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Resonant-test-field model of fluctuating nonlinear waves

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A Hamiltonian system of nonlinear dispersive waves is used as a basis for generalizing the test-wave model to a set of resonantly interacting waves. The resonant test field (RTF) is shown to obey a nonlinear generalized Langevin equation in general. In the Markov limit a Fokker-Planck equation is obtained and the *exact* steady-state solution is determined. An algebraic expression for the power spectral density is obtained in terms of the number of resonantly interacting waves (n) in the RTF, the interaction strength (V_k), and the dimensionality of the wave field (d). For gravity waves on the ocean surface a k^{-4} spectrum is obtained, and for capillary waves a k^{-8} spectrum, both of which are in essential agreement with data.

I. INTRODUCTION

Studies of the properties of nonlinear wave fields have determined that fluctuations in the physical observables can often be one of the dominant features of the evolving system.[1–4] In the dynamic description of such systems a great deal of attention is generally given to constructing the deterministic equations of motion. However, the statistics often enter as an assumption on the inital conditions of the observables or as an *ad hoc* attachment of a fluctuating force to the equations of motion. A number of noteworthy exceptions exist within the statistical mechanic literature where serious attempts at establishing the evolution of the statistical properties of a physical system have been made.[5] The formal intricacy of the problem often obscures the salient features of these analyses, however, and makes their application to particular physical systems quite difficult. In this paper we restrict our discussion to a model Hamiltonian system. The analysis of this model system suggests how one might proceed in describing the generation and evolution of the fluctuations in a physical nonlinear wave field. The model Hamiltonian we choose is fairly general and it is anticipated that the analysis given here will be calculationally useful in applications to a variety of nonlinear wave fields.

In many physical problems the strategy that one adopts is to assume a weak-interaction theory. By this is meant that the observables of the system are expressed in series expansions of the eigenmodes of the linearized system. The expansion coefficients in such series are constant in the linearized system, but are variable in the nonlinear system. If the linear system is harmonic, as it is for a wave field, the eigenfunctions are sines and cosines and the series expansion is just the Fourier series. The expansion coefficients are then referred to as the mode amplitudes and are interpreted as the amplitudes of independent waves in a linear wave field. Correspondingly, the nonlinear system is referred to as a nonlinear wave field and the nonlinearities are interpreted as couplings or scatterings of the linear waves. The Hamiltonian for this system is a series in which the nonlinear terms appear as products of the mode amplitudes. These interactions induce a variation in both the amplitudes and phases of the linear waves in the equations of motion. For a weakly nonlinear system such as water waves or plasmas, this induced variation is much slower than the harmonic variation of the linearized system.[1–4]

Hamilton's equations of motion provide a deterministic description of the evolution of the wave field considered here. If we assume that this field is well represented by N degrees of freedom, where for the moment N is large but finite, the system can be represented by N-coupled, deterministic, nonlinear rate equations for the mode amplitudes. Moser[6] gives a general mathematical discussion of the separation of the interactions into resonant and nonresonant groups for arbitrary Hamiltonian systems. Part of his interest in the partitioning of the interactions is that the nonresonant interactions lead to a stable evolution of the system, whereas resonant interactions lead to instabilities. Here we are not interested in the detailed stable phase-space evolution of the system due to the nonresonant interactions, but rather in the more dramatic process of resonances. It is usually argued that because of the separation in the interaction time scales between resonant and nonresonant interactions that the former dominate the macroscopic evolution of the wave field. In a Fourier representation we index a wave with a wave vector $\vec{k}$ and frequency ω_k, and the nonlinear interactions couple these modes together thereby providing a mechanism for energy exchange. The nonresonant interactions provide a periodic exchange of energy between the mode of interest, $\vec{k}$, for example, and the other modes in the wave field. The frequency mismatch in this interaction results in many changes in sign of this term in the characteristic time interval for the mode amplitude to sensibly change. It is for this reason that such interactions are usually neglected in the equations of motion for the wave field. The weak-interaction theory, therefore, is described by a system of nonlinear mode rate equations in which the sum of the wave vectors $\vec{k}$ and frequencies ω_k in each interaction vanish. In quantum systems this indicates the conservation of momentum and energy, respectively, during an interaction and is called on-the-energy-shell scattering. Correspondingly, the nonresonant interactions are off-the-energy-shell or virtual scatterings.

