Electron collision frequency in plasmas

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In strongly coupled, degenerate plasmas, the electron collision frequency has been described by the Ziman formula with the ion-ion correlations modeled by the classical one-component plasma (OCP). However, this model fails to reproduce the correct quantum Lenard-Balescu result in the weak-coupling limit. It is demonstrated here that a recently obtained correlation-function expression for the collision frequency reduces to the Ziman and Lenard-Balescu results in the appropriate limits. In addition, it is shown that an extension of the Lenard-Balescu result to include strong coupling can be interpreted as the Ziman collision frequency with the OCP structure factor replaced by the ion-ion structure factor for a two-component system. Numerical estimates of this structure factor are used to calculate the electrical conductivity in moderately coupled ($\Gamma \leq 2$) hydrogen plasmas.

I. INTRODUCTION

A reliable picture of electron transport in plasmas requires an accurate model for the electron collision frequency. In strongly coupled, degenerate plasmas the collision frequency has been described by the Ziman formula[1,2] with the ion correlations modeled by the classical one-component plasma (OCP). The primary difficulty with this approach is that it treats the electrons and ions as separate, independent subsystems. As a result, this model fails in the weak-coupling limit where the appropriate collision frequency is that found from the quantum generalization of the linearized Lenard-Balescu equation.[3,4] The latter equation treats electron and ion screening effects symmetrically, and includes correlations between the electrons and ions.

Recently, there has been progress in unifying these two points of view. It has been shown that the electron collision frequency in a neutral plasma containing only a single ion species may be approximated by[5]

$$\nu=\frac{\beta}{6m_e n_e}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int\frac{d^3k}{(2\pi)^3}v_{ei}^2(k)k^2[(S_{ee}(k,-\omega)S_{ii}(k,\omega)-S_{ei}(k,-\omega)S_{ie}(k,\omega)], \tag{1}$$

where $v_{ei}(k)$ is the electron-ion potential, β^{-1} is the temperature, m_e and n_e are the electron mass and number density, respectively, and the units are selected so that $\hbar=1$. As noted in Ref. 5, the electron-electron collisions do not contribute to Eq. (1), because they conserve the total electron momentum.

The functions, $S_{ab}(k,\omega)$, are the dynamic structure factors in a true two-component quantum system, and they are defined in terms of the density-density time-correlation functions

$$S_{ab}(k,\omega)\equiv\int_{-\infty}^{\infty}dt\, e^{i\omega t}\langle\rho_b(-k)\rho_a(k,t)\rangle_0 , \tag{2}$$

where $\langle\cdots\rangle_0$ indicates an average over the equilibrium grand canonical ensemble. The operator $\rho_a(k)$ represents the Fourier transformed density of species a and is defined in terms of the quantum position operators $\vec{r}_a$,

$$\rho_a=\Omega^{-1/2}\sum_{i=1}^{N_a}e^{i\vec{k}\cdot\vec{r}_a(i)},$$

where N_a is the number of particles of species a and Ω is the volume of the system. The structure factors are quite difficult to calculate in general, but a considerable simplification results if the electrons and ions are viewed as two independent, one-component systems. In this approximation $S_{ei}(k,\omega)$ vanishes, and it has been shown[5] that if $S_{ee}(k,\omega)$ is calculated in the random-phase approximation, and $S_{ii}(k,\omega)$ is calculated from a generalization of this same approximation that includes strong ion-ion correlations, the Ziman formula mentioned above is obtained,

$$\nu = -\frac{m_e}{12\pi^3 Z}\int_0^\infty dp \frac{\partial f_e(p)}{\partial p}\int_0^{2p} dk\ k^3 \left|\frac{v_{ei}(k)}{\epsilon_{\rm RPA}(k,0)}\right|^2 S_{\rm OCP}(k)$$
$$= \frac{m_e}{12\pi^3 Z}\int_0^\infty dk\ f_e(k/2)k^3 \left|\frac{v_{ei}(k)}{\epsilon_{\rm RPA}(k,0)}\right|^2 S_{\rm OCP}(k)\ . \quad (3)$$

Here, $f_e(p)$ is the electron momentum distribution, Z is the net ionization of the ion, $\epsilon_{\rm RPA}$ is the random-phase dielectric function for a (one-component) electron gas, and $S_{\rm OCP}(k)$ is the exact static-structure factor for the classical one-component plasma.[6] Notice that while Eq. (3) accounts for strong correlations among the ions, it allows only weak electron-ion collisions through an effective Born cross section.

