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K-shell ionization during α decay

R. Anholt
Department of Physics, Stanford University, Stanford, California 94305

P. A. Amundsen*
Department of Physics, University of Oslo, Oslo, Norway

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Quantum-mechanical calculations of the probability of K-shell ionization in the α decay of Po and in α+Pb collisions are made. The initial and final electronic wave functions are approximated by Dirac hydrogenic wave functions centered on the Pb nucleus. Also, we include the contribution from the part of the trajectory where the α particle tunnels through the Coulomb barrier. The contribution from tunneling is small except at small α-particle velocities.

I. INTRODUCTION

Recent "time-delay" experiments which measure K-shell ionization probabilities in collisions where long-lived compound nuclei are formed are sensitive to the amplitude or probability of forming a K-shell vacancy in half of a collision, i.e., while the projectile is on the way in or the way out.[1-5] Aside from these scattering experiments, the only other way we have of measuring these quantities is by measuring internal ionization during the proton or α decay of a nucleus. Several measurements have been made of K- and L-shell ionization during the α decay of various polonium nuclei.[6-8] New semiclassical calculations of those ionization probabilities are in reasonable agreement with experiment.[9,10]

One aspect of internal ionization which has not been investigated thoroughly is the possible contribution to ionization while the α particle is tunneling through the Coulomb barrier. In the semiclassical approximation the probability of exciting an electron from an initial state i to a final state f in half of a collision is given by the square of the amplitude

$$a_{fi} = \int_0^\infty dt\, e^{i\omega t} \langle \phi_f \mid H' \mid \phi_i \rangle \ , \tag{1}$$

where H' is the perturbation Hamiltonian, and $\hbar\omega$ is the difference in the electronic binding energies. The time integral in Eq. (1) is done along a Coulomb trajectory starting at an internuclear distance $R = D$, where the repulsion between the α particle and daughter nucleus is equal to the final center-of-mass kinetic energy. In the decay of ^{210}Po, D is approximately 50 fm, thus the α particle must tunnel a distance of $\sim$40 fm through the Coulomb barrier. (The Po nuclear radius R_N is approximately 10 fm.)

Although no quantitative calculations have been made of the tunneling contribution to K-shell ionization, Law[9] has pointed out that this contribution might be sensitive to the α-decay mechanism. He compared the ionization probabilities calculated by assuming that the α particle tunnels through the barrier instantly and infinitely slowly. The probabilities differed considerably for high kinetic energy of the ionized electron. However, calculations incorporating α-decay dynamics have not yet been done.

In this paper we make quantum-mechanical (QM) calculations of ionization during α decay. To describe the motion of the α particle, we use the standard particle-in-a-well picture.[11,12] Inside and outside of the Coulomb Barrier, WKB α-particle wave functions are used, which are matched to the wave functions inside of the nucleus at the nuclear radius R_N. Using these wave functions we show that the amplitude for K-shell ionization is given by the amplitude in Eq. (1) plus a "tunneling-correction" amplitude. The properties of this amplitude are discussed in Sec. II D, and the results are compared with measurements of Lund *et al.*[6,8] and Fischbeck and Freedman[7] in Sec. III.

Atomic units are used throughout this paper.

II. QUANTUM-MECHANICAL THEORY OF IONIZATION DURING α DECAY

A. Boundary conditions

Normally one calculates the probability of exciting an electron from state ϕ_i to state ϕ_f in a collision where the incoming particle with momentum $\vec{K}$ scatters to the final momentum $\vec{K}'$. The probability is proportional to the square of the matrix element of the perturbing Hamiltonian H' (Ref. 13)

$$\langle \Psi^-(\vec{K}',\vec{R})\phi_f \,|\, H'(\vec{r},\vec{R}) \,|\, \phi_i \Psi^+(\vec{K},\vec{R})\rangle \ , \qquad (2)$$

where $\Psi^{\pm}$ is given by

$$\Psi^{\pm}(\vec{K},\vec{R}) = \frac{4\pi}{KR}\sum_{LM}(i)^L e^{\pm i\delta_L}\mathcal{F}_L(KR)Y^*_{LM}(\hat{K})Y_{LM}(\hat{R}) \ , \qquad (3)$$

δ_L is the Coulomb + nuclear phase shift, and $\mathcal{F}_L(KR)$ is the exact solution to the radial Schrödinger equation for the projectile-target nucleus potential $V(R)$. $\mathcal{F}_L$ goes to zero at $R=0$ and asymptotically approaches a sine wave.