A given wave in a physical wave field can, in general, participate in both resonant interactions with some waves and nonresonant interactions with others. To maintain the formal simplicity of weak-interaction theory and at the same time include the *effects* of nonresonant interactions we introduce a two-field model for the physical wave field. The first field is a generalization of the test wave model that has been used with some success in both plasma physics[7] and geophysics.[8,9] This more general model replaces the test wave by a set of test waves having nonlinear interactions that are solely resonant. We refer to this set of waves as the resonant test field (RTF). The second field of waves, referred to as the ambient wave field, are noninteracting except through a member of the resonant test field. In the terminology of statistical mechanics this linear ambient wave field provides a "heat bath" for the RTF.

In Sec. II we construct the equations of motion for both the heat bath and RTF degrees of freedom. By adiabatically eliminating the heat-bath variables from the RTF equations of motion we find that the ambient waves provide (1) a source of fluctuating flux driving the resonant waves, (2) a dissipative current to balance the fluctuations, and (3) a modification in the interaction strength among the RTF waves. The equation of motion for the RTF waves when linearly coupled to the heat bath is found to be a nonlinear, generalized Langevin equation with a corresponding generalized fluctuation-dissipation relation.[10,11] For a Markov process the RTF equations of motion reduce to a Langevin equation with a delta-correlated fluctuating flux so that a Fokker-Planck equation for the probability density of the RTF mode amplitudes can be constructed. In Sec. III the steady-state solution of the Fokker-Planck equation is found to be a local equilibrium distribution clearly indicating the non-Gaussian behavior of the RTF waves. The physical interpretation of these results is discussed in detail. In Sec. IV the interaction potential for water waves on the deep ocean is used; for capillary waves a k^{-8} spectrum is calculated, and for gravity waves a k^{-4} spectrum is calculated. This is the first *dynamic model* to yield these observed steady-state spectra.[4]

II. DYNAMICS OF THE RESONANT TEST FIELD

The model wave field we are interested in here is represented by a Hamiltonian consisting of three pieces. The first piece, H_R, consists of the resonant test waves with mode amplitude a_k and frequency ω_k and can be written

$$H_R = \sum_k \omega_k a_k a_k^* + V_R \ , \tag{2.1}$$

where V_R is the nonlinear resonant interaction potential. The heat bath in this model consists of a system of linear waves with mode amplitudes $b_\nu(t)$. We write the Hamiltonian for the noninteracting waves of the heat bath and the coupling between the heat bath and the RTF waves as

$$H_B + H_{RB} = \sum_\nu \omega_\nu [b_\nu + iB_\nu(\vec{a},\vec{a}^{\,*})] \times [b_\nu^* - iB_\nu^*(\vec{a},\vec{a}^{\,*})] \ , \tag{2.2}$$

where $B_\nu(\vec{a},\vec{a}^{\,*})$ is a function describing the modulation of the heat bath by the RTF waves.

We leave this function unspecified for the time being and find that we can go quite far in the analysis without specifying B_ν.

Hamilton's equations for the heat bath can be written as

$$\dot{b}_\nu = -i\frac{\partial H}{\partial b_\nu^*}, \quad \dot{b}_\nu^* = i\frac{\partial H}{\partial b_\nu} \quad , \tag{2.3}$$

where H is the total Hamiltonian

$$H = H_R + H_B + H_{RB} \ , \tag{2.4}$$

and similarly for the RTF waves. The waves in the heat bath satisfy the dynamic equations

$$\dot{b}_\nu + i\omega_\nu b_\nu = \omega_\nu B_\nu(\vec{a}, \vec{a}^*) \ , \tag{2.5}$$

clearly indicating the linear nature of this wave field. Similarly the equations of motion for the RTF waves are given by

$$\dot{a}_k + i\omega_k a_k = -i\frac{\partial V_R}{\partial a_k^*} + \sum_\nu \omega_\nu (b_\nu + iB_\nu)\frac{\partial B_\nu^*}{\partial a_k^*} - \sum_\nu \omega_\nu (b_\nu^* - iB_\nu^*)\frac{\partial B_\nu}{\partial a_k^*} \ , \tag{2.6}$$

which in turn are driven by the heat-bath variables. Taken together (2.5) and (2.6) constitute a feedback system between two fields of waves. We construct a description of the evolution of the resonant test field alone by solving (2.5) and using the solution to eliminate the dependence of (2.6) on the bath variables. We find that just as in the theories of Mori[10] and Zwanzig,[11] that this elimination leads to a generalized Langevin equation.