This result requires that the electron-ion correlations be neglected while the electron-electron and ion-ion correlations are retained. Whenever this is inappropriate a different approach is necessary. In particular, we demonstrate in Sec. II that for weak coupling, the functions, $S_{ab}(k,\omega)$, in Eq. (1) can be evaluated in the two-component random-phase approximation to reproduce the correct quantum Lenard-Balescu result. By exploiting the large ion-electron mass ratio, we show that the Lenard-Balescu result is equivalent to Eq. (3) with $S_{\rm OCP}(k)$ replaced by the weak-coupling limit of the two-component ion-ion structure factor. In the third section, we use this result and the strong-coupling classical result to suggest a strong-coupling extension of the quantum Lenard-Balescu result for nondegenerate plasmas, and show that it can also be put into a Ziman-like form with the structure factor generalized to include electron-ion correlations. In the final two sections we present some numerical results for the electrical conductivity of hydrogen and a critique of the theory.

II. WEAKLY COUPLED QUANTUM SYSTEMS

In general, the structure factors $S_{ab}(k,\omega)$ are related to the response functions,

$$\tilde{X}_{ab}(k,z) \equiv -i\int_0^\infty dt\ e^{izt}\langle[\rho_a(k,t),\rho_b(-k)]\rangle_0, \quad {\rm Im}z>0 \quad (4)$$

through a form of the fluctuation-dissipation theorem,[7]

$$S_{ab}(k,\omega) = \frac{2}{(1-e^{\beta\omega})}{\rm Im}[\lim_{\eta\to 0+}\tilde{X}_{ab}(k,\omega+i\eta)]\ . \quad (5)$$

The procedure followed here will be to solve for the structure factors from Eq. (5) by writing the response functions in the form

$$\tilde{X}_{ab}(k,z) = \frac{1}{\Omega}{\rm Tr}_1 e^{i\vec{k}\cdot\vec{r}_a(1)}\tilde{Y}_{ab}(1;kz) \quad (6)$$

and calculating the single-particle operators, $\tilde{Y}_{ab}$, from the weak-coupling kinetic equations for two components[8]

$$[z-L_a(1)]\tilde{Y}_{ab}(1;kz) - \sum_{a'} B_{aa'}(1)\tilde{Y}_{a'b}(1;kz) = Y_{ab}(1;k)\ . \quad (7)$$

The Liouville operator, $L_a(1)$, represents commutation with the kinetic energy of a particle of species a, and the initial condition, $Y_{ab}(1;k)$, is defined by

$$Y_{ab}(1;k) \equiv [e^{-i\vec{k}\cdot\vec{r}_b(1)},\hat{f}_a(1)]\ , \quad (8)$$

where $\hat{f}_a(1)$ is the quantum operator whose momentum matrix elements give $f_a(p)$. In the weak-coupling limit, the mean-field operators B_{ab}, defined in Ref. 5, may be evaluated to lowest order in the interactions to give

$$B_{aa'}(1)\tilde{Y}_{a'b}(1;kz) = \int\frac{d^3k'}{(2\pi)^3}v_{aa'}(k')[e^{-i\vec{k}'\cdot\vec{r}_a(1)},f_a(1)]{\rm Tr}_2 e^{i\vec{k}'\cdot\vec{r}_{a'}(2)}\tilde{Y}_{a'b}(2;kz)$$
$$= v_{aa'}(k)Y_{aa}(1;k)\tilde{X}_{a'b}(k,z)\ , \quad (9)$$

which is the quantum generalization of the linearized Vlasov operator for two components. Solving Eqs. (7)–(9) gives

$$\tilde{X}_{ab}(k,z) = \tilde{\chi}_a(k,z)\left[\delta_{ab} + \sum_{a'} v_{aa'}(k)\tilde{X}_{a'b}(k,z)\right], \tag{10}$$

with

$$\tilde{\chi}_a(k,z) \equiv (2s_a+1)\int \frac{d^3p}{(2\pi)^3}\frac{f_a(|\vec{p}+\vec{k}|)-f_a(p)}{z+\Delta E_a}. \tag{11}$$