The boundary conditions on Ψ^+ and Ψ^- arise from different sources. Ψ^+ corresponds to the preparation of the initial state as a monoenergetic beam of large cross section with a definite direction $\vec{K}$. Ψ^- corresponds to the detector detecting at $R=\infty$ particles of a definite momentum $\vec{K}'$. Note that the demand for a stationary solution in practice means that any experiment must run longer than the lifetimes of the nuclear or electronic states.

Ψ^- and Ψ^+ are regular solutions, which approach zero at $R=0$. In the case of α decay the physical situation is different, since the experiment is carried out on a time scale much shorter than the nuclear lifetime. The stationary wave describing the α particle must therefore be a purely outgoing spherical wave. It will consequently have a singularity at the origin, corresponding to a source of α particles there, which ensures probability conservation. The relevant matrix element in the case of $L=0$ α decay is therefore

$$\langle \Psi^-(\vec{K}',\vec{R})\phi_f \,|\, H'(\vec{r},\vec{R}) \,|\, \phi_i \mathscr{R}_0^+(\vec{K},\vec{R})/R\rangle \ . \qquad (4)$$

Asymtotically $\mathscr{R}_0^+$ is given by [13]

$$\mathscr{R}_0^+ = \exp[i(KR-\eta\ln 2KR)] \ , \qquad (5)$$

where $\eta = Z_1Z_2/v$. Note that $\Psi^-(\vec{K}',\vec{R})$ is unchanged, corresponding to the same detector detecting particles of a definite momentum $\vec{K}'$. Furthermore this wave function, being regular at the origin, behaves very differently within the Coulomb barrier than the resonant wave $\mathscr{R}_0^+$. The resonant wave is matched to a bound-state wave function inside the nucleus, hence decays exponentially with increasing R until it arrives at the Coulomb barrier D, where it becomes a free-particle wave function. Since the energy E' of the final-state wave function Ψ^- differs from the initial energy E by the ionization energy $\hbar\omega$ needed to excite the K electron, the final-state wave function cannot match to the same bound-state wave function inside the nucleus, hence it decays exponentially with decreasing R. These wave functions are derived in the following section.

B. α-particle wave functions

In the standard particle-in-a-well picture, the potential on the α particle is given by[11,12]

$$V(R)=\begin{cases} V_C & R>R_N \ \text{ region II} \\ V_N & R<R_N \ \text{ regionI} \end{cases} \ , \qquad (6)$$

where V_C is the Coulomb potential Z_1Z_2/R, and V_N ($\sim$const) is the nuclear potential. The regular and irregular solutions in region II for a given angular momentum L can be written as

$$F_L^{\rm II} = F_L^C\cos\eta_L + G_L^C\sin\eta_L$$

and (7)

$$G_L^{\rm II} = G_L^C\cos\eta_L - F_L^C\sin\eta_L \ ,$$

where F_L^C and G_L^C are Coulomb wave functions. We require that at R_N,

$$F_L^{\rm I}(R_N)=F_L^{\rm II}(R_N) \ , \quad G_L^{\rm I}(R_N)=G_L^{\rm II}(R_N) \ ,$$

$$f_L^{\rm I}(R_N)=f_L^{\rm II}(R_N) \ , \quad g_L^{\rm I}(R_N)=g_L^{\rm II}(R_N) \ , \qquad (8)$$

where

$$f_L^{\rm II}(R) = \frac{R}{F_L^{\rm II}}\frac{d}{dR}F_L^{\rm II} \ ,$$

etc.

The solutions in region I are restricted by the requirement that $F_L^{\rm I}(0)=0$. By using Eqs. (7) and (8) we can then determine η_L uniquely. $G_L^{\rm I}$ is then the unique irregular solution for V_N that can be

fitted to G_L^{II} at R_N once η_L is determined.

As is usual in studies of α decay, we approximate the Coulomb wave functions F_L^C and G_L^C with WKB wave functions[14]

$$G_L^C=\phi_L^{-1/4}e^{W_L},\quad F_L^C=\tfrac{1}{2}\phi_L^{-1/4}e^{-W_L},$$
$$\phi_L(R)=\frac{D}{R}+\frac{(L+\frac{1}{2})^2}{(KR)^2}-1, \tag{9}$$
$$W_L(R)=K\int_R^D\phi_L^{1/2}dR',$$

where D is the classical turning point (we neglect its dependence on L), K is the α-particle momentum, and L is its angular momentum. This gives

$$\frac{d}{dR}F_L^C=K(1-\gamma_L)\phi_L^{1/2}F_L^C,$$
$$\frac{d}{dR}G_L^C=-K(1+\gamma_L)\phi_L^{1/2}G_L^C, \tag{10}$$

