p}{2}\right]\Gamma\left[\frac{q}{2}\right]\Gamma\left[\frac{d}{2}\right]} \; . \tag{4.15}$$

Equation (4.13) gives a spectral representation of the correlation function in terms of exponentials decaying in time. We have therefore succeeded in reordering the formal series (3.14) into a new series and an integral where each term of the new series and the integrand remain meaningful for $t \to \infty$.

The same spectral representation can also be obtained from the Fokker-Planck description along very different lines, cf. Appendix B. Therefore, we have shown that the present description and the Fokker-Planck description are entirely equivalent, contrary to what has recently been claimed in the paper by Brenig and Banai.[8]

B. Asymptotic results for $t \to \infty$

The long-time behavior of the correlation function is easily extracted from the general results of the preceding section. From the general form of the result of (4.13) it is clear that the long-time behavior is dominated by the contribution to the finite sum over n from $n=0$ and 1, and by the contribution to the integral over κ^2 from the vicinity of the lower bound $\kappa^2=0$. We discuss the various contributions separately:

1. Contribution from $n=0$

Since $\lambda_0=0$ is the only vanishing eigenvalue for $d>0$ the contribution to the correlation function from $n=0$ must give

$$c_0(p,q) = \lim_{t\to\infty}\langle x^p(t)x^q(0)\rangle \tag{4.16}$$

which, due to the mixing property of the Markovian process under study, must equal

$$\lim_{t\to\infty}\langle x^p(t)x^q(0)\rangle = \langle x^p\rangle\langle x^q\rangle \; . \tag{4.17}$$

Indeed, our result for $c_0(p,q)$,

$$c_0(p,q) = \left[2^{p/2}\frac{\Gamma\left[\frac{p+d}{2}\right]}{\Gamma\left[\frac{d}{2}\right]}\right]\left[2^{q/2}\frac{\Gamma\left[\frac{d+q}{2}\right]}{\Gamma\left[\frac{d}{2}\right]}\right] , \tag{4.18}$$

satisfies this relation, if it is compared with the moments of the steady-state distribution (3.11) (cf. Appendix B). We note, that the term with $n=0$ already exhausts the entire sum over n in the domain $n_0=0$, i.e., $d<4$ [for the definition of n_0, cf. text after Eq. (4.6)].

2. Contribution from $n=1$

The finite sum over n in Eq. (4.13) contains a term with $n=1$ only if $d>4$. In that case the relaxation of $\langle x^p(t)x^q(0)\rangle$ towards $\langle x^p\rangle\langle x^q\rangle$ for long times is given by the exponential decay

$$\langle x^p(t)x^q(0)\rangle-\langle x^p\rangle\langle x^q\rangle\simeq c_1(p,q)e^{-\lambda_1 t} \quad (4.19)$$

with

$$c_1(p,q)=2^{(p+q)/2-1}\frac{\left[\frac{d}{4}-1\right]}{\Gamma\left[\frac{d}{2}\right]^2}pq \times\Gamma\left[\frac{d+p}{2}-1\right]\Gamma\left[\frac{d+q}{2}-1\right], \quad (4.20)$$

$$\lambda_1=4\left[\frac{d}{2}-1\right], \quad (4.21)$$

i.e., the correlation functions at long times relax exponentially for $d>4$. At $d=4$, c_1 approaches zero and the long-time behavior is changed.

3. Contribution from small κ^2

In the domain $0\le d\le 4$ the relaxation of the correlation function is entirely determined by the contribution from the integral over κ^2. For long times, only the contributions from κ^2 near zero survive. We have to distinguish two cases:

a. *$d\neq 4n$, i.e., in particular, $0<d<4$.* The integral over κ^2 in Eq. (4.13) for long times then reduces to

$$\int_0^\infty d\kappa^2 c(\kappa^2,p,q)e^{-\lambda(\kappa^2)t}\simeq\sqrt{\pi}\Gamma^2\left[-\frac{d}{4}\right]\tilde{c}(p,q)e^{-d^2t/4}\int_0^\infty d\kappa^2(\kappa^2)^{1/2}e^{-4\kappa^2 t}=\frac{1}{2}\frac{\tilde{c}(p,q)}{(4t)^{3/2}}\Gamma^2\left[-\frac{d}{4}\right]e^{-d^2t/4}, \quad (4.22)$$

with

$$\tilde{c}(p,q)=\frac{2^{(p+q)/2}\Gamma^2\left[\frac{p}{2}+\frac{d}{4}\right]\Gamma^2\left[\frac{q}{2}+\frac{d}{4}\right]}{\sqrt{\pi}\Gamma\left[\frac{d}{2}\right]\Gamma\left[\frac{p}{2}\right]\Gamma\left[\frac{q}{2}\right]}. \quad (4.23)$$

Therefore, we have for $0<d<4$,

$$\langle x^p(t)x^q(0)\rangle-\langle x^p\rangle\langle x^q\rangle =\tfrac{1}{2}\Gamma^2\left[-\frac{d}{4}\right]\frac{\tilde{c}(p,q)}{(4t)^{3/2}}e^{-d^2t/4}. \quad (4.24)$$

We see that for $0<d<4$ the relaxation of the correlation functions for long times is dominated by the lowest eigenvalue $\lambda(0)=d^2/4$ of the continuum. However, the continuum above $\lambda(0)$ modifies the purely exponential decay by the algebraic prefactor $t^{-3/2}$. If d approaches the value $d\to 4n$, the coefficient of $t^{-3/2}\exp-d^2t/4$ diverges like $(d-4n)^{-2}$. This divergence has to be taken as a signal that the two limits involved, $t\to\infty$ and $d\to 4n$, cannot be interchanged. Therefore, we have to evaluate the asymptotic behavior for these cases separately.

b. $d=4n$, i.e., in particular, $d=4$. If we insert $d=4n$ into the integrand of the integral over κ^2 in Eq. (4.13), its behavior for $\kappa^2\to 0$ is modified, since, for $\kappa^2\to 0$,

$$|\Gamma(-n+i\kappa)|^2\to\left[\frac{1}{n!}\right]^2\kappa^{-2}. \quad (4.25)$$

Hence, we obtain

$$\int_0^\infty d\kappa^2 c(\kappa^2,p,q)e^{-\lambda(\kappa^2)t}\simeq\frac{1}{\sqrt{\pi}}\tilde{c}(p,q,4n)e^{-4n^2t}\left[\frac{1}{n!}\right]^2 \times\int_0^\infty d\kappa^2\frac{1}{(\kappa^2)^{1/2}}e^{-4\kappa^2 t} =\frac{\tilde{c}(p,q,4n)}{\sqrt{4t}}e^{-4n^2t},\left[\frac{1}{n!}\right]^2 \quad (4.26)$$

with $\tilde{c}(p,q,d)$ given by Eq. (4.23). We note that for

$d=0$, $\tilde{c}$ vanishes. For $d=4$, $\tilde{c}$ has a finite, nonzero value for $p,q>0$. Thus, the point $d=4$ is special in at least two ways: (i) if d passes the point $d=4$ from below, the number of discrete eigenvalues is changed from zero to one and (ii) the asymptotic form of the correlation function

$$\langle [x^p(t)-\langle x^p\rangle][x^q(0)-\langle x^q\rangle]\rangle$$

is changed from

$$t^{-3/2}\exp^{-d^2t/4} \text{ for } d<4 ,$$

to

$$t^{-1/2}\exp^{-4t} \text{ for } d=4 ,$$

to

$$\exp -4\left[\frac{d}{2}-1\right]t \text{ for } d>4 .$$

The long-time behavior of the correlation function for $d=4$ is, of course, not directly observable, since exact equality in $d=4$ is required. However, the divergence of the coefficient of $t^{-3/2}\exp^{-d^2t/4}$ as $d=4$ is approached from below is, in principle, observable, and would be very interesting to look for in experiments on systems described by Eq. (1.11).