In Eq. (11), s_a is the spin of species a, and

$$\Delta E_a = \frac{|\vec{p}+\vec{k}|^2}{2m_a} - \frac{p^2}{2m_a}$$

is the kinetic energy change of a particle of species a. For two components, Eq. (10) represents a set of four equations which may be solved simultaneously to give

$$\tilde{X}_{ee}(k,z) = \epsilon^{-1}(k,z)\epsilon_i(k,z)\tilde{\chi}_e(k,z), \tag{12}$$

$$\tilde{X}_{ei}(k,z) = \tilde{X}_{ie}(k,z) = \epsilon^{-1}(k,z)v_{ei}(k)\tilde{\chi}_e(k,z)\tilde{\chi}_i(k,z), \tag{13}$$

$$\tilde{X}_{ii}(k,z) = \epsilon^{-1}(k,z)\epsilon_e(k,z)\tilde{\chi}_i(k,z), \tag{14}$$

where $\epsilon(k,z)$ is the random-phase dielectric function for a two-component system,

$$\begin{aligned}\epsilon(k,z) &= \epsilon_e(k,z)\epsilon_i(k,z) - v_{ei}^2(k)\tilde{\chi}_e(k,z)\tilde{\chi}_i(k,z)\\ &= 1 - v_{ee}(k)\tilde{\chi}_e(k,z) - v_{ii}(k)\tilde{\chi}_i(k,z),\end{aligned} \tag{15}$$

and $\epsilon_a(k,z)$ is the corresponding one-component result,

$$\epsilon_a(k,z) = 1 - v_{aa}(k)\tilde{\chi}_a(k,z). \tag{16}$$

The second line in Eq. (15) occurs because $v_{ee}v_{ii} = v_{ei}^2$ for the Coulomb interaction. By taking the imaginary parts of Eqs. (12)–(14), the structure factors may be found through use of Eq. (5). Combining the results then yields, after a few manipulations,

$$S_{ee}(k,-\omega)S_{ii}(k,\omega) - S_{ei}(k,-\omega)S_{ie}(k,\omega) = \frac{4e^{\beta\omega}}{(e^{\beta\omega}-1)^2}|\epsilon(k,\omega)|^{-2}\,\mathrm{Im}\tilde{\chi}_e\,\mathrm{Im}\tilde{\chi}_i. \tag{17}$$

By noting from Eq. (11) that

$$\mathrm{Im}\tilde{\chi}_a(k,\omega+i\eta) = \pi(2s_a+1)\int \frac{d^3p}{(2\pi)^3}\delta(\omega+\Delta E_a)(1-e^{-\beta\Delta E_a}) \times f_a(p)[1-f_a(\vec{p}+\vec{k})],$$

Eq. (17) may be put into Eq. (1) and the frequency integral performed to give the desired result,

$$\nu = \frac{2\pi\beta}{3m_e n_e}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3\bar{p}}{(2\pi)^3}k^2\left|\frac{v_{ei}(k)}{\epsilon(k,\Delta E_e)}\right|^2 f_e(p)[1-f_e(|\vec{p}+\vec{k}|)]f_i(\bar{p})\delta(\Delta E_e+\Delta E_i), \tag{18}$$

where it has been assumed the ions obey Maxwell-Boltzmann statistics. This is precisely the collision frequency associated with the linearized form of the quantum Lenard-Balescu equation studied by Lampe.[3]

In order to compare Eqs. (18)–(3) we have to perform the integration over the ion momentum. To lowest order in the electron-ion mass ratio this may be done just as in Ref. 5 to obtain

$$\nu = -\frac{m_e}{12\pi^3 Z}\int_0^\infty dp\frac{\partial f_e}{\partial p}\int_0^{2p} dk\, k^3 v_{ei}^2(k)\bar{S}(k), \tag{19}$$

where the cutoff of the k integration at $2p$ is due to the energy conserving delta function in Eq. (18). The function $\bar{S}(k)$ is defined by