where

$$\gamma_L=\frac{1}{4K}\phi_L^{-3/2}\frac{d\phi_L}{dR}\bigg|_{R_N}. \tag{11}$$

Inserting into Eq. (8), we can express η_L in terms of $f_L^I(R_N)$ as

$$\tan\eta_L=-\tfrac{1}{2}e^{-2W_L}\frac{f_L^I-KR\phi_L^{1/2}(1-\gamma_L)}{f_L^I+KR\phi_L^{1/2}(1+\gamma_L)}, \tag{12}$$

where the functions on the right-hand side are all evaluated at $R=R_N$. $e^{-2W(R_N)}$ is a very small quantity, therefore $\eta_L\approx 0$ except where $f_L^I+KR\sqrt{\phi}(1+\gamma_L)$ is equal to zero. Indeed

$$f_L^I=-KR\phi_L^{1/2}(1+\gamma_L) \tag{13}$$

is the usual quantization condition.[11,12] When this equation is fulfilled η_L is equal to $\pi/2$. We can interpret η_L as a resonance phase shift which passes through $\pi/2$ as one varies K. Although we are not interested in calculating the α-decay rate in this work, we note that the lifetime can be obtained from the derivative of η_L with respect to K,[13]

$$\tau=\frac{1}{2v}\frac{\partial\eta_L}{\partial K}=\frac{1}{2v}\cos^2\eta_L\frac{\partial\tan\eta_L}{\partial K}\bigg|_{\eta_L=\pi/2}. \tag{14}$$

It can be shown after some algebra that the lifetime obtained from this method is identical to that obtained by Winslow [Eq. (16) of Ref. 12] using the complex energy eigenvalue method.

Inside the nucleus, the wave function is again a combination of F_L^I and G_L^I. Since only outgoing initial $L=0$ waves are needed, only G_0 is required. We have

$$F_L^I=N_{FL}^I Rj_L(\bar{K}R)$$

and

$$G_0^I=\frac{N_G^I}{\bar{K}}\cos(\bar{K}R+\zeta_0). \tag{15}$$

Using Eq. (8), the normalization constants are given by

$$N_{Fl}^I=\frac{e^{W_L}\sin\eta_L(1+\frac{1}{2}\cot\eta_L e^{-2W_L})}{\phi_L^{1/4}Rj_L(\bar{K}R)}\bigg|_{R=R_N}, \tag{16}$$

$$N_G^I=\frac{\bar{K}e^{W_L}\cos\eta_L(1-\frac{1}{2}\tan\eta_L e^{-2W_L})}{\phi_L^{1/4}\cos(\bar{K}R+\zeta_0)}\bigg|_{R=R_N}, \tag{17}$$

and $\bar{K}$ is calculated from Eq. (13) with $F_L=F_0$,

$$\bar{K}\cot\bar{K}R=-K\phi_0^{1/2}(1+\gamma_0)\,|_{R=R_N}. \tag{18}$$

It will turn out that the phase factor ζ_0 is not required. It can be calculated from matching g_L^I and g_L^{II}, Eq. (8). One finds $\zeta_0\neq\pm\pi/2$ so that F_0^I and G_0^I are indeed linearly independent.

We now have all of the formulas necessary to calculate the initial and final wave functions $\mathcal{F}_L$ and $\mathcal{U}_0^+$. $\mathcal{U}_0^+$ is to be evaluated on the resonance ($\eta_0=\pi/2$) using

$$\mathcal{U}_0^+=e^{-i(\sigma_0+\eta_0)}(G_0+iF_0), \tag{19}$$

where σ_0 is the Coulomb phase shift. Inside the nucleus we have, inserting Eqs. (15)–(17),

$$\mathcal{U}_0^+=e^{-i(\sigma_0+\eta_0)}\frac{e^{W_0(R_N)}}{\phi_0^{1/4}(R_N)}\left[\cos\eta_0(1-\tfrac{1}{2}\tan\eta_0 e^{-2W_0})\frac{\cos(\bar{K}R+\zeta_0)}{\cos(\bar{K}R_N+\zeta_0)}+i\sin\eta_0(1+\tfrac{1}{2}\cot\eta_0 e^{-2W_0})\frac{\sin\bar{K}R}{\sin\bar{K}R_N}\right]$$
$$\cong e^{-i\sigma_0}\frac{\sin\bar{K}R}{\sin\bar{K}R_N}e^{W_0(R_N)}[\phi_0(R_N)]^{-1/4}, \tag{20}$$