V. TRANSIENT MOMENTS

A. General formulas

The discussion of the moments requires the formal summation of Eq. (3.8). Using the same idea which was successful in Sec. IV, we replace the sum (3.8) by a contour integral in the complex s plane around the poles of $\Gamma(-s)$. We obtain immediately,

$$y_p(t)=x_0^p\int_{-i\infty}^{+i\infty}\frac{ds}{2\pi i}\exp\left[-\frac{d^2t}{4}+4\left[s+\frac{d}{4}+\frac{p}{2}\right]^2 t\right]\Gamma(-s)\left[-\frac{x_0^2}{2}\right]^s\frac{\Gamma\left[\frac{p}{2}+s\right]\Gamma\left[p+\frac{d}{2}+s\right]}{\Gamma\left[\frac{p}{2}\right]\Gamma\left[p+\frac{d}{2}+2s\right]}$$

$$\times {}_1F_1\left[\frac{p}{2}+s;p+\frac{d}{2}+2s+1;\frac{x_0^2}{2}\right], \tag{5.1}$$

where the contour has all poles of $\Gamma(-s)$ to its right and all other poles to its left (cf. Fig. 3). We may now go through the same steps as in Sec. IV A in order to derive a spectral decomposition of the moments. First we change the integration variable

$$s=i\kappa-\frac{d}{4}-\frac{p}{2} \tag{5.2}$$

and then we deform the resulting contour in the complex κ plane until it coincides with the real κ axis. Since a finite number of poles changes the sides of the contour due to this deformation, their contribution has to be added (if they change from left to right) or substracted (in the opposite case) from the integral along the real κ axis. We have to distinguish the following three cases.

(i) $-2p<d<0$. In this case the contour in the κ plane can be deformed into the real axis, without any pole changing sides (cf. Fig. 4). Therefore, we obtain

$$y_p(t)=y_p^c(t) , \tag{5.3}$$

with

$$y_p^c(t)=\frac{2^{(p/2)-1}}{\pi^2}\int_{-\infty}^{\infty}d\kappa\, e^{-\lambda(\kappa^2)t}\kappa\sinh 2\pi\kappa\frac{\left|\Gamma\left[i\kappa-\frac{d}{4}\right]\Gamma\left[i\kappa+\frac{d}{4}+\frac{p}{2}\right]\right|^2}{\Gamma\left[\frac{p}{2}\right]}{}_2F_0\left[i\kappa-\frac{d}{4},-i\kappa-\frac{d}{4};-\frac{2}{x_0^2}\right], \tag{5.4}$$

where some algebraic rearrangement of the integrand has produced the manifestly symmetrical form in κ. $y_p(t)$ is given by an integral over the continuous spectrum with eigenvalues $\lambda(\kappa^2)$ given by Eq. (4.14). The asymptotic form of this result for long times will be discussed further in Sec. V B.

(ii) $d>0$. In this case the finite number of poles of

$$\Gamma\left[\frac{p}{2}+s\right]=\Gamma\left[i\kappa-\frac{d}{4}\right]$$

satisfying

$$\kappa=i\left[n-\frac{d}{4}\right] \text{ with } 0\le n\le n_0 , \tag{5.5}$$

where n_0 is the largest integer smaller than $d/4$, changes from the left to the right of the contour, cf. Fig. 5. In this case we obtain from Eq. (5.1),

$$y_p(t)=y_p^c(t)+x_0^p\sum_{n=0}^{n_0}\exp(-\lambda_n t)\frac{(-1)^n}{n!}\left[\frac{x_0^2}{2}\right]^{-n-p/2}\frac{\Gamma\left[n+\frac{p}{2}\right]\Gamma\left[-n+\frac{d}{2}+\frac{p}{2}\right]}{\Gamma\left[\frac{p}{2}\right]\Gamma\left[-2n+\frac{d}{2}\right]}{}_1F_1\left[-n;-2n+\frac{d}{2}+1;\frac{x_0^2}{2}\right] \tag{5.6}$$

We see therefore that the discrete spectrum with eigenvalues λ_n given by Eq. (4.8) contributes. It should be noted that all discrete eigenvalues which contribute are lower than the continuum, and that the contribution from the eigenvalue λ_n disappears continuously at $n=d/4$.

(iii) $d<-2p$. This case corresponds to sufficiently large negative values of the pump parameter d. In this domain poles of the two functions

$$\Gamma\left[i\kappa+\frac{d}{4}+\frac{p}{2}\right],\quad \Gamma\left[-i\kappa+\frac{d}{4}+\frac{p}{2}\right],$$

satisfying $\kappa=\pm i(n+d/4+p/2)$, with $0\le n\le n_0$, where n_0 is the largest integer smaller than $-p/2+|d/4|$, change sides by the deformation of the contour (cf. Fig. 6). We obtain

$$y_p(t)=y_p^c(t)+\frac{2^{(p/2)+1}}{\pi}\sum_{n=0}^{n_0}e^{-\lambda_n(p)t}\left[n+\frac{d}{4}+\frac{p}{2}\right]\frac{\Gamma\left[n+\frac{d}{2}+p\right]\Gamma\left[n+\frac{p}{2}\right]\Gamma\left[-n-\frac{p+d}{2}\right]}{n!\Gamma\left[\frac{p}{2}\right]}\sin\pi\left[\frac{d}{2}+p\right]$$
$$\times{}_2F_0\left[n+\frac{p}{2},-n-\frac{d}{2}-\frac{p}{2};-\frac{2}{x_0^2}\right], \tag{5.7}$$

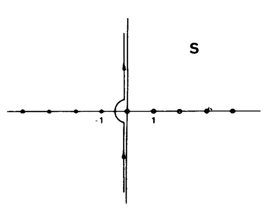

FIG. 3. Contour of the integral in Eq. (5.1) in the complex s plane for $p=2$, $d=6$. Open circles, poles of $\Gamma(-s)$; full circles, poles of $\Gamma[(p/2)+s]$ and $\Gamma[p+(d/2)+s]$.

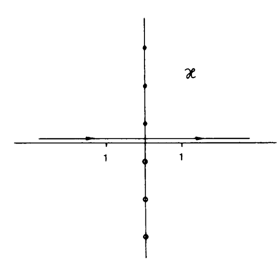

FIG. 4. Contour in the complex κ plane for $-2p<d<0$ drawn for $p=2$, $d=-2$. Open circles, poles of $\Gamma(-s)$; full circles, poles of $\Gamma[(p/2)+s]$ and $\Gamma[p+(d/2)+s]$ with s given by Eq. (5.2).