To obtain the solution to (2.5) in a more convenient form we rewrite that equation as

$$\frac{d}{dt}(b_\nu + iB_\nu) = -i\omega_\nu (b_\nu + iB_\nu) + i\dot{B}_\nu \ , \tag{2.7}$$

and solve it as an inhomogenous equation to obtain

$$b_\nu(t) + iB_\nu(t) = e^{-i\omega_\nu t}[b_\nu(0) + iB_\nu(0)] + i\int_0^t e^{-i\omega_\nu(t-t')}\frac{d}{dt'}B_\nu(t-t')dt' \ , \tag{2.8}$$

with initial conditions $b_\nu(0)$ and $B_\nu(0)$. Since B_ν is only a function of the RTF mode amplitude we use the chain condition for derivatives to write

$$\frac{dB_\nu(t)}{dt} = \sum_l \left[\frac{\partial B_\nu}{\partial a_l}\dot{a}_l + \frac{\partial B_\nu}{\partial a_l^*}\dot{a}_l^*\right] \tag{2.9}$$

so that interchanging derivatives with respect to t and t' in (2.8) and substituting this solution (2.8) into (2.6) we obtain

$$\dot{a}_k + i\omega_k a_k = -i\frac{\partial V_R}{\partial a_k^*} + \sum_\nu \left[F_\nu(t)\frac{\partial B_\nu^*}{\partial a_k^*} - F_\nu^*(t)\frac{\partial B_\nu}{\partial a_k^*}\right] + i\sum_{\nu,l}\left[\frac{\partial B_\nu}{\partial a_k^*}\int_0^t \lambda_\nu(t-t')\left[\frac{\partial B_\nu^*}{\partial a_l^*}\dot{a}_l^* + \frac{\partial B_\nu^*}{\partial a_l}\dot{a}_l\right] - \frac{\partial B_\nu^*}{\partial a_k^*}\int_0^t \lambda_\nu^*(t-t')\left[\frac{\partial B_\nu}{\partial a_l^*}\dot{a}_l^* + \frac{\partial B_\nu}{\partial a_l}\dot{a}_l\right]\right] \ , \tag{2.10}$$

where we have introduced the functions

$$F_\nu(t) \equiv \omega_\nu [b_\nu(0) + iB_\nu(0)]e^{-i\omega_\nu t} \ , \tag{2.11}$$

$$\lambda_\nu(t) \equiv \omega_\nu e^{-i\omega_\nu t} \ . \tag{2.12}$$

Equation (2.10) has the form of a nonlinear, generalized Langevin equation if the initial conditions for the bath variables are selected appropriately. We assume that for the RTF variables held fixed at time $t=0$ that we can select the mode amplitudes $b_\nu(0)$ from an equilibrium ensemble of initial waves specified by the "canonical distribution,"

$$P(\vec{b}(0) \mid \vec{a}(0)) \sim \exp\left[-\sum_\nu (H_B + H_{BR})(\nu)/\Lambda_\nu\right] \ . \tag{2.13}$$

The parameter Λ_ν is the level of excitation of the ambient wave field and can be interpreted as the "temperature of the heat bath." This ensemble of initial conditions yields $\langle b_\nu(0)\rangle = -iB_\nu(0)$, where $\langle --- \rangle$ denotes an average with respect to (2.13), so that from (2.11)

$$\langle F_\nu(t)\rangle = 0 \; . \tag{2.14}$$

Also, the variance of $b_\nu(0) + iB_\nu(0)$ is given by

$$\langle [b_\nu(0)+iB_\nu(0)][b_{\nu'}(0)+iB_{\nu'}(0)]^* \rangle = 2\delta_{\nu\nu'}\Lambda_\nu/\omega_\nu \; ,$$

so that the correlation of the fluctuations is given by

$$\langle F_\nu(t)F^*_{\nu'}(t')\rangle = 2\delta_{\nu\nu'}\Lambda_\nu\omega_\nu e^{-i\omega_\nu(t-t')} = 2\delta_{\nu\nu'}\Lambda_\nu\lambda_\nu(t-t') \; . \tag{2.15}$$

Equation (2.15) is the generalized fluctuation-dissipation relation for the wave field relating the memory kernel in (2.10), $\lambda_\nu(t)$, to the correlations in the fluctuating flux $F_\nu(t)$. It is quite clear from the form of the Hamiltonian $H_B + H_{BR}$ that the distribution of mode amplitudes $b_\nu(0) + iB_\nu(0)$ is Gaussian so that $F_\nu(t)$ is a homogenous, Gaussian process with finite time correlations.