where we have used the fact that $\cos\eta_0=0$, $\sin\eta_0=1$, and $e^{-2W_0}<<1$. For $R>R_N$, we have, inserting the WKB wave functions and using standard connection formulas[14] for $R=D$,

$$\mathscr{U}_0^+=\begin{cases} e^{-i\sigma_0}(e^{W_0(R)}+i\frac{1}{2}e^{-W_0(R)})\phi_0^{-1/4}, & R_N<R<D\ , \\ i\exp\left[-i\sigma_0+i\overline{W}_0(R)-i\dfrac{\pi}{4}\right](-\phi_0)^{-1/4}, & R>D\ , \end{cases} \tag{21}$$

where

$$\overline{W}_0(R)=\int_D^R(-\phi_0)^{1/2}dR'\ .$$

For the final-state wave function $\mathscr{F}_L$, the corresponding particle energy is smaller by at least the K-shell binding energy, which is much greater than the width of the resonance ($\hbar/\tau$). Thus the resonance condition Eq. (13) is not satisfied and $\eta_0\approx 0$. In this case $\mathscr{F}_L$ is given by

$$\mathscr{F}_L=\begin{cases} \frac{1}{2}e^{-W_L'(R_N)}[\phi_L'(R_N)]^{-1/4}\dfrac{j_L(\bar{K}'R)R}{j_L(\bar{K}'R_N)R_N}, & R<R_N\ , \\ \frac{1}{2}e^{-W_L'(R)}[\phi_L'(R)]^{-1/4}, & R_N<R<D\ , \\ \sin(\overline{W}_L'+\pi/4)[\phi_L'(R)]^{-1/4}, & R>D\ , \end{cases} \tag{22}$$

where $\bar{K}'^2=\bar{K}^2-2\omega/\mu$.

We note that $W(R)$ in Eq. (9) is a positive quantity that decreases with increasing R. Hence the first term of $\mathscr{U}_0^+$ corresponds to a wave which decays exponentially with increasing R. The second term, which decays with decreasing R, will play no role. $\mathscr{F}_L$, however, is a wave which decays exponentially with decreasing R, as expected from the considerations discussed at the end of the last section.

C. *K*-shell ionization probability

Using Eq. (4) we can write the probability of exciting a K electron into a continuum state with energy E_f and angular-momentum quantum numbers λ and μ as

$$\frac{dP_{\lambda\mu}(E_f)}{dE_f}=|A_{E_f\lambda\mu}|^2(2\lambda+1)^{-1}\ , \tag{23}$$

where

$$A_{E_f\lambda\mu}=\frac{K'}{2\pi v}\langle\Psi^-(\vec{\mathbf{K}}',\vec{\mathbf{R}})\phi_{E_f\lambda\mu}|H'|\phi_{1s}\mathscr{U}_0^+/R\rangle \times(2\lambda+1)^{1/2} \tag{24}$$

and v is the alpha-particle velocity.

For the electronic wave functions we will use Dirac hydrogenic wave functions of the united atom centered on the daughter nucleus. In principle, since the velocity of the outgoing α particle is much smaller than the K-electron velocity, diatomic molecular wave functions which evolve continuously from united-atom wave functions to wave functions of the daughter nucleus should be used.[15] The use of united-atom wave functions for all R here is consistent with the prescription used by Andersen *et al.*[16] to describe K-shell excitation in slow ion-atom collisions. The perturbation Hamiltonian H' is given by[17,18]

$$H'=-\frac{Z_1}{|\vec{\mathbf{r}}-\vec{\mathbf{R}}|}+\frac{Z_1}{r}+\frac{1}{M_2}\vec{\nabla}_R V(R)\cdot\vec{\mathbf{r}}\ , \tag{25}$$

where M_2 is the target nuclear mass (in atomic units). The first term here is just the Coulomb potential between the projectile nucleus and electron. The term Z_1/r is present because the wave functions are eigenfunctions of the Hamiltonian with the potential $(Z_1+Z_2)/r$ hence this term must be included to avoid double counting the projectile-nucleus-electron Coulomb potential. The final

term is the recoil contribution.[17,18] The electronic matrix elements are readily evaluated.[17,19] One obtains

$$(2\lambda+1)^{1/2}\langle\phi_{E_f\lambda\mu}\,|\,H'\,|\,\phi_{1s}\rangle = G_\lambda(R)\left[\frac{4\pi}{2\lambda+1}\right]^{1/2} Y^*_{\lambda\mu}(\hat{R})\,, \quad (26)$$

where

$$G_\lambda(R)=Z_1 S_\lambda(R)-\delta_{\lambda 1}Q\frac{1}{M_2}\frac{\partial V}{\partial R}\,, \quad (27)$$