$$\begin{aligned}\bar{S}(k) &\equiv \pi^{-1/2}\int_{-\infty}^{\infty} dY\, e^{-Y^2}\left|\epsilon\left[k,\left[\frac{2}{\beta m_i}\right]^{1/2}kY\right]\right|^{-2}\\ &\approx \pi^{-1/2}\int_{-\infty}^{\infty} dY\, e^{-Y^2}|1-v_{ee}(k)\tilde{\chi}_e(k,0) - k_i^2/k^2\psi(Y)|^{-2},\end{aligned} \tag{20}$$

where $k_i^2 = 4\pi\beta n_i Z^2 e^2$, and

$$\psi(Y)=i\pi^{1/2}Ye^{-Y^2}-\pi^{-1/2}PP\int_{-\infty}^{\infty}dx\ e^{-x^2}\frac{x}{x-Y}\ . \tag{21}$$

In the second line of Eq. (20), we have used the classical limit of $\tilde{\chi}_i$, and the zero-frequency limit of $\tilde{\chi}_e$. These approximations are correct to lowest order in m_e/m_i. The integral may now be performed using contour integration[4] to obtain

$$\begin{aligned}\bar{S}(k)&=\epsilon_e^{-1}(k,0)\epsilon^{-1}(k,0)\\&=\epsilon_{\mathrm{RPA}}^{-1}(k,0)\epsilon^{-1}(k,0)\ ,\end{aligned} \tag{22}$$

where the subscript "e" has been replaced with "RPA" in order to conform to the notation used in Eq. (3). Therefore, after a partial integration, Eq. (19) becomes

$$\begin{aligned}\nu&=\frac{m_e}{12\pi^3Z}\int_0^{\infty}dk\ f_e(k/2)k^3\frac{v_{ei}^2(k)}{\epsilon_{\mathrm{RPA}}(k,0)\epsilon(k,0)}\\&=\frac{m_e}{12\pi^3Z}\int_0^{\infty}dk\ f_e(\tfrac{k}{2})k^3\left|\frac{v_{ei}(k)}{\epsilon_{\mathrm{RPA}}(k)}\right|^2S_{ii}^{w}(k)\ ,\end{aligned} \tag{23}$$

where

$$S_{ii}^{w}(k)\equiv\frac{1}{1+\beta n_i v_{ii}\epsilon_{\mathrm{RPA}}^{-1}(k,0)}$$

is the weak-coupling structure factor for ions interacting through the Coulomb potential screened by the electrons through ϵ_{RPA}. Equation (23) reduces to the weak-coupling form of Eq. (3) only when $\epsilon_{\mathrm{RPA}}\simeq 1$ for the values of k of most significance in Eq. (23). This can occur in an extremely degenerate plasma in which the Thomas-Fermi screening length becomes very large, but in such a case the screening of v_{ei} by the electrons would also be negligibly small. Therefore, it appears that it is inconsistent to screen v_{ei} without also screening v_{ii} in the calculation of the ion structure factor.[9]

It is interesting at this point to note that in the Debye-Hückel limit, Eq. (23) may also be written as[10]

$$\begin{aligned}\nu&=\frac{m_e}{12\pi^3Z}\int_0^{\infty}dk\ f_e(k/2)k^3\left[\frac{4\pi Ze^2}{k^2+k_e^2}\right]^2\frac{k^2+k_e^2}{k^2+k_e^2+k_i^2}\\&=\frac{m_en_e}{3Z}\left[\frac{\beta}{2\pi m_e}\right]^{3/2}\int_0^{\infty}dk\ e^{-\beta k^2/8m_e}k^3\left|\frac{v_{ei}}{\epsilon_e^{\mathrm{DH}}(k,0)}\right|^2S_{ii}^{\mathrm{DH}}(k)\ ,\end{aligned} \tag{24}$$

which is of the same form as Eq. (3) with the Debye-Hückel estimate to the two-component ion-ion static structure factor and to the electron screening function.

III. THE CLASSICAL LIMIT AND STRONG COUPLING

In Sec. II we demonstrated that the collision frequency predicted by the linearized, quantum Lenard-Balescu equation is equivalent to the Ziman formula with the correct weak-coupling ion-ion structure factor. The purpose of this section is to discuss an approximation which will extend this result to strong couping. However, to motivate this approximation, we will first discuss the strongly coupled classical limit of Eq. (1).