If we now iterate (2.10) by replacing $\dot{a}_l$ under the integral by the first two terms in the evolution equation for $\dot{a}_k$, we obtain

$$\begin{aligned}
\dot{a}_k + i\omega_k a_k \cong & -i\frac{\partial V_R}{\partial a_k^*} + \sum_\nu \left[F_\nu(t)\frac{\partial B_\nu^*}{\partial a_k^*} - F_\nu^*(t)\frac{\partial B_\nu}{\partial a_k^*}\right] \\
& + \sum_{\nu,l}\int_0^t dt' \left[-\lambda_\nu(t-t')\frac{\partial B_\nu(t)}{\partial a_k^*}\frac{\partial B_\nu^*(t')}{\partial a_l^*}\left[\omega_l a_l^*(t') + \frac{\partial V_R(t')}{\partial a_l}\right]\right. \\
& \qquad + \lambda_\nu(t-t')\frac{\partial B_\nu(t)}{\partial a_k^*}\frac{\partial B_\nu^*(t')}{\partial a_l}\left[\omega_l a_l(t') + \frac{\partial V_R(t')}{\partial a_l^*}\right] \\
& \qquad + \lambda_\nu^*(t-t')\frac{\partial B_\nu^*(t)}{\partial a_k^*}\frac{\partial B_\nu(t')}{\partial a_l^*}\left[\omega_l a_l^*(t') + \frac{\partial V_R(t')}{\partial a_l}\right] \\
& \qquad \left. - \lambda_\nu^*(t-t')\frac{\partial B_\nu^*(t)}{\partial a_k^*}\frac{\partial B_\nu(t')}{\partial a_l}\left[\omega_l a_l(t') + \frac{\partial V_R(t')}{\partial a_l^*}\right]\right] \; .
\end{aligned} \tag{2.16}$$

To proceed beyond this expression we must specify a choice for the function $B_\nu(\vec{a},\vec{a}^*)$. We choose a linear modulation of the ambient waves by writing

$$B_\nu(\vec{a},\vec{a}^*) \equiv \sum_p \Gamma_{\nu p} a_p \; , \tag{2.17}$$

where $\Gamma_{\nu p}$ is a complex coupling coefficient. With this choice of B_ν the expression (2.16) is greatly simplified and yields

$$\dot{a}_k + i\omega_k a_k \cong -i\frac{\partial V_R}{\partial a_k^*} + \sum_\nu F_\nu(t)\Gamma_{\nu k}^* - \sum_{\nu,l}\int_0^t dt' \Gamma_{\nu k}^*\Gamma_{\nu l}\lambda_\nu^*(t-t')\omega_l \left[a_l(t') + \frac{1}{\omega_l}\frac{\partial V_R(t')}{\partial a_l^*}\right] \; . \tag{2.18}$$

We can now introduce the zero-centered fluctuating force

$$f_k(t) = \sum_\nu F_\nu(t)\Gamma_{\nu k}^* \; . \tag{2.19}$$

and the memory kernel

$$M_{kl}(t-t') = \sum_\nu \Gamma_{\nu k}^*\Gamma_{\nu l}\omega_l\omega_\nu e^{i\omega_\nu(t-t')} \; , \tag{2.20}$$

to rewrite (2.18) as

$$\dot{a}_k + i\omega_k a_k + \sum_l \int_0^t M_{kl}(t-t')a_l(t')dt' = -i\frac{\partial V_R}{\partial a_k^*} - \sum_l \int_0^t M_{kl}(t-t')\omega_l^{-1}\frac{\partial V_R(t')}{\partial a_l^*}dt' + f_k(t) \; . \tag{2.21}$$

If we can further approximate (2.21) by a Markovian equation, i.e., one for which the correlation time of the fluctuations approaches a delta function and the interaction strength is peaked at $k=l$, i.e.,

$$M_{kl}(t-t')\rightarrow\delta_{kl}\lambda_k\delta(t-t') , \tag{2.22}$$

then (2.21) becomes

$$\dot{a}_k(t)+(\lambda_k+i\omega_k)a_k(t)=-\left[\frac{\lambda_k+i\omega_k}{\omega_k}\right]\frac{\partial V_R}{\partial a_k^*}+f_k(t) . \tag{2.23}$$

Thus the coupling of the RTF to the ambient wave field has modified the original Hamiltonian equations in three ways: (1) There is a zero-centered, Gaussian fluctuating flux driving the wave field; (2) the Hamiltonian character of the system is lost due to the dissipative flux of action (energy) to the ambient waves; and (3) there is a modification of the nonlinear interactions due to a back reaction of the ambient waves to the nonlinear interactions among the test waves.