$$S_\lambda(R)=\int_0^\infty r^2 dr\, R_{E_f\lambda}R_{1s}(r)\left[\frac{r_<^\lambda}{r_>^\lambda+1}-\delta_{\lambda 0}\frac{1}{r}\right], \quad (28)$$

and

$$Q=\int_0^\infty r^3 dr\, R_{E_f\lambda}R_{1s}\,. \quad (29)$$

[Equations (26)–(29) assume for simplicity the use of nonrelativistic radial electronic wave functions $R_{E_f\lambda}$ and R_{1s}, but are valid in form for Dirac wave functions also.[19] For excitation to p continuum states one must separately calculate radial matrix elements for $p_{\frac{1}{2}}$ and $p_{\frac{3}{2}}$ continuum states, and to obtain the total excitation probability, one must weight the two contributions by $J+\frac{1}{2}$.]

Taking the quantization axis along K′ so that

$$Y^*_{\lambda\mu}(\hat{K}')=\delta_{\mu 0}\left[\frac{2\lambda+1}{4\pi}\right]^{1/2}, \quad (30)$$

$A_{E_f\lambda\mu}$ is given by

$$A_{E_f\lambda\mu}=\frac{2}{v}(i)^\lambda e^{i\delta_\lambda}\delta_{\mu 0} \times\int dR\,\mathcal{F}_\lambda(K'R)G_\lambda(R)\mathcal{W}_0^+(KR)\,. \quad (31)$$

Since the phase shifts will drop out when we square $A_{E_f\lambda\mu}$, they will henceforth be omitted. We will also omit the $\delta_{\mu 0}$ factor, and abbreviate $A_{E_f\lambda\mu}$ as A_λ.

Using the wave functions derived in the previous section, A_λ is given by

$$A_\lambda=-i\int_0^{R_N} dR\, G_\lambda(R)\frac{\sin(\bar{K}R)}{\sin(\bar{K}R_N)}\frac{j_\lambda(\bar{K}'R)}{j_\lambda(\bar{K}'R_N)}\frac{R}{R_N}\frac{\exp[W_0(R_N)-W'_\lambda(R_N)]}{[\phi_0(R_N)\phi'_\lambda(R_N)]^{1/4}}$$
$$-i\int_{R_N}^D dR\, G_\lambda(R)\frac{\exp[W_0(R)-W'_\lambda(R)]}{[\phi_0(R)\phi'_\lambda(R)]^{1/4}}$$
$$+\int_D^\infty dR\, G_\lambda(R)\frac{\exp\{i[\bar{W}_0(R)-\bar{W}'_\lambda(R)]\}}{[\phi_0(R)\phi'_\lambda(R)]^{1/4}}\,, \quad (32)$$

where we have neglected small terms of the form

$$\int_{R_N}^D dR\, G_\lambda(R)\exp[-W_0(R)-W'_\lambda(R)](\phi_0\phi'_\lambda)^{-1/4} \quad (33)$$

and

$$\int_D^\infty dR\, G_\lambda(R)\exp[i\bar{W}_0(R)+i\bar{W}_\lambda(R)](\phi_0\phi'_\lambda)^{-1/4}\,.$$

Neglect of the first term is justified because $\exp(-W_0-W_\lambda)$ is much less than $\exp(+W_0-W_\lambda)$. Neglect of the second term is justified because $\exp[i(\bar{W}_0+\bar{W}_\lambda)]\approx\exp(2iKR)$ oscillates very rapidly, hence the integral gives approximately zero.

Equation (32) can be simplified if we approximate

$$\phi'_\lambda=\frac{D'}{R}-1+\left[\frac{\lambda+1/2}{K'R}\right]^2\approx\frac{D'}{R}-1=\phi'_0\,. \quad (34)$$

This approximation should be valid because the term neglected is less than a factor of $[(\lambda+1/2)/\eta]^2\,D/R_N$ smaller than the D/R term. For ^{210}Po α decay, $\eta\approx 23$ and $D/R_N\approx 5$, hence this term should be negligible.