It has already been shown[5] that for strongly coupled classical systems, Eq. (1) can be generalized to

$$\nu=\frac{\beta}{6m_en_e}\int\frac{d\omega}{2\pi}\int\frac{d^3k}{(2\pi)^3}k^2u_{ei}(k)v_{ei}(k)\left[S_{ee}(k,-\omega)S_{ii}(k,\omega)-S_{ei}(k,-\omega)S_{ie}(k,\omega)\right]\ , \tag{25}$$

where the generalized potentials, u_{ab}, are defined in terms of the direct correlation functions, c_{ab},

$$u_{ab}(k)\equiv-\beta^{-1}c_{ab}(k)\ . \tag{26}$$

The direct correlation functions are defined by the Ornstein-Zernike relations

$$c_{ab}(k)=h_{ab}(k)+\sum_{a'}c_{aa'}(k)n_a'h_{a'b}(k)\ , \tag{27}$$

$$h_{ab}(k)=\int d^3r\ e^{i\vec{k}\cdot\vec{r}}\left[g_{ab}(r)-1\right]\ . \tag{28}$$

In terms of these functions, the static structure factors are

$$S_{ab}(k) \equiv \delta_{ab} + n_b h_{ab}(k) = \frac{\delta_{ab}(1+\beta n_i u_{ii} + \beta n_e u_{ee}) - \beta n_b u_{ab}}{(1+\beta n_i u_{ii})(1+\beta n_e u_{ee}) - \beta^2 n_e n_i u_{ei}^2} . \tag{29}$$

Equation (25) differs from Eq. (1) primarily due to the replacement of one v_{ei} with u_{ei}. This replacement, which can be done exactly only in the classical limit, preserves the exact short-time behavior of the memory operator from which the collision frequency is calculated.[5,11]

It is also possible in the classical limit to evaluate the mean-field operators in Eq. (9) exactly in terms of the u_{ab}.[12] As a result, the response functions may now be determined from the classical limit of Eq. (6):

$$\tilde{X}_{ab}(k,z) = \int \frac{d^3p}{(2\pi)^3} \int d^3r \, e^{i\vec{k}\cdot\vec{r}} \tilde{Y}_{ab}(pr;z) = \int \frac{d^3p}{(2\pi)^3} \tilde{Y}_{ab}(pk;z) , \tag{30}$$

where $\tilde{Y}_{ab}(pk;z)$ is a function[13] which satisfies

$$\left[z - \frac{\vec{k}\cdot\vec{p}}{m_a} \right] \tilde{Y}_{ab}(pk;z) + \sum_{a'} u_{aa'}(k) \frac{\beta\vec{k}\cdot\vec{p}}{m_a} f_a(p) \tilde{X}_{a'b}(k,z) = -\frac{\beta\vec{k}\cdot\vec{p}}{m_a} f_a(p)\delta_{ab} . \tag{31}$$

Solving Eq. (31) and using the classical form of Eq. (5) then yields

$$\nu = \frac{2\pi\beta}{3m_e n_e} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3P}{(2\pi)^3} k^2 u_{ei}(k) v_{ei}(k) \, | \, \epsilon(k, \vec{k}\cdot\vec{P}/M) \, |^{-2} \times f_e(p) f_i(P) \delta\left[\frac{\vec{p}\cdot\vec{k}}{\mu} \right] , \tag{32}$$

where both the electron and ion momentum distributions are Maxwell-Boltzmann and a transformation to relative and center-of-mass coordinates has been made. The reduced mass and total mass are $\mu(\simeq m_e)$ and $M(\simeq m_i)$, respectively. The screening factors are defined just as before, but with v_{ab} replaced by u_{ab},

$$\epsilon(k,\omega) = \epsilon_e(k,\omega)\epsilon_i(k,\omega) - u_{ei}^2(k)\tilde{\chi}_e(k,\omega)\tilde{\chi}_i(k,\omega) , \tag{33}$$

$$\epsilon_a(k,\omega) = 1 - u_{aa}(k)\tilde{\chi}_a(k,\omega) , \tag{34}$$

and with $\tilde{\chi}_a$ given by the classical limit of Eq. (11). This result is a multicomponent generalization of the effective interaction approximation described by Gould and Mazenko.[11,14]