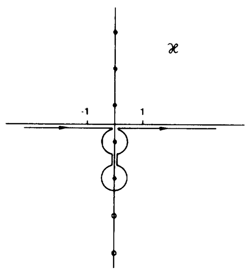

FIG. 5. Contour in the complex κ plane for $d>0$ before deformation into the real κ axis, drawn for $p=2$, $d=6$. Open circles poles of $\Gamma(-s)$; full circles, poles of $\Gamma[(p/2)+s]$ and $\Gamma[p+(d/2)+s]$ with s given by Eq. (5.2). In this example $n_0=1$.

with

$$\lambda_n(p)=\frac{d^2}{4}-4\left[n+\frac{d}{4}+\frac{p}{2}\right]^2 . \tag{5.8}$$

It is not possible to compare this result with the solution of the eigenvalue problem associated with the Fokker-Planck equation. The difficulty with the Fokker-Planck method is that integration over the eigenfunctions and their summation over the eigenvalues cannot be interchanged for $d<0$. The appearance of discrete eigenvalues for $d<-p/4$ in Eq. (5.7) seems puzzling in view of the fact that the spectrum of the Fokker-Planck equation is purely continuous for $d<-p/4$. However, it would be very misleading, to infer the Fokker-Planck spectrum from the moments alone. This is made obvious by a discussion of the Fokker-Planck equation and the moments of the linear equation

$$\dot{x}=-dx+xf(t) \tag{5.9}$$

in Appendix A.

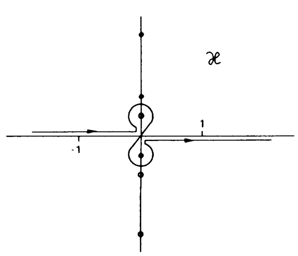

FIG. 6. Contour in the complex κ plane for $d<-2p$ before deformation into the real κ axis, drawn for $p=2$, $d=-5$. Open circles, poles of $\Gamma(-s)$; full circles, poles of $\Gamma[(p/2)+s]$ and $\Gamma[p+(d/2)+s]$ with s given by (5.2). In this example $n_1=0$.

Furthermore, it should be noted that the eigenvalues $\lambda_n(p)$, as given by Eq. (5.8), explicitly depend on p, i.e., on the particular moment which is calculated. They cannot, therefore, represent an intrinsic property of the stochastic process studied here, quite different from the true eigenvalues (4.8) and (4.14), which are independent of the evaluated moments.

B. Asymptotic results for long times

For long times, the general formulas obtained in the last section can be evaluated further. We investigate separately the contribution from the finite sum over $\exp-\lambda_n(p)t$ or $\exp-\lambda_n t$ and the integral over $\exp-\lambda(\kappa^2)t$.

(i) Contribution from $n=0$ for $d>0$. In the domain $d>0$ the moments $y_p(t)=\langle x^p(t)\rangle$ for long times relax towards a finite value. Because of the ergodicity of the process for $x\neq 0$, this finite value must coincide with the corresponding moment $\langle x^p\rangle$ of the steady-state distribution. Indeed, we obtain from our result (5.6) with $\lambda_n=0$ for $n=0$, and $\lambda_n>0$, $n>0$, and $\lambda(\kappa^2)>0$,

$$\lim_{t\to\infty} y_p(t)=2^{p/2}\frac{\Gamma\left[\frac{p+d}{2}\right]}{\Gamma\left[\frac{d}{2}\right]}, \quad d>0 \tag{5.10}$$

which satisfies the above requirement. For $d<0$ we obtain from Eqs. (5.7) and (5.4),

$$\lim_{t\to\infty} y_p(t)=0 , \tag{5.11}$$

i.e., the moments relax to zero for $d<0$.

(ii) Contribution from $n=1$ for $d>4$. In the domain $d>4$ the relaxation of the moments towards their value in the steady state is dominated by $\exp-\lambda_1 t$. From Eq. (5.6) we obtain

$$\langle x^p(t)\rangle-\langle x^p\rangle \simeq 2^{(p/2)-2}p(d-4)\left[\frac{d-2}{x_0^2}-1\right] \times\frac{\Gamma\left[\frac{p+d}{2}-1\right]}{\Gamma\left[\frac{d}{2}\right]} \times\exp-4\left[\frac{d}{2}-1\right]t . \tag{5.12}$$

For $d=4$ the purely exponential decay disappears. The singular dependence of the amplitude on x_0 should be noted. It indicates, that the limits $t\to\infty$ and $x_0\to 0$ may not be interchanged.

(iii) Contribution from $n=0$ for $d<-2p$. For $d<-2p$ the relaxation of the moment $\langle x^p(t)\rangle$ to zero is dominated by the exponential $\exp-\lambda_0(p)t$ in Eq. (5.7). We obtain

$$\langle x^p(t)\rangle \simeq 2^{(p/2)-1}(d+2p)\frac{\Gamma\left[-\frac{p}{2}-\frac{d}{2}\right]}{\Gamma\left[1-\frac{d}{2}-p\right]} \times {}_2F_0\left[\frac{p}{2},-\frac{d+p}{2};-\frac{2}{x_0^2}\right]e^{-\lambda_0(p)t}. \tag{5.13}$$

For $d=-2p$ the purely exponential decay again disappears.

(iv) Contribution from the continuum for $-2p<d<4$, $d\neq 0$. In the domain $-2p<d<4$ the relaxation of the moment $\langle x^p(t)\rangle$ towards its steady-state value (which is zero for $d<0$ and finite for $d>0$) is dominated by the lower part of the continuum. From Eq. (5.4) for $d\neq -2p,0,4$ we obtain

$$\langle x^p(t)\rangle - \langle x^p\rangle \simeq \frac{1}{2}\left[\Gamma\left[-\frac{d}{4}\right]\Gamma\left[\frac{d}{4}+\frac{p}{2}\right]\right]^2 M(p,d) \times (4t)^{-3/2}\exp-\frac{d^2}{4}t \tag{5.14}$$

with

$$M(p,d)=\frac{2^{p/2}}{\sqrt{\pi}}\Gamma^{-1}\left[\frac{p}{2}\right]{}_2F_0\left[-\frac{d}{4},-\frac{d}{4};-\frac{2}{x_0^2}\right]. \tag{5.15}$$

Thus, the decay of the moments is dominated by the eigenvalue at the lower boundary of the continuum. The algebraic prefactor $t^{-3/2}$ which modifies the exponential decay represents the influence from the continuum. Owing to the prefactors in Eq. (5.14) the coefficient of $t^{-3/2}\exp-(d^2/4)t$ diverges at the three points $d=d_0$ like $(d-d_0)^{-2}$ where $d_0=0,4$ and $-2p$, respectively. A similar divergence for $d=4$ was encountered in Sec. IV B in the study of the asymptotic behavior of the correlation functions which are nonzero only for $d>0$. It is interesting to note, that the asymptotic results for the moments have a similar divergence also for vanishing and negative d. The divergence signals, that the relaxation of the moments at long times for $d=0,4,-2p$ is different from the $t^{-3/2}e^{-d^2t/4}$ behavior. As for the correlation functions we expect a $t^{-1/2}e^{-d^2t/4}$ behavior.