In the above Markov approximation it is possible to replace the dynamic equations (2.23) by the phase-space equation of evolution for the probability density. In Sec. III we show that this equation is the Fokker-Planck equation and discuss the asymptotic properties of the RTF.

III. STEADY-STATE RESONANT TEST FIELD

The dynamic equations (2.23) describe the development of the set of dynamic variables $\vec{a}(t)=\{a_k(t)\}$ for a particular realization of the set of fluctuations $\{f_k(t)\}$. In the phase space for the dynamical system the coordinate axes are labeled by the values $\vec{A}$ that the dynamic vector $\vec{a}(t)$ can assume. For each realization of the additive fluctuations $\vec{f}(t)$ there corresponds a unique trajectory in this phase space which describes the evolution of the wave field. A large number of realizations of $\vec{f}(t)$ defines an ensemble of trajectories in phase space described by the phase-space distribution function $\rho_f(\vec{A},t)$. This ensemble of test wave fields can be described by a probability density obtained by averaging $\rho_f(\vec{A},t)$ over an ensemble of realizations of $\vec{f}(t)$, i.e.,

$$P(\vec{A},t\mid\vec{A}_0)=\langle\rho_f(\vec{A},t)\rangle_f , \tag{3.1}$$

where each member of the ensemble is initiated from the point $\vec{A}_0$. $P(\vec{A},t\mid\vec{A}_0)d\vec{A}$ is the probability that the dynamic variable $\vec{a}(t)$ has a value in the interval $(\vec{A},\vec{A}+d\vec{A})$ at time t given an initial value $\vec{A}_0$.

The arguments leading from the dynamic equations (2.23) to the equation of evolution for the probability density are standard and will not be reviewed here.[12,13] We merely record that the statistical properties of $\vec{f}(t)$ are

$$\langle f_k(t)\rangle_f=0 , \tag{3.2a}$$

$$\langle f_k(t)f_{k'}^*(t')\rangle_f=2D_k\delta_{kk'}\delta(t-t') , \tag{3.2b}$$

$$\langle f_k(t)f_{k'}(t')\rangle_f=0 , \tag{3.2c}$$

in the Markov approximation of the preceding section. Thus for a homogeneous, delta-correlated Gaussian process describing the fluctuating flux we obtain the Fokker-Planck equation as the phase-space equation of evolution for the probability density, i.e.,

$$\frac{\partial P}{\partial t}(\vec{A},t\mid\vec{A}_0)=\sum_k\left[\frac{\partial}{\partial A_k}\left[\frac{\lambda_k+i\omega_k}{\omega_k}\frac{\partial H_R}{\partial A_k^*}P(\vec{A},t\mid\vec{A}_0)\right]+\text{c.c.}+2D_k\frac{\partial^2}{\partial A_k\partial A_k^*}P(\vec{A},t\mid\vec{A}_0)\right] , \tag{3.3}$$

where H_R is the RTF Hamiltonian given by (2.1) and c.c. denotes the complex conjugate of the preceding term in the series.

It is notoriously difficult to solve (3.3) in general so here we will restrict our discussion to the steady-state properties of the test wave field. The steady-state solution to (3.3), denoted by $P_{ss}(\vec{A})$, is specified by the condition

$$\frac{\partial P_{ss}(\vec{A})}{\partial t}=\lim_{t\rightarrow\infty}\frac{\partial}{\partial t}P(\vec{A},t\mid\vec{A}_0)=0 , \tag{3.4}$$

and is independent of both time and the initial conditions of the RTF. The steady-state equation is therefore

$$\sum_k \left[\frac{\partial}{\partial A_k} \left[\frac{\lambda_k + i\omega_k}{\omega_k} \frac{\partial H_R}{\partial A_k^*} \right] + \text{c.c.} + 2D_k \frac{\partial^2}{\partial A_k \partial A_k^*} \right] P_{\text{ss}}(\vec{\mathrm{A}}) = 0 \; . \tag{3.5}$$