The ion momentum integrations may now be performed just as before with the result

$$\nu = \frac{m_e n_e}{3Z} \left[\frac{\beta}{2\pi m_e} \right]^{3/2} \int_0^\infty dk \, k^3 \frac{u_{ei}(k) v_{ei}(k)}{\epsilon_e(k,0)\epsilon(k,0)} = \frac{m_e n_e}{3Z} \left[\frac{\beta}{2\pi m_e} \right]^{3/2} \int_0^\infty dk \, k^3 \frac{u_{ei}(k) v_{ei}(k)}{|\, \epsilon_e(k,0) \,|^2} S_{ii}(k) , \tag{35}$$

where use has been made of Eq. (29).

This result is of the same general form as Eq. (23) and Eq. (24), but with the ion structure factor and electron screening function generalized to strong coupling. However, the exponential, which is responsible for the convergence of the integral in Eq. (24), is absent from Eq. (35). Nevertheless, Eq. (35) will still be finite because u_{ei} decreases more rapidly than v_{ei} at large k.[11] For weak coupling, however, u_{ei} will cut off the integration more slowly than the exponential, with the result that Eq. (35) should give significantly larger answers than Eq. (24) in the weak-coupling limit. The exponential does not appear in Eq. (35) because the classical form of the response functions leads to a delta function in Eq. (32) which conserves energy only to lowest order in the momentum transfer, $\vec{k}$.

This comparison of the quantum weak-coupling and classical results suggests using a "hybrid" approach in which the structure factors in Eq. (25) are evaluated by using Eqs. (5)–(9), but with the potentials appearing in the mean-field operator replaced with the classical generalized potentials in Eq. (26). Such an approach should provide a strong-coupling extension of Eq. (23) for nondegenerate plasmas that includes the quantum cutoff.

Following the procedure outlined above yields,

$$\nu=\frac{2\pi\beta}{3m_e n_e}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3P}{(2\pi)^3}k^2u_{ei}(k)v_{ei}(k)\,|\,\epsilon(k,\vec{k}\cdot\vec{P}/M)\,|^{-2}f_e(p)f_i(P)\delta\left[\frac{\vec{p}\cdot\vec{k}}{\mu}+\frac{k^2}{2\mu}\right], \quad (36)$$

which is identical to Eq. (32) except that the delta function conserves energy, and the response functions are given by their quantum forms in Eq. (11). Performing the ion momentum integrations now leads to

$$\nu=\frac{m_e n_e}{3Z}\left[\frac{\beta}{2\pi m_e}\right]^{3/2}\int_0^\infty dk\; e^{-\beta k^2/8m_e}k^3\frac{u_{ei}(k)v_{ei}(k)}{\epsilon_e(k,0)\epsilon(k,0)}$$

$$=\frac{m_e n_e}{3Z}\left[\frac{\beta}{2\pi m_e}\right]^{3/2}\int_0^\infty dk\; e^{-\beta k^2/8m_e}k^3\frac{u_{ei}v_{ei}(k)}{|\,\epsilon_e(k,0)\,|^2}S_{ii}(k)\,, \quad (37)$$

where the second line uses the classical (low-k) limit of $\tilde{\chi}_e(k,0)$ in ϵ and ϵ_e. This formula combines the most important features of Eqs. (24) and (35). The exponential provides the correct quantum cutoff, while the screening factors incorporate the short-range (local-field) corrections to the mean fields, which are important in strongly coupled plasmas. Nevertheless, Eq. (37) still contains only weak, Born-like collisions, even though the particle correlations have modified the Born cross section in Eqs. (3) and (23) through the appearance of the product $u_{ei}(k)v_{ei}(k)$.