(v) The case $d=4n$. In this case the integrand near $\kappa^2=0$ is modified due to a singular contribution from

$$|\Gamma(i\kappa-n)|^2\sim\frac{1}{\kappa^2}\left[\frac{1}{n!}\right]^2 \tag{5.16}$$

and we obtain

$$\langle x^p(t)\rangle \simeq \left[\frac{\Gamma\left[n+\frac{p}{2}\right]}{n!}\right]^2 M(p,4n)\frac{e^{-4n^2t}}{(4t)^{1/2}} + \langle x^p\rangle . \tag{5.17}$$

In particular, for $n=0$, i.e., $d=0$:

$$\langle x^p(t)\rangle \simeq 2^{(p/2)-1}\frac{\Gamma\left[\frac{p}{2}\right]}{\sqrt{\pi t}} . \tag{5.18}$$

This result is remarkable, since it does not depend on the initial value x_0, as long as $x_0\neq 0$. Thus, at $d=0$ the moments relax by the power law $t^{-1/2}$ with a universal amplitude. For $n=1$, i.e., $d=4$ we obtain

$$\langle x^p(t)\rangle - \langle x^p\rangle \simeq \left[\frac{p}{2}\right]^2\Gamma\left[\frac{p}{2}\right]\left[1-\frac{2}{x_0^2}\right]2^{(p/2)-1} \times \frac{1}{\sqrt{\pi t}}e^{-4t} . \tag{5.19}$$

(vi) The case $d=-2p-4n$. In this case, which requires strong linear damping, the integrand near $\kappa^2=0$ is again modified due to a singular factor of the form

$$|\Gamma(i\kappa-n)|^2\simeq\frac{1}{\kappa^2}\left[\frac{1}{n!}\right]^2 . \tag{5.20}$$

We obtain, therefore,

$$\langle x^p(t)\rangle \simeq \left[\frac{\Gamma\left[n+\frac{p}{2}\right]}{n!}\right]^2 M(p,-2p-4n) \times \frac{1}{\sqrt{4t}}\exp-4\left[n+\frac{p}{2}\right]^2 t . \tag{5.21}$$

In particular, for $n=0$, i.e., $d=-2p$, we obtain

$$\langle x^p(t)\rangle \simeq 2^{p/2-1}\Gamma\left[\frac{p}{2}\right]{}_2F_0\left[\frac{p}{2},\frac{p}{2};-\frac{2}{x_0^2}\right]\frac{e^{-p^2t}}{\sqrt{\pi t}}\ . \tag{5.22}$$

Cases (v) and (vi) are not observable, in principle, since values of d cannot be prescribed to arbitrary precision. However, the crossover from the $t^{-3/2}e^{-d^2t/4}$ behavior to the $t^{-1/2}e^{-d^2t/4}$ behavior manifests itself in an enhancement of the amplitude of $t^{-3/2}e^{-d^2t/4}$ which is, in principle, observable, and would be interesting to look for in experiments.

An asymptotic evaluation of the transient moments was attempted before in the paper of Brenig and Banai,[8] starting from Eq. (3.8). Their results differ from ours due to an erroneous evaluation of the limit $t\rightarrow\infty$ in their paper.

VI. CONCLUSIONS

We now briefly summarize our results and present our final conclusions. Carleman's method of linear imbedding has been applied in this paper to the solution of the stochastic problem (1.10) and its associated deterministic problem. For the applicability of the method, the triangular form of the matrix representing the linear problem, Eqs. (2.6) and (3.3), is crucial. While deterministic rate equations always lead to such a triangular form, it is not obtained in stochastic processes, unless they contain the noise sources in a multiplicative way. Hence, the applicability of the method for stochastic processes is at present much more restricted than for deterministic processes.

A particular difficulty of the method lies in the fact, that an artificial time dependence is introduced via the linear imbedding. This time dependence is directly related to the linear part of the given nonlinear problem, and has nothing to do with the real physical time dependence. In particular, the artificial time dependence depends on and changes with the coordinates used to represent the problem. In the processes studied here, the linear part for $d>0$ leads to unlimited amplification, while the complete nonlinear process saturates and relaxes towards a steady state. The physical time dependence has to be extracted from the artificial time dependence introduced via linear imbedding by exact summation of an infinite series. This summation is the crucial step of the method. It appears that any approximation made before this step has been carried out introduces errors in a completely uncontrolled way. Hence, it seems that the method remains restricted to exactly solvable problems, unless one can find further criteria, on how to choose the linear imbedding in order to control errors introduced by later approximations.

The model studied in this paper is exactly solvable. Therefore, the above-mentioned difficulty could be overcome. The infinite series of the deterministic problem, Eq. (2.13), was summed up exactly in Eq. (2.14). The corresponding infinite series of the stochastic problem, Eqs. (3.8) and (3.14), were summed up exactly in Eqs. (5.1) and (4.4), respectively, and represented by contour integrals. In this way, exact integral representations of the stationary two-time correlation functions and the transient moments of the stochastic models have been obtained for $d\geq 0$ and $d\leq 0$.

A simple deformation of the contour in the integral representation was enough in order to reproduce the spectral representation of the correlation function and moments which follow from the solution of the Fokker-Planck equation for $d\geq 0$. The results of the Fokker-Planck analysis are thereby completely confirmed. In particular, correlation functions and moments for long times relax exponentially for $d>4$, with a rate $\lambda_1=2(d-2)$.

One remarkable result of the Fokker-Planck analysis[6] was that this eigenvalue disappears continuously from the spectrum at $d=4$, thus preventing the system from critical slow down and undergoing a noise-induced transition at $d=2$. This result emerged from the Fokker-Planck analysis by using the boundary condition at $x=0$. As mentioned in Sec. I this boundary condition and the result it implies was challenged by Suzuki *et al.*[7] However, in the analysis given here this result is completely confirmed, without the necessity to ever invoke explicitly a boundary condition for the probability density at $x=0$. Instead, the appearance or disappearance of the eigenvalue λ_1 is here due to the crossing of the real axis of a corresponding pole in the integral representation of the correlation functions and moments. Therefore, our results definitely show that the boundary conditions used in the Fokker-Planck analysis were indeed adequate, and that a noise-induced transition due to critical slowing down at some finite positive value of d does not exist in the model studied here.

The integral representation for the transient moments obtained here has also been used to obtain results for $d<0$. If d deeply enters the negative domain ($d<-2p$, where p is the order of the moment), new poles cross the real axis from both sides leading again to purely exponential decay at long times. The decay rates $\lambda_n(p)$, Eq. (5.8), depend explicitly on p. The Fokker-Planck spectrum in the same domain consists only of a continuum, which is higher than all $\lambda_n(p)$, for $0\le n\le n_1$, n_1 being the largest integer smaller than $|d/4|-p/2$, without any trace of the discrete values $\lambda_n(p)$. This raises the question of how the two results are reconciled.

The relaxation of the moments for $d<0$ cannot, so far, be calculated by the Fokker-Planck method, since the expansion in eigenfunctions is not possible for $d<0$ (cf. Appendix B). However, an analogous problem is already posed by the linear equation studied in Appendix A. The Fokker-Planck spectrum of that linear problem is purely continuous, while the moments relax exponentially at long times, with rates which depend on the order of the moment. Thus, this example demonstrates that our results for the transient moments for $d<0$ do not contradict the Fokker-Planck spectrum.