We assume a solution to (3.5) of the form

$$P_{\text{ss}}(\vec{\mathrm{A}}) = Z^{-1} \exp\left[-\sum_k \beta_k Q_k(\vec{\mathrm{A}}) \right] , \tag{3.6}$$

where Z is the partition function, $\{ \beta_k \}$ is a set of unknown parameters, and $\{ Q_k(\vec{\mathrm{A}}) \}$ is a set of unknown functions. Substituting (3.6) into (3.5) we obtain

$$\sum_k \left\{ -\beta_k \left[\frac{\lambda_k}{\omega_k} \left[\frac{\partial H_R}{\partial A_k^*} \frac{\partial Q_k}{\partial A_k} + \frac{\partial H_R}{\partial A_k} \frac{\partial Q_k}{\partial A_k^*} \right] + i \left[\frac{\partial H_R}{\partial A_k^*} \frac{\partial Q_k}{\partial A_k} - \frac{\partial H_R}{\partial A_k} \frac{\partial Q_k}{\partial A_k^*} \right] - 2D_k \beta_k \frac{\partial Q_k}{\partial A_k} \frac{\partial Q_k}{\partial A_k^*} \right] + \frac{2\lambda_k}{\omega_k} \frac{\partial^2 H_R}{\partial A_k \partial A_k^*} - 2\beta_k D_k \frac{\partial^2 Q_k}{\partial A_k \partial A_k^*} \right\} = 0 \; . \tag{3.7}$$

By inspection we observe that with

$$\frac{1}{\beta_k} = \frac{\omega_k D_k}{\lambda_k} \tag{3.8}$$

and

$$\frac{\partial Q_k}{\partial A_k} = \frac{\partial H_R}{\partial A_k} \; , \tag{3.9}$$

Eq. (3.7) is satisfied, i.e., a solution to (3.5) of the form (3.6) can be found.

The condition (3.9) is analogous to a fluctuation-dissipation relation for the resonant test field of waves. Integrating (3.9) we obtain (up to a function whose divergence is zero)

$$Q_k(\vec{\mathrm{A}}) = \omega_k A_k A_k^* + V_R(k) \; , \tag{3.10}$$

where by $V_R(k)$ we mean

$$V_R(k) = \int \frac{\partial V_R}{\partial A_k} dA_k \; . \tag{3.11}$$

For a potential of the form

$$V_R = \sum_{lmpq} V_{pq}^{lm} A_l A_m A_p^* A_q^* \delta_{l+m-p-q} \; . \tag{3.12}$$

We obtain from (3.11) (up to an unimportant constant)

$$V_R(k) = \sum_{lpq} 2 V_{pq}^{kl} A_k A_l A_p^* A_q^* \delta_{k+l-p-q} \tag{3.13}$$

for which (3.9) is clearly satisfied. The quantity $Q_k(\vec{\mathrm{A}})$ is therefore a type of single "particle" energy which includes all the interactions of the k wave with the other test waves. The steady-state distribution (3.6) is now

$$P_{\text{ss}}(\vec{\mathrm{A}}) = Z^{-1} \exp\left[-\sum_k \beta_k \omega_k A_k A_k^* - \sum_{k>l,pq} 2\beta_k V_{pq}^{kl} A_k A_l A_p^* A_q^* \right] , \tag{3.14a}$$

or removing the restriction on the l summation, i.e., $\sum_{l>k} = \frac{1}{2} \sum_{lk}$, yields

$$P_{\text{ss}}(\vec{\mathrm{A}}) = Z^{-1} \exp\left[-\sum_k \beta_k \left[\omega_k A_k A_k^* + \sum_{lpq} V_{pq}^{kl} A_k A_l A_p^* A_q^* \right] \right] . \tag{3.14b}$$

We now denote the quantity in large parentheses as $H_R(k)$ and write

$$P_{ss}(\vec{A})=Z^{-1}\exp\left[-\sum_k \beta_k H_R(k)\right] . \tag{3.15}$$

It is clear that since $\sum_k H_R(k)=H_R$ that if β_k were independent of k it could be interpreted as the scalar temperature of the resonant test field and (3.15) would be a canonical distribution. A steady-state distribution of the form (3.15) can be shown to satisfy (3.5) for any symmetric interaction of resonant test waves.

The set of parameters $\{\beta_k\}$ can be calculated in two different ways; either by using the properties of the heat bath, i.e., the coupling coefficients, or self-consistently by means of the fluctuation-dissipation relation (3.8). In the Markov approximation, the spectral strength of the energy flux to the RTF averaged over one period of the k-test wave is obtained using (2.19) and (3.2b) to be

$$D_k=\sum_\nu |\Gamma_{\nu k}|^2\Lambda_\nu \sin\left[\frac{\pi\omega_\nu}{\omega_k}\right] , \tag{3.16}$$

and the dissipation strength is obtained from (2.20) and (2.22) to be

$$\lambda_k=\omega_k\sum_\nu |\Gamma_{\nu k}|^2 \sin\left[\frac{\pi\omega_\nu}{\omega_k}\right] . \tag{3.17}$$