Using this approximation $(\phi_0\phi_0')^{1/4}$ is approximately given by $(\pm\phi)^{1/2}$, where $\phi=1-D/R$. Thus

$$W_0(R)-W_\lambda'(R)\approx\int_R^D dR(-K^2\phi_0)^{1/2}-(-K'^2\phi_0)^{1/2}$$
$$\approx q\int_D^R dR(-\phi)^{-1/2}\,, \tag{35}$$

where $q=K-K'=\omega/v$. Similarly,

$$\bar{W}_0(R)-\bar{W}_\lambda'(R)=q\int_D^R dR(+\phi)^{-1/2}=\omega t\,. \tag{36}$$

We therefore have

$$A_\lambda=\int_0^\infty dt\,e^{i\omega t}G_\lambda(R)-i\int_{R_N}^D\frac{dR}{v\sqrt{-\phi}}G_\lambda(R)\exp\left[q\int_D^R dR(-\phi)^{-1/2}\right]$$
$$-i\int_0^{R_N}\frac{dR}{v}[-\phi(R_N)]^{-1/2}G_\lambda(R)\exp\left[q\int_D^{R_N}dR(-\phi)^{-1/2}\right]\frac{\sin(\bar{K}R)}{\sin(\bar{K}R_N)}\frac{j_\lambda(\bar{K}'R)}{j_\lambda(\bar{K}'R_N)}\frac{R}{R_N}\,. \tag{37}$$

The first term in this expression is just the semiclassical amplitude given by Eq. (1). We shall henceforth denote this term by a_λ. The last two terms are the tunneling correction, which we shall call a_λ'.

We note that the contribution to a' coming from inside the barrier ($R_N<R<D$) is what one would obtain from elementary considerations. For the semiclassical contribution R is greater than D, hence $\phi=1-D/R$ is positive. To extend the integral to $R<D$, one realizes that ϕ is negative, so that $\sqrt{\phi}$ will yield $i\sqrt{-\phi}$. Hence one would expect

$$a_\lambda'=-i\int_0^D\frac{dR}{v\sqrt{-\phi}}G_\lambda(R)$$
$$\times\exp\left[q\int_D^R dR(-\phi)^{-1/2}\right]\,. \tag{38}$$

Except for the contribution from inside the nucleus, this is the same as Eq. (37).

D. Tunneling corrections

Since the contribution to a_λ' from inside the nucleus turns out to be numerically small, we can discuss the magnitude of the tunneling correction by considering just the contribution from within the Coulomb barrier. Using Dirac wave functions, it can be shown that for small R, $G_0(R)$ varies as $C_0R^{2\gamma}$, where C_0 is constant.[15] Similarly $G_1(R)$ can be written as[18,20]

$$G_1(R)\approx Z_1C_1\left[1-\frac{Z_2\mu}{Z_1M_2}\right]R\,,$$

where $C_1=Q\omega^2/Z_2$, μ is the nuclear reduced mass, and Q is given by Eq. (29). With these approximations a_λ' is given by

$$a_0'=-i\frac{Z_1}{v}C_0D^{2\gamma+1}I_{2\gamma}(\xi,x_0)\,,$$
$$a_1'=-i\frac{Z_1}{v}C_1\left[1-\frac{Z_2\mu}{Z_1M_2}\right]D^2I_1(\xi,x_0)\,, \tag{39}$$

where

$$I_n(\xi,x_0)=\int_{X_0}^1\frac{X^{n+1}dX}{(X^2-X)^{1/2}}$$
$$\times\exp\left[2\xi\int_1^X\frac{X'dX'}{(X'^2-X')^{1/2}}\right]\,, \tag{40}$$

$2\xi=qD$, and $X_0=R_N/D$.

We can immediately draw some conclusions about the magnitude of the tunneling corrections. First, since the integrand in Eq. (40) peaks at $X=1$, the value of I_n will be nearly independent of X_0 for $X_0\le0.5$. Unlike studies of α-decay rates,[21] studies of α-decay ionization yield no information on nuclear radii R_N. For the same reason, we doubt if a_λ' is sensitive to more refined models of the nuclear potential. We assume a sharp-cutoff model where $V(R)$ is a constant for $R<R_N$ and $V(R)=Z_1Z_2e^2/R$ for $R>R_N$. More refined theories of the $V(R)$ use different potentials near $R=R_N$. However, since a_λ' is sensitive only to the region around $R\approx D$, this should not change the results.

One finds that the tunneling correction is very small. However, it falls off slowly with ξ or E_f, because ξ is small, and because the integrand peaks at $X=1$ where

$$\exp\left[2\xi\int_1^X X\,dX(X^2-X)^{-1/2}\right]\approx 1\ .$$

Therefore, although the tunneling correction is much smaller than the semiclassical amplitude at $E_f=0$, it does not fall off rapidly with E_f unlike the SCA parts, hence the tunneling correction is important at large E_f.