In the domains $0<d<4$ and $-2p<d<4$ the correlation functions and transient moments, respectively, relax nonexponentially at long times like $t^{-3/2}\exp-(d^2/4)t$. The algebraic prefactor is due to the continuum of eigenvalues above $d^2/4$. The correction to this behavior is smaller by a factor t^{-1}. Near the points $d=d_0$ with $d_0=-2p,0,4$ the coefficients in front of $t^{-3/2}\exp-(d^2/4)t$ are enhanced like $(d-d_0)^{-2}$. If $d=d_0$, the long-time behavior of correlation functions ($\neq 0$ only for $d_0=4$) and moments is like $t^{-1/2}\exp-(d^2/4)t$. For $d=0$, the moments at long times relax with an amplitude which is universal, i.e., independent of the initial condition and the strength of the nonlinearity [cf. Eq. (1.12)]. However, since d cannot be precisely prescribed, only the enhancement of the prefactor of $t^{-3/2}\exp-(d^2/4)t$ near $d=d_0$ is observable, and presents a challenge for observation in experiments with systems described by Eq. (1.11).[17]

APPENDIX A: LINEAR PROCESSES

1. The linear process $\dot{x}=-|d|x+xF(t)$

In this appendix we want to study the process

$$\dot{x}=-|d|x+xF(t) \tag{A1}$$

with Gaussian $F(t)$. Equation (A1) is to be interpreted in the sense of Stratonovich. This process is obtained from the nonlinear process Eq. (1.10) by dropping the nonlinear term, and choosing $d<0$. Equation (A1) is interesting to study because it is solvable in closed form, but displays nevertheless, some of the peculiarities of the nonlinear process for $d<0$, which were found in Sec. V. In particular, we wish to understand how a purely exponential decay of the transient moments for long times can be reconciled with a Fokker-Planck spectrum which consists *only* of a continuum above the exponential decay rate.

First we solve Eq. (A1) by the same method as described in Sec. III. The linear infinite hierarchy of equations associated with (A1) via

$$y_n(t)=\langle x^n(t)\rangle,\quad y_n(0)=x_0^n\ , \tag{A2}$$

is

$$\dot{y}_n(t)=n(-|d|+n)y_n\ , \tag{A3}$$

with the solution

$$y_n(t)=x_0^n\exp-\lambda(n)t\ ,\qquad \lambda(n)=n(|d|-n)\ . \tag{A4}$$

According to Eq. (A4) all moments $\langle x^n(t)\rangle$ with $n<|d|$ relax to zero, while all other moments with $n>|d|$ increase with time, due to the fluctuations in Eq. (A1). For all finite times $t<\infty$, all moments $\langle x^n(t)\rangle$ of the process exist.

Let us now see, ow these results emerge from the Fokker-Planck analysis. The Fokker-Planck equation of this process reads

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial x}\left[|d|-1\right]xP+\frac{\partial^2}{\partial x^2}x^2P\ , \tag{A5}$$

where $P(x\,|\,x_0,t)$ is the conditional probability density satisfying

$$P(x\,|\,x_0,t=0)=\delta(x-x_0)\ . \tag{A6}$$

Equation (A5) with (A6) is easily solved in closed form yielding

$$P(x\,|\,x_0,t)=\frac{1}{\sqrt{4\pi t}}\frac{1}{x}\exp\left[-\frac{\left[\ln\frac{x}{x_0}+|d|t\right]^2}{4t}\right]. \tag{A7}$$

It is easily checked, that despite the prefactor x^{-1}, the probability density (A7) is normalized for all finite times $t<\infty$, and that its moments $\langle x^n(t)\rangle$ for all finite times are given by Eq. (A4). Thus, the moment $\langle x(t)\rangle$, e.g., relaxes to zero for $|d|>1$ via

$$\langle x(t)\rangle = x_0 \exp-(|d|-1)t \tag{A8}$$

with a relaxation rate $\lambda(1)=|d|-1$. Let us now see, how this rate $\lambda(1)$ compares with the long-time behavior of $P(x\,|\,x_0,t)$. Rewriting Eq. (A7) we obtain

$$P(x\,|\,x_0,t) = \frac{1}{x}\left[\frac{x}{x_0}\right]^{-|d|/2} \frac{1}{\sqrt{4\pi t}} e^{-d^2t/4} e^{-\{[\ln(x/x_0)]^2/4t\}} . \tag{A9}$$

Expanding the second exponential into a Taylor series, which is allowed for $x\neq 0$, we obtain

$$P(x\,|\,x_0,t)=\frac{1}{x}\left[\frac{x}{x_0}\right]^{-|d|/2}\frac{e^{-d^2t/4}}{\sqrt{4\pi t}} \times \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(4t)^{-n}\left[\ln\frac{x}{x_0}\right]^{2n} . \tag{A10}$$

For $t\to\infty$, we obtain asymptotically

$$P(x\,|\,x_0,t)\to\frac{1}{x}\left[\frac{x}{x_0}\right]^{-|d|/2}\frac{e^{-d^2t/4}}{\sqrt{4\pi t}} . \tag{A11}$$

Three features of this result are worth noting:

(i) The decay of $P(x\,|\,x_0,t)$ at long times is an exponential, modified by an algebraic prefactor. This result has also been encountered for the nonlinear process and was associated there with a continuum of eigenvalues above the rate in the exponential $d^2/4$. Thus, we expect (and this expectation is confirmed below) that such a continuum is also present here. No discrete eigenvalues smaller than $d^2/4$ can be present in the spectrum of the Fokker-Planck operator, because they would otherwise dominate the long-time behavior of P.

(ii) The lower boundary of the continuum at $d^2/4$ is equal or above the largest decay rate $\lambda(n)$ [Eq. (A9) of the moments]. This shows us, that in the solution (A7) an exponential decay of the moments is reconciled with a continuous Fokker-Planck spectrum above the decay rate.

(iii) We will see below, that this reconciliation is closely related with a third feature of Eq. (A11); while $P(x\,|\,x_0,t)$, according to Eq. (A7), is normalizable at all finite times t, its asymptotic expansions around infinite t, as seen in Eqs. (A10) and (A11), are no longer normalizable at $x=0$. This shows us that the integration of P over x cannot be exchanged with the limit $t\to\infty$. It also warns us that the eigenfunctions of the Fokker-Planck operator, which dominate the long-time behavior of the conditional probability at $t\to\infty$, will not be integrable at $x=0$.

Let us now turn to the eigenvalue problem

$$\frac{\partial}{\partial x}(|d|-1)xP_\lambda(x)+\frac{\partial^2}{\partial x^2}x^2P_\lambda=-\lambda P_\lambda \tag{A12}$$

associated with Eq. (A5). First, we look at its solution for $\lambda=0$ and require that its probability current

$$j_0(x)=(|d|-1)xP_0+\frac{\partial}{\partial x}x^2P_0 \tag{A13}$$

vanished at infinity, $j_0(\infty)=0$, and hence everywhere, $j_0\equiv 0$. We obtain

$$P_0(x)=x^{-1-|d|} \tag{A14}$$

which is not normalizable, and therefore, not the steady-state distribution. In addition, Eq. (A13) admits the solution

$$w_0(x)=\lim_{\epsilon\to 0}\frac{2}{\sqrt{\pi}}\frac{\Gamma\left[\frac{1}{2}+\frac{|d|}{2}\right]}{\Gamma\left[\frac{|d|}{2}\right]} \times\frac{\epsilon^{|d|}}{(x^2+\epsilon^2)^{1/2+|d|/2}}=\delta(x) , \tag{A15}$$

which is the proper steady-state distribution in our case. The explicit representation of the δ function in Eq. (A15) is obtained if additive Gaussian white noise with intensity $\sim\epsilon^2$ is included in Eq. (A13) and the limit $\epsilon\to 0$ is taken. Equation (A15) shows that $w_0(x)$ reduces to a δ function only for test functions increasing slower than $x^{|d|}$ for $x\to\infty$. In particular, only the moments x^n of $w_0(x)$ exist with $n<|d|$. These moments, of course, all vanish.