Thus using (3.8) the parameter β_k is given by

$$\frac{1}{\beta_k}=\frac{\sum_\nu |\Gamma_{\nu k}|^2\Lambda_\nu \sin\left[\frac{\pi\omega_\nu}{\omega_k}\right]}{\sum_\nu |\Gamma_{\nu k}|^2 \sin\left[\frac{\pi\omega_\nu}{\omega_k}\right]} , \tag{3.18}$$

so that β_k *could* be evaluated from a knowledge of the coupling parameters $\Gamma_{\nu k}$ and the power spectral density of the heat bath Λ_ν alone. However, since we have explicitly eliminated the bath degrees of freedom from the RTF equations of motion we use a self-consistency procedure to evaluate β_k.

If we use the fluctuation-dissipation relation for a Markov process, in the asymptotic steady state, and write

$$\langle |a_k|^2\rangle_{ss}=\frac{D_k}{\lambda_k} , \tag{3.19}$$

then (3.18) yields the relation

$$\frac{1}{\beta_k}=\omega_k\langle |a_k|^2\rangle_{ss} . \tag{3.20}$$

Thus, using the probability density (3.15) to evaluate the average in (3.20) we obtain the implicit relation for β_k,

$$\frac{1}{\beta_k}=\omega_k\frac{\int A_k A_k^* \exp\left[-\sum_l \beta_l H_R(l)\right]d\vec{A}}{\int \exp\left[-\sum_l \beta_l H_R(l)\right]d\vec{A}} . \tag{3.21}$$

Equation (3.21) is the type of self-consistency relation that is often encountered in classical statistical mechanics for a two-body Hamiltonian H_R. It is well known that the averages in such cases can only be expressed in terms of an infinite sum of linked-cluster diagrams to determine the energy spectrum of the resonant test wave field.

The interpretation of β_k in terms of the steady-state spectral density of the RTF is clearly indicated in (3.20) and (3.21). The thermodynamic interpretation of β_k is less obvious but is also quite interesting. Although the form of the steady-state distribution (3.15) is more general than the canonical distribution of classical statistical mechanics, we can still interpret β_k as the thermodynamic potential required to maintain the integral constraint on the total energy of the RTF. To understand β_k we apply Parseval's theorem to the expression in the exponent and obtain

$$\sum_k \beta_k H_R(k)=\int_{V_0}\beta(\vec{x}-\vec{x}')H_R(\vec{x}')d\vec{x}' , \tag{3.22}$$

where V_0 is the "volume" of the integral. Equation (3.22) indicates that the integral constraint is nonlocal in configuration space, i.e., the coupling of the test waves to the heat bath is spatially dependent. A distribution similar in form to (3.15) was first obtained for hydrodynamic systems by Piccirelli[16] and is called a local equilibrium distribution. The spatial dependence of the thermodynamic potential, in this case, the "temperature" β_k, indicates that the level of excitation of the ambient wave field is not homogeneous. The inhomogeneity arises from the modulation of the heat bath by the RTF. However, the back reaction of the RTF to the fluctuations in the heat bath is such as to maintain the integral constraint on the energy, but only locally. Thus each test wave experiences a different temperature, or stated somewhat differently, each location on the ocean surface has a different temperature (level of excitation of the ambient wave field).

Another property of the parameter β_k is that for a given spectral level of fluctuations D_k, it is the relative magnitudes of λ_k and ω_k that determine the temperature of the heat bath. For weak linear dissipation, $\lambda_k << \omega_k$, the coefficient of the nonlinear term in (2.23), i.e., $(\lambda_k + i\omega_k)/\omega_k$, is small and the temperature [cf. (3.8)] is low. As the linear dissipation increases, the contribution of the nonlinear term correspondingly increases yielding higher temperatures of the ambient wave field.

IV. DISCUSSION AND CONCLUSIONS

The resonant test field model is seen to give rise to a probability density which explicitly depends on the nonlinear interactions in the wave field [cf. Eq. (3.15)]. Thus the statistics of the RTF are non-Gaussian and the spectrum of the mode amplitudes can be calculated using the derived probability density for the statistical steady state, i.e.,

$$\langle |a_k|^2\rangle_{ss} = Z^{-1}\int A_k A_k^* \exp\left[-\sum_l \beta_l H_R(l)\right] d\vec{A} . \tag{4.1}$$

The integral in (4.1) is the same as that in (3.21).