The transport properties of nondegenerate electron-ion plasmas have also been studied recently using theoretical[15] and numerical simulation[16] techniques by Hansen and his co-workers. In order to make contact with their results we note that Eq. (37) may also be written as

$$\nu=\frac{m_e n_e}{3Z}\left[\frac{\beta}{2\pi m_e}\right]^{3/2}\int_0^\infty dk\; k^3e^{-\beta k^2/8m_e}\frac{u_{ei}(k)v_{ei}(k)}{\epsilon_e(k,0)}[S_{ee}(k)S_{ii}(k)-S_{ei}(k)S_{ie}(k)]\,. \quad (38)$$

The corresponding result from Ref. 15 is

$$\nu_{\rm BHS}=\frac{m_e n_e}{3Z}\left[\frac{\beta}{2\pi m_e}\right]^{3/2}\int_0^\infty dk\; k^3w_{ei}^2(k)[S_{ee}(k)S_{ii}(k)-S_{ei}(k)S_{ie}(k)]\,, \quad (39)$$

where $w_{ei}(k)$ is a pseudopotential chosen to simulate quantum effects at short range and keep Eq. (39) finite. This result was obtained by retaining the exact initial values for $S_{ab}(k)$, but assuming the time dependence corresponding to the classical propagation of free particles. By comparison, the result in Eq. (38) assumes the time dependence corresponding to the quantum propagation of particles through a generalized mean field. The additional physics in Eq. (38) shows up as the exponential cutoff in the integration and the additional electron screening factor. Because no direct correlation function appears in Eq. (39), it fails to include the exact short-time behavior of the memory operator.

IV. NUMERICAL RESULTS

In this section, we will use Eq. (37) to estimate the electrical conductivity of a fully ionized hydrogen plasma. The collision frequency is related to the conductivity through

$$\sigma=\lambda(\Gamma)\frac{e^2n_e}{m_e\nu}\,, \quad (40)$$

where $\Gamma\equiv\beta e^2(4\pi n_i/3)^{1/3}$ is the plasma-coupling parameter. The factor $\lambda(\Gamma)$ is a correction for the fact that using Eq. (37) for ν is equivalent to a single-Sonine polynomial approximation. For very small Γ, the two-polynomial results for hydrogen are produced by taking $\lambda=1.93$.[4] However, for larger values of Γ this factor is greatly reduced, so that using the value 1.93 in Eq. (40) will only place an upper bound on the conductivity.

If we nonetheless select λ to be 1.93, Eqs. (40) and (37) may be combined to give

$$\sigma^*\equiv\frac{\sigma}{\omega_p}=\frac{1.93(3\pi/2)^{1/2}}{4\pi\Gamma^{3/2}\Lambda}\approx\frac{1}{3\Gamma^{3/2}\Lambda}\,, \quad (41)$$

where ω_p is the electron plasma frequency, and Λ is a generalization of the Coulomb logarithm defined by

$$\Lambda\equiv\frac{-1}{4\pi e^2}\int_0^\infty dk\; e^{-\beta k^2/8m_e}k\frac{u_{ei}(k)}{|\,\epsilon_e(k,0)\,|^2}S_{ii}(k)\,. \quad (42)$$

In the Debye-Hückel limit, Λ may be evaluated analytically, with the result

$$\Lambda \to \Lambda^{\mathrm{DH}} = e^{1/\alpha} E_1(1/\alpha) - \tfrac{1}{2} e^{1/2\alpha} E_1(1/2\alpha)$$
$$\to \tfrac{1}{2}(\ln\alpha - \gamma - \ln 2) \approx \tfrac{1}{2}(\ln\alpha - 1.27) , \tag{43}$$

where $\alpha = (\beta k_D^2/8m_e)^{-1}$, $E_1(x)$ is the exponential integral, and γ is Eulers constant.[18] The limiting expression in the second line of Eq. (43) corresponds precisely to that obtained by Williams and DeWitt[4] for their dynamic Born term. For comparison, we note that the corresponding result from Eq. (39) is[15]

$$\Lambda^{\mathrm{BHS}} = \frac{\ln\alpha + \ln(\pi/4) - 1 + (4/\pi\alpha)}{2[1-(4/\pi\alpha)]^2}$$
$$\to \tfrac{1}{2}[\ln\alpha - 1 - \ln(4/\pi)]$$
$$\approx \tfrac{1}{2}(\ln\alpha - 1.24) . \tag{44}$$

It seems remarkable that the numerical values of the terms following the logarithms in Eqs. (43) and (44) are so nearly equal.