We now turn to the eigenfunctions with $\lambda\neq 0$. In the usual way,[5] the eigenfunctions of Eq. (A13) can be used as an orthogonal basis in a function space with the scalar product

$$\langle\lambda\,|\,\lambda'\rangle=\int dx\frac{P_\lambda(x)P_{\lambda'}(x)}{P_0(x)}$$
$$=\begin{cases}\delta_{\lambda,\lambda'} & \text{discrete spectrum}\\ \delta(\lambda-\lambda') & \text{continuous spectrum}\,.\end{cases} \tag{A16}$$

The only condition on the weight function $P_0(x)$ is that the probability current associated with $P_0(x)$ vanishes identically. The boundary conditions required for the $P_\lambda(x)$, $\lambda\neq 0$ now follow from the definition of the scalar product (A16). Equation (A12) is solved by the functions

$$P_\lambda^{(\pm)}(x)=x^{-1-|d|/2\pm[(d^2/4)-\lambda]^{1/2}}\,. \tag{A17}$$

Integrability at $x=\infty$ and $x=0$ in Eq. (A16) requires

$$\mathrm{Re}\left[\frac{d^2}{4}-\lambda\right]^{1/2}\leq 0,\quad \mathrm{Re}\left[\frac{d^2}{4}-\lambda\right]^{1/2}\geq 0\,, \tag{A18}$$

respectively, which forces us to choose

$$\mathrm{Re}\left[\frac{d^2}{4}-\lambda\right]^{1/2}=0,\quad \lambda>\frac{d^2}{4}\,. \tag{A19}$$

The corresponding eigenfunction P_λ are then given by those two linear combinations of $P_\lambda^{(+)}$ $P_\lambda^{(-)}$ which are normalized onto the δ function in the scalar product (A16). Two appropriate linear combinations are given by

$$P_\lambda^{(1)}=\frac{1}{\sqrt{8\pi}(\lambda-d^2/4)^{1/4}}[e^{-i\pi/4}P_\lambda^{(+)}(x)+e^{i\pi/4}P_\lambda^{(-)}(x)]\,,$$
$$P_\lambda^{(2)}=\frac{1}{\sqrt{8\pi}(\lambda-d^2/4)^{1/4}}[e^{i\pi/4}P_\lambda^{(+)}(x)+e^{-i\pi/4}P_\lambda^{(-)}(x)]\,. \tag{A20}$$

Hence, we have found the continuous spectrum predicted above. Every eigenvalue is doubly degenerate. None of the eigenfunctions $P_\lambda^{(1,2)}(x)$ is integrable over x.

It is important to note that $P_0(x)$ itself is not a member of the function space defined by (A16), since the scalar product (A16) is ill defined if $P_0(x)$ is substituted for one or both $P_\lambda(x)$ in (A16). Let us now discuss the asymptotic behavior for long times of the conditional probability density and its moments in terms of the eigenfunction expansion, and see how their different behavior is reconciled in this description. The expansion of $P(x\,|\,x_0,t)$ reads

$$P(x\,|\,x_0,t)=\sum_{\nu=1,2}\int_{d^2/4}^{\infty}d\lambda\,e^{-\lambda t}\frac{P_\lambda^{(\nu)}(x)P_\lambda^{(\nu)}(x_0)}{P_0(x_0)}\,. \tag{A21}$$

Inserting Eq. (A20), the integral may be carried out and Eq. (A7) is reobtained. For $t\to\infty$ a saddle-point evaluation of Eq. (A21) is appropriate where only the eigenvalues λ near the lower boundary $d^2/4$ contribute, and we indeed obtain (A11) as the leading term. For the moments of $P(x\,|\,x_0,t)$:

$$\langle x^n(t)\rangle=\int dx\,P(x\,|\,x_0,t)x^n\,, \tag{A22}$$

a dilemma seems to appear, since, if we use Eq. (A21) in (A22) and if we would exchange the order of the integrals over λ and x, we would obtain

$$\langle x^n(t)\rangle=\sum_{\nu=1}^{2}\int_{d^2/4}^{\infty}d\lambda\,e^{-\lambda t}\frac{P^{(\nu)}(x_0)}{P_0(x_0)}\int dx\,x^nP_\lambda^{(\nu)}(x)\,. \tag{A23}$$

If Eq. (A23) were correct, we would have to conclude in the usual way that the long-time behavior of $\langle x^n(t)\rangle$ is *also* determined by the low lying eigenvalues λ in the vicinity of $d^2/4$. This would contradict our earlier result, that the moments relax with a *slower* rate than $d^2/4$. Why is Eq. (A23) wrong? The error was made, when the order of the integrals over x and over λ were interchanged. This exchange is not allowed, since the integral over x in Eq. (A23) does not converge. We therefore reach the important conclusion: The fact that the moments of the eigenfunctions do not exist allows the moments of the conditional probability density to decay with rates which do not belong to the Fokker-Planck spectrum. For the same reason, the expansion in eigenfunctions of the Fokker-Planck operators appears to be useless for the calculation of the moments in such cases.

For the nonlinear problem (1.10) the moments of the eigenfunction in the domain $d<0$ do not exist. Hence, the moments can relax with rates outside the Fokker Planck spectrum.

2. The linear process as a limit of the nonlinear process

In this section we ask how the results of (A4) for the transient moments of the process (A1) for

$d<0$, may be obtained from the results of Sec. V A. This is instructive for two reasons. First, by considering the limit of vanishing nonlinearity we can check whether the results of Sec. V A are consistent with the results of Sec. A 1 of this appendix. Secondly, it will be interesting to see explictly, how the two different regions $-2p<d<0$ and $d<-2p$, which we had to distinguish for the nonlinear results in Sec. V, merge and are connected in the linear case.

In order to be able to send the coefficient b of the nonlinearity to zero we go back to Eq. (1.12) and reintroduce b by replacing

$$x \to \sqrt{b}\,x,\quad x_0 \to \sqrt{b}\,x_0\ , \tag{A24}$$

in Eqs. (5.3), (5.4), (5.7), and (5.8). In the resulting expressions we take the limit $b \to 0$. We have to consider the following two cases:

(i) $-2p<d<0$. In this case Eq. (5.4) in the limit $b \to 0$ asymptotically reduces to

$$y_p(t)=y_p^c(t)\ , \tag{A25}$$

$$y_p^c(t)=x_0^p\int_{-\infty}^{\infty}\frac{d\kappa}{2\pi}\exp[-\lambda(\kappa^2)t]\left[\frac{bx_0^2}{2}\right]^{i\kappa-d/4-p/2}\times\frac{\left|\Gamma\left[-i\kappa+\frac{d}{4}+\frac{p}{2}\right]\right|^2\Gamma\left[i\kappa-\frac{d}{4}\right]}{\Gamma\left[\frac{p}{2}\right]\Gamma(2i\kappa)}\ , \tag{A26}$$

where the limit $b \to 0$ has already been taken in terms which converge to a finite constant. The contour of the κ integral along the real axis may be closed at infinity in the lower half κ plane. Then only the poles of $\Gamma(-i\kappa+d/4+p/2)$ at

$$\kappa=-in-i\frac{d}{4}-i\frac{p}{2},\quad n=0,1,2,\ldots, \tag{A27}$$

contribute. The residue of the integrand in (A26) at the nth pole contains a factor $(bx_0^2/2)^n$. Thus, in the limit $b \to 0$, only the pole with $n=0$,

$$\kappa=-i\frac{d}{4}-i\frac{p}{2}\ , \tag{A28}$$

retains a finite residuum. With

$$\lambda\left[-\left[\frac{p}{2}+\frac{d}{4}\right]^2\right]=p(\,|\,d\,|\,-p) \tag{A29}$$

we obtain

$$y_p(t)=x_0^p\exp-p(\,|\,d\,|\,-p)t\ . \tag{A30}$$

Thus, the result (A4) in the region $-2p<d<0$ has been obtained from Eq. (5.4) in the limit $b \to 0$.