Although we cannot integrate (4.1) in general, we can get an idea of the spectrum by restricting the integral to a weighted self-interaction of the test k wave. At this discrete value the weight of the interaction is zero, therefore to account in part for the interactions which are being omitted, we weight the diagonal interaction strength V_k by an element of volume αk^d in d dimensions, where α is a constant. Then by introducing polar conditions

$$a_k = \sqrt{J_k}\, e^{-i\theta_k} , \tag{4.2}$$

and integrating over all modes except k, we obtain

$$\langle |a_k|^2\rangle_{ss} = Z_k^{-1}\int_0^\infty J_k dJ_k \times \int_0^{2\pi} d\theta_k e^{-\beta_k(\omega_k J_k + \alpha k^d V_k J_k^{n/2})} , \tag{4.3}$$

where n is the number of waves involved in an interaction and

$$Z_k = 2\pi \int_0^\infty dJ_k e^{-\beta_k(\omega_k J_k + \alpha k^d V_k J_k^{n/2})} . \tag{4.4}$$

Thus in this approximation (4.3) may be written as

$$\langle |a_k|^2\rangle_{ss} \cong -\frac{1}{\beta_k}\frac{\partial}{\partial\omega_k}\ln Z_k . \tag{4.5}$$

We can integrate (4.4) by expanding the interaction terms to obtain

$$\begin{aligned} Z_k &= 2\pi \sum_{l=0}^\infty \frac{(-\beta_k \alpha k^d V_k)^l}{l!} \int_0^\infty J_k^{nl/2} e^{-\beta_k\omega_k J_k} dJ_k \\ &= 2\pi \sum_{l=0}^\infty \frac{(-\beta_k \alpha k^d V_k)^l}{l!} \frac{\Gamma(nl/2+1)}{(\beta_k\omega_k)^{nl/2+1}} . \end{aligned} \tag{4.6}$$

Inserting (4.6) into (4.5) and using (3.20) we then obtain

$$\frac{1}{\beta_k\omega_k} = -\frac{1}{\beta_k}\frac{\partial}{\partial\omega_k}\ln\left[2\pi\sum_{l=0}^\infty \frac{(-\beta_k\alpha k^d V_k)^l}{l!} \times \frac{\Gamma(nl/2+1)}{(\beta_k\omega_k)^{nl/2+1}}\right] \tag{4.7}$$

which from the terms $l=0$, 1, 2 yields

$$\frac{1}{\beta_k^{n/2-1}} \cong \frac{\omega_k^{n/2}}{\alpha k^d V_k}\frac{\Gamma(n/2+1)}{\Gamma(n+1)} . \tag{4.8}$$

Thus for a four-wave interaction ($n=4$) we obtain the steady-state action spectrum

$$\langle |a_k|^2\rangle_{ss} \sim \frac{\omega_k^2}{k^d V_k} ,\quad n=4 \tag{4.9}$$

and for a three-wave interaction we obtain

$$\langle |a_k|^2\rangle_{ss} \sim \frac{\omega_k}{k^{2d} V_k^2} ,\quad n=3 . \tag{4.10}$$

We observe that if the wave field under discussion is that for deep water gravity waves, then the dominant interaction is a four-wave resonance. The diagonal interaction strength for these waves is $V_k \sim k^3$ and $d=2$, so that

$$\langle |a_{\vec{k}}|^2\rangle_{ss} \sim \frac{\omega_k}{k^5} , \tag{4.11}$$

and the energy spectral density is given by

$$F_{ss}(k) = \frac{\langle |a_{\vec{k}}|^2\rangle_{ss}}{\omega_k/k} \sim k^{-4} . \tag{4.12}$$

The spectrum (4.12) has also been obtained by Phillips using a scaling argument.[14]

If the water waves are high-frequency capillary waves then the dominant interaction is a three-wave resonance. The diagonal interaction strength for these waves is $V_k \sim k^3(\omega_k/k)^{1/2}$ and $d=2$ so that

$$\langle |a_{\vec{k}}|^2\rangle_{ss} \sim \frac{\omega_k}{k^4 k^5} , \tag{4.13}$$

and the energy spectral density is given by

$$F_{ss}(k) \sim k^{-8} . \tag{4.14}$$

The spectrum (4.14) is nearly that observed in the viscous dissipation range of the high-frequency water waves.[15]

The agreement between the energy spectral density predicted by the RTF model applied to water wave fields and that obtained using other techniques gives us confidence in the veracity of the model. Note that this agreement comes about because of the weakly interacting nature of water waves (except near breaking) which gives rise to a rapid convergence of the series (4.6). The agreement may not be as satisfactory in other wave fields.

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