Finally it is interesting to compare the magnitude of the dipole and monopole tunneling corrections. Roughly speaking I_1 and $I_{2\gamma}$ are equal, hence

$$\frac{a_1'}{a_0'}=\frac{C_1}{C_0}(D)^{1-2\gamma}\left[1-\frac{Z_2\mu}{Z_1M_2}\right]\ . \tag{41}$$

For ^{210}Po α decay at $E_f\approx 0, C_1=0.75\ Z_u^2$ and $C_0=-0.94\ Z_u^{2\gamma}$, where $Z_u=Z_1+Z_2$. We obtain $|a_1'/a_0'|\sim 0.76$. In this case, the dipole correction is smaller than the monopole correction. This will not always be true, however. If we consider a proton tunneling through a barrier instead of an α particle, the factor $1-Z_2\mu/Z_1M_2$ ($=0.21$ for ^{210}Po) can be as large as 0.6. s/Z_u^2 and $C_0Z_u^{-2\gamma}$ are relatively independent of Z_u. However, if we decrease Z_u so that $1-2\gamma\approx -1$, and increase the projectile velocity, $|a_1'/a_0'|$ will increase as $(DZ_u)^{-1}$ or approximately as v^2. Thus for lighter target atoms at higher projectile velocities, the dipole correction is expected to be the dominant one. Nevertheless, because the SCA amplitude increases rapidly with the projectile velocity and the tunneling amplitude decreases with the projectile velocity (as v^{-5} or $v^{-4\gamma-3}$ for dipole and monopole excitation), the tunneling correction is expected to be more negligible in higher-velocity collisions.

III. RESULTS

Figure I shows calculations of the differential excitation probabilities dP_λ/dE_f for exciting K-shell electrons into continuum states with kinetic energy E_f during the decay of ^{210}Po. Only probabilities for monopole and dipole excitation are shown. Quadrupole excitation probabilities are less than 10^{-8} for $E_f=0$ and 10^{-9} for $E_f=2(Z^2\mathrm{Ry})$. We show separately the semiclassical probability $|a_\lambda|^2$, the probability calculated including tunneling $|a_\lambda+a_\lambda'|^2$, and the "tunneling correction" $|a_\lambda'|^2$. The tunneling correction falls off very

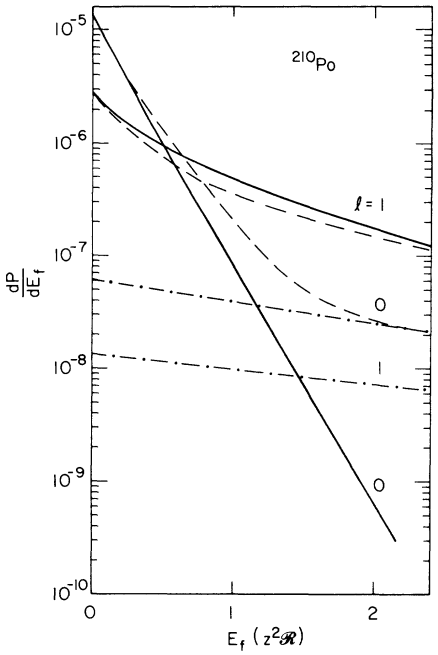

FIG. 1. Calculated differential probability for K-shell ionization in the α decay of ^{210}Po in units of $(Z^2\mathrm{Ry})^{-1}$ plotted against the kinetic energy of the ionized electron. Solid line: probability calculated neglecting the tunneling contribution; dashed line: including tunneling. The chain curve shows the tunneling corrections separately. Contributions from monopole ($l=0$) and dipole ($l=1$) excitation are shown separately.

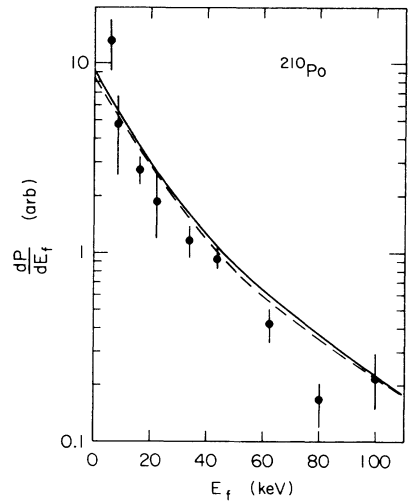

FIG. 2. Total differential ionization probability in ^{210}Po α decay. Measured points from Fischbeck and Freedman,[7] arbitrarily normalized to the theoretical calculation near $E_f=10$ keV. (Several measured points for $E_f<20$ keV have been omitted.) Comparison is made with the calculations including (dashed line) and neglecting (solid line) tunneling.