(ii) $d<-2p$. In this case we have to consider, according to Eq. (5.7), the contribution from the integral (A26) and from the sum in Eq. (5.7). We first consider the integral. Again we close the contour along the real axis through the lower half κ plane at ∞. Because d has dropped below $-2p$, the pole at

$$\kappa=-i\frac{d}{4}-i\frac{p}{2}\ ,$$

which contributed in case (i), has crossed into the upper half plane. Hence, this pole no longer contributes. Instead, new poles have crossed the real axis into the lower half plane. These are the poles of

$$\Gamma\left[i\kappa+\frac{d}{4}+\frac{p}{2}\right],$$

which satisfy

$$i\kappa+\frac{d}{4}+\frac{p}{2}=-n \tag{A31}$$

with

$$0\le n\le n_1 \tag{A32}$$

where n_1 is the smallest nonnegative integer smaller than $|d/4|-p/2$. The residua of the integrand of (A26) at these poles contain a factor $(bx_0^2/2)^{-n-d/2-p}$. The exponent is positive since

$$n+\frac{d}{2}+p\le n_1+\frac{d}{2}+p<\frac{d}{4}+\frac{p}{2}<0\ . \tag{A33}$$

Hence, all residua vanish in the limit $b \to 0$, and

$$\lim_{b\to 0}y_p^c(t)=0\ . \tag{A34}$$

Thus, for $b \to 0$, only the sum in Eq. (5.7) remains. Reexpressing ${}_2F_0$ in terms of ${}_1F_1$ and taking $b \to 0$, it is easy to show that only the term with $n=0$ survives, and we obtain again

$$y_p(t)=x_0^pe^{-p(\,|\,d\,|\,-p)t}$$

for $d<-2p$ in agreement with Eq. (A4). Thus the two domains $-2p<d<0$ and $d<-2p$ of the nonlinear process merge in the linear limit, as had to be expected from the results of Sec. A 1 of this appendix. We conclude that the results for the nonlinear and the linear process are consistent with each other.

APPENDIX B: FOKKER-PLANCK TREATMENT OF THE PROCESS OF EQ. (1.10)

The stochastic process defined through the Langevin equation (2.17) can be characterized as well by the stochastically equivalent Fokker-Planck equation

$$\frac{\partial}{\partial t}P(x,t)=-\frac{\partial}{\partial x}[(d+1)x-x^3]P+\frac{\partial^2}{\partial x^2}x^2P \tag{B1}$$

for the probability density $P(x,t)$. This equation has been solved exactly by analytical methods.[6] It is the purpose of this appendix, to summarize briefly the previously derived Fokker-Planck results along with some new asymptotic limits in order to allow a direct and easy comparison with the results derived from the method of linear imbedding presented in this paper.

1. Stationary solution

From an arbitrary initial condition, the system relaxes towards the stationary distribution

$$\lim_{t\to\infty}P(x,t)=w_0(x)$$

$$=\begin{cases}N_0x^{-1+d}e^{-x^2/2}, & d>0\\ \delta(x), & d\leq 0\end{cases} \tag{B2}$$

where

$$N_0=2^{1-d/2}\Gamma^{-1}\left[\frac{d}{2}\right]$$

guarantees the normalization of the probability density. The stationary moments assume the following form:

$$\lim_{t\to\infty}\langle x^n(t)\rangle=2^{n/2}\Gamma\left[\frac{d+n}{2}\right]\Big/\Gamma(d/2)\,. \tag{B3}$$

We note, that $\delta(x)$ also solves the stationary Fokker-Planck equation for $d>0$ but this solution is not approached asymptotically from an arbitrary initial condition as long as $d>0$.

2. Solution of the eigenvalue problem

The Fokker-Planck equation (B1) is solved in terms of an eigenfunction expansion which consists of a discrete as well as a continuum branch

$$P(x,t)=\sum\!\!\!\!\!\!\int_n c_nP_n(x)e^{-\lambda_n t}\,. \tag{B4}$$

By imposing L_2 integrability we construct a linear unitary function space which allows us to associate the expansion coefficients c_n with the projections of the initial distribution onto the basis set P_n.

(a) Discrete branch of the spectrum. We impose the rigorous condition of L_2 integrability[5]

$$\int P_n^2(x)P_0^{-1}(x)dx<\infty\,, \tag{B5}$$

where the function $P_0(x)$ is a solution of (B1) with vanishing probability current, i.e.,

$$[-(d+1)x+x^3]P_0(x)+\frac{\partial}{\partial x}x^2P_0=0\,.$$

For $d>0$, we can identify P_0 with w_0 given by (B2). For $d<0$ we also find

$$P_0=N_0x^{-1+d}\exp-\frac{x_0^2}{2}\,.$$

In the latter case, P_0 is different from the steady-state distribution. Imposing (B5) we obtain the discrete branch of the eigenvalue spectrum which contains only a finite number of eigenvalues. With n_0 equaling the largest positive integer smaller than $d^2/4$ we find

$$\lambda_n=2n(d-2n),\quad 0\leq n\leq n_0 \tag{B6}$$

$$P_n(x)=N_n^{1/2}x^{-1+d-2n}L_n^{(d/2)-2n}\left[\frac{x^2}{2}\right]e^{-x^2/2}\,, \tag{B7}$$

where

$$N_n=2^{2n+1-d}\frac{n!(d-4n)}{\Gamma(d/2)\Gamma(d/2+1-n)}\,.$$

L_n^m is the associated Laguerre polynomial.

(b) Continuous branch of the spectrum. The finite number of eigenfunctions Eq. (B7) obviously does not exhaust the entire Hilbert space and we have to somewhat relax the condition of (B5) in order to include the continuous branch as well.

If we replace Eq. (B5) by

$$\int P_s(x)P_{s'}(x')P_0^{-1}(x)dx=\delta(s-s') \tag{B8}$$

we obtain the following continuum of eigenfunctions:

$$\lambda(s)=\left[\frac{d^2}{4}+s^2\right],\quad s\geq 0 \tag{B9}$$

$$P_s(x)=N_s^{1/2}x^{(d/2)-2}e^{-x^2/4}W_{1/2+d/4,is/2}\left[\frac{x^2}{2}\right] \tag{B10}$$

with[18]

$$N_s=\frac{2^{1-d/2}}{\pi^2}s\sinh\pi s\left|\Gamma\left[-\frac{d}{4}+\frac{is}{2}\right]\right|^2\Gamma^{-1}(d/2)\,, \tag{B11}$$

where $W_{\lambda,\mu}(x)$ is the Whittaker function. Under the assumption that the basis set (B7) and (B10) spans the entire Hilbert space, we obtain a complete description of the stochastic process in terms of the transient probability $P(x,t)$, subject to an arbitrary initial condition. It also becomes possible to characterize the stochastic process by its stationary correlation functions or its transient moments.

3. The stationary correlation function

With the eigenfunction expansion Eq. (B4) we are in a position to calculate explicitly the stationary correlation function via

$$\lim_{t\to\infty}\langle x^p(t+\tau)x^q(t)\rangle=\sum\!\!\!\!\!\!\int_n g_n^p g_n^q e^{-\lambda_n\tau}\,, \tag{B12}$$

where

$$g_n^p=\int x^p P_n(x)dx\ .$$

Equation (B12) is valid under the assumption that integration over x can be exchanged with the summation over n. This assumption is not satisfied for $d<0$, as our discussion of the linear process in Appendix A has indicated. For $d>0$, we restrict ourselves here to the special case $p=q=1$ and obtain the following result:

$$\lim_{t\to\infty}\langle x(t+\tau)x(t)\rangle-\langle x^2(t)\rangle=\pi\sum_{n=1}^{n_0}\frac{(d-4n)\Gamma^2\left[\frac{d+1}{2}-n\right]}{n!\Gamma^2(\frac{1}{2}-n)\Gamma\left[\frac{d}{2}+1-n\right]\Gamma\left[\frac{d}{2}\right]}e^{-4n[(d/2)-n]\tau}$$
$$+\frac{e^{-d^2\tau/4}}{2\pi^3\Gamma\left[\frac{d}{2}\right]}\int_0^\infty s\sinh\pi s\left|\Gamma\left[\frac{d}{4}+\frac{1}{2}+\frac{is}{2}\right]\right|^4\left|\Gamma\left[-\frac{d}{4}+i\frac{s}{2}\right]\right|^2 e^{-s^2\tau}ds \tag{B13}$$

for $d>0$; n_0 was defined in Eq. (B6).

For $d\leq 0$ all correlation functions vanish identically. This is easily understood when we recall that the stationary distribution for $d\leq 0$ is given by $P_0(x)=\delta(x)$ [cf. Eq. (B2)]. The discrete eigenfunctions contribute a finite number of exponentially decaying terms, while the continuous branch leads to a more involved analytical structure. These two contributions can be separated due to their asymptotic time dependence.

a. For $0\leq d\leq 4$. The correlation functions is entirely determined by the continuous branch of the spectrum and, for large τ, assumes asymptotically the following form:

$$\lim_{t\to\infty}\langle x(t+\tau)x(t)\rangle-\langle x(t)\rangle^2\simeq\begin{cases}\dfrac{\Gamma^4(n+\frac{1}{2})}{\pi n!}\dfrac{e^{-4n^2\tau}}{\Gamma(2n)}[(\pi\tau)^{-1/2}+O(\tau^{-3/2})], & d=4n\\[2ex] \dfrac{1}{8}\dfrac{\Gamma^2\left[-\frac{d}{4}\right]}{\Gamma\left[\frac{d}{2}\right]}\Gamma^4\left[\dfrac{d/2+1}{2}\right]e^{-d^2\tau/4}[(\pi\tau)^{-3/2}+O(\tau^{-5/2})], & 0<d<4\end{cases}\ . \tag{B14}$$

The appearance of a nonexponential decay is not surprising in the regime where the eigenvalue spectrum is purely continuous.

b. For $d>4$. The asymptotic time dependence is dominated by the lowest discrete eigenvalue λ_1 and the correlation function can be written in the form[6]

$$\lim_{t\to\infty}\langle x(t+\tau)x(t)\rangle-\langle x(t)\rangle^2\simeq\left[\frac{d}{4}-1\right]\left[\frac{\Gamma\left[\frac{d-1}{2}\right]}{\Gamma\left[\frac{d}{2}\right]}\right]^2\exp-4\left[\frac{d}{2}-1\right]\tau\,. \tag{B15}$$

4. The transient moments

The explicit form of the transient time evolution of the moments depends on the choice of the initial condition for the probability density. A rather natural choice is the δ distribution located at an arbitrary value $x=x_0$. The corresponding time-dependent probability density is the conditional probability

$$P(x\mid x_0,t)=\sum\!\!\!\!\!\!\int_n P_0^{-1}(x_0)P_n(x_0)P_n(x)e^{-\lambda_n t}\,. \tag{B16}$$

The transient moments resulting from Eq. (B16) for $d>0$ can be written in the form

$$\langle x^m(t)\rangle-\langle x^m\rangle=\sum_{n=1}^{n_0}(-1)^n 2^{n-1+m/2}\left[\frac{m}{2}\right]_n\frac{\Gamma\left[\frac{d}{2}-n+\frac{m}{2}\right](d-4n)}{\Gamma\left[\frac{d}{2}-n+1\right]x_0^{2n}}L_n^{(d/2)-2n}\left[\frac{x_0^2}{2}\right]e^{-\lambda_n t}$$

$$+\Lambda_m\int_0^\infty s\sinh\pi s\left|\Gamma\left[-\frac{d}{4}+\frac{is}{2}\right]\right|^2\left|\Gamma\left[\frac{d}{4}+\frac{m}{2}+\frac{is}{2}\right]\right|^2 W_{(1/2)+d/4,is/2}\left[\frac{x_0^2}{2}\right]e^{-s^2t}ds \tag{B17}$$

with

$$\Lambda_n=\frac{x_0^{(d/2)-2}2^{-(d/4)-(1/2)+(n/2)}}{\pi^2\Gamma\left[\frac{n}{2}\right]\Gamma\left[\frac{d}{2}\right]}\times e^{-x_0^2/4}e^{-d^2\tau/4}P_0^{-1}(x_0)\,.$$

This result can be shown to coincide with (5.6). For $d<0$ integration over x and integration over s cannot be exchanged and hence the transient moments cannot be calculated this way.

In the asymptotic time regime for the case $m=2$ we obtain the following simplified relations: (a) In the regime $0\le d\le 4$ we obtain for $d=4n$,

$$\langle x^2(t)\rangle-\langle x^2\rangle\simeq\frac{2^n}{\sqrt{\pi t}}e^{-4n^2t}\times\sum_{m=0}^{n}\frac{(-1)^{n-4m}}{(n-m)!}\left[\frac{x_0^2}{2}\right]^{m-n} \tag{B18}$$

and for $0<d<4$,

$$\langle x^2(t)\rangle-\langle x^2\rangle\simeq\frac{1}{4}\left[\frac{\pi}{t}\right]^{3/2}\left[\frac{x_0^2}{2}\right]^{d/4}\times\frac{{}_2F_0\left[-\frac{d}{4},-\frac{d}{4};-\frac{2}{x_0^2}\right]}{1-\cos\frac{d}{2}\pi}\,. \tag{B19}$$

(b) For $d>4$ the asymptotic time dependence is governed by the lowest discrete eigenvalue λ_1 and leaves us with the simple exponential decay

$$\langle x^2(t)\rangle-\langle x^2\rangle\simeq(d-4)\left[1-\frac{d-2}{x_0^2}\right]e^{-4[(d/2)-1]t}\,. \tag{B20}$$

These results coincide with those of Sec. V B for $d\ge 0$.

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