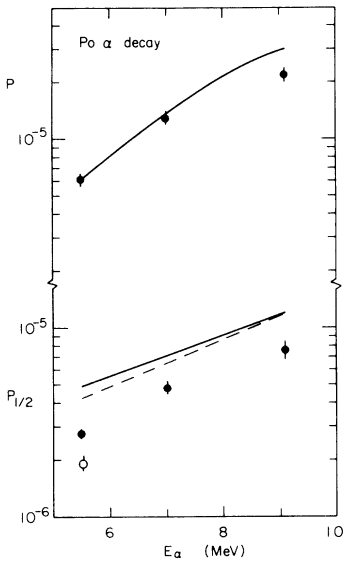

FIG. 3. Total K-shell ionization probability $P_{1/2}$ in the α decay of ^{210}Po (5.3 MeV), ^{216}Po (6.7 MeV), and ^{212}Po (8.8 MeV) and the probability P measured in the α backscattering from Pb. Solid line calculated neglecting tunneling; dashed line includes tunneling. Data from Lund *et al.*[6] The probability in ^{210}Po decay has been measured many times. The 0 point is the average of several measurements taken by Fischbeck and Freedman.[7]

slowly with E_f or the momentum transfer q, as expected from the considerations of Sec. II D. At $E_f=0$, the tunneling correction is much smaller than the semiclassical probability, and the dipole tunneling probability is much smaller than the monopole one. At larger E_f, the monopole probability including tunneling is larger than the semiclassical probability, because C_0 in Eq. (39) is negative, thus a_0' adds to the imaginary part of a_0. The dipole tunneling amplitude subtracts from a_1 because C_1 is positive.

The total differential K-shell ionization probability for ^{210}Po is compared with measurements of Fischbeck and Freedman[7] in Fig. 2. In this figure, the data points have been arbitrarily normalized to our calculations at $E_f=10$ keV. Although tunneling greatly increases the monopole excitation probability at $E_f\sim 100$ keV, and reduces the dipole probability, these two contributions nearly cancel out so that the total differential probabilities with and without tunneling are almost identical. Unfortunately, the tunneling contribution can only be measured at very large electron kinetic energies $E_f>200$ keV.

Figure 3 compares the integrated α-decay probabilities $P_{1/2}$ with measurements of Lund *et al.*[6] Lund *et al.* also measured the probability P of K-shell ionization in backscattering $\alpha+$Pb collisions with the same relative velocity. The energy scale in Fig. 3 refers to the bombarding energy in the backscattering experiments. Although the calculated backscattering probability agrees well with experiment except at 9.2 MeV, the decay ionization probability is higher than experiment at all α-decay energies. Introducing the tunneling corrections improves the agreement between the measured and calculated internal ionization probabilities, but not by much. The tunneling correction is a larger fraction of the total K-shell ionization probability at lower α-decay energies. For $E_\alpha>10$ MeV, the tunneling correction should be negligible. The results for the semiclassical contribution in Fig. 3 are the same as those obtained by Kocbach.[10]

IV. CONCLUSION

This paper has been concerned with the contribution to the α-decay K-shell ionization probability due to the part of the trajectory where the particle tunnels through the Coulomb barrier. From the point of view of the electron this tunneling is sudden. This is shown by the result that the ionization probability during tunneling is nearly independent of the momentum transfer q needed to ionize the K electron. The calculated tunneling probability falls off very slowly with q or the kinetic energy E_f of the ionized electron. The usual semiclassical contribution to K-shell ionization, due to the part of the particle trajectory starting at the Coulomb barrier and going to infinity, is much larger than the tunneling probability at $E_f=0$ and falls off very rapidly with the momentum transfer or electron kinetic energy. Consequently, the tunneling probability can only be directly observed at large E_f.

For the tunneling contribution to be a large fraction of the total K-shell ionization probability, the semiclassical probability should be very small, and tunneling distance D should be as large as possible. This situation is rare in α or proton decay and is nearly unobtainable in nuclear reactions. A large D implies that the particle decay rate or reaction cross section will be very small. In studies of internal ionization, the small decay rate poses less of a problem because one can always study a large number of unstable nuclei and have many nuclei decaying per second. In a nuclear reaction, however, one is usually limited by the number of beam

particles (or by γ-ray background associated with the beam) so one needs a large projectile energy and a small D to initiate the reaction. In ^{210}Po decay ($E_\alpha \sim 5.3$-MeV) tunneling affected the K-shell ionization probability by only 30%. For increasing particle energies, the semiclassical K-shell ionization probabilty increases while the tunneling contribution decreases.

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*Present address: Physik-Department, Technischen Universität München, D-8046 Garching, West Germany.

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