We have also evaluated the integral in Eq. (42) numerically using direct correlation functions and structure factors found from the solution to the HNC equation.[19] The collapse of electron-ion pairs is prevented through the use of a pseudopotential, which at these temperatures and densities is the same as that employed in Ref. 15. The results of these calculations are presented in column A in Tables I and II. Column B contains the results obtained in the same manner, but with u_{ei} replaced by v_{ei} in Eq. (42). The conductivities found from the Debye-Hückel result, Eq. (43), are presented in column C. In columns D and E, we present, for the purposes of comparison, the numerical results of Ref. 15. Column D shows the conductivity from numerical integration of Eq. (39), with the corresponding Debye-Hückel results in column E. Finally, the columns labeled MD are the molecular dynamics results of Hansen and McDonald.[16]

Columns A, B, and C all show the same qualitative behavior. Initially, the conductivity decreases with increasing Γ, reaches a minimum somewhere near $\Gamma \simeq 1$, and then starts to rise again for large Γ. Columns D and E show the same decreasing behavior, but appear to reach their minimum value at larger Γ.

The difference between columns A and B in Tables I and II is due to the different behavior of u_{ei} and v_{ei} at large k. As mentioned before, u_{ei} falls off with k rapidly enough that it can cause the integral in Eq. (42) to converge even without the exponential. However, for low Γ the exponential appears to cutoff the integration before the u_{ei} can deviate significantly from k^{-2} behavior. For higher Γ the u_{ei} seems to gain more control over the integral, and presumably at large enough Γ would effectively cut the integration off before the exponential could have a serious effect. It is possible that this "competition" is related to the appearance of "high-temperature" and "low-temperature" cutoffs in the early theories of electrical conductivity.[20]

V. DISCUSSION

The primary objective of this paper has been to use Eq. (1) to explore the connection between the Ziman formula and the linearized quantum Lenard-Balescu equation. In Sec. II we explicitly demonstrate that the Lenard-Balescu collision fre-

TABLE I. Comparisons of σ^* for various values of Γ at $r_s = 0.4$ ($r_s \equiv a_e/a_0$, where $a_e^3 \equiv (4\pi n_e/3)^{-1}$ and a_0 is the Bohr radius). Column A gives the results for Eq. (41) using the numerical evaluation of Eq. (42); column B uses Eq. (42) with u_{ei} replaced by v_{ei}; column C gives the corresponding Debye-Hückel result, Eq. (43). Columns D, E, and MD are the results from Refs. 15 and 16.

Γ	A	B	C	D	E	MD
0.05	16.2	14.3	14.6	14.18	14.27	
0.1	8.61	7.22	7.69	7.13	7.28	
0.2	5.33	4.07	4.52	3.99	4.22	
0.5	4.13	2.51	3.55	2.30	2.71	3.6
1.0	5.29	2.43	4.88	1.87	2.48	
2.0	12.3	3.86	11.6			

TABLE II. Same as Table I, but with $r_s = 1.0$.

Γ	A	B	C	D	E	MD
0.05	12.7	11.8	11.9	11.72	11.78	
0.1	6.16	5.61	5.77	5.57	5.65	
0.2	3.36	2.91	3.12	2.87	2.99	
0.5	2.07	1.49	1.87	1.43	1.63	2.15
1.0	2.13	1.14	1.88	0.99	1.29	
2.0	3.72	1.25	3.11	0.82	1.27	1.1

quency is equivalent to the Ziman result with the weak-coupling structure factor for screened ions. This result is not equivalent to the weak-coupling limit of the OCP result in Eq. (3).

In the third section, we use the classical form of Eq. (1) to suggest a way to extend the result of Sec. II to strongly coupled nondegenerate plasmas. This extension is obtained by solving for the structure factors in Eq. (25) from the linearized, quantum Vlasov equation for two components with the potentials, $v_{ab}(k)$, in Eq. (9) replaced by the generalized potentials, $u_{ab}(k)$, defined in Eq. (26). In this approach, the local-field corrections to the mean fields governing the motions of the electrons and ions are related to the equilibrium correlation functions in the same way as in classical kinetic theory.[11,12,14] However, the quantum-diffraction effects upon the time development are included, just as in the quantum weak-coupling case through the commutator appearing in Eq. (9). This last feature retains the energy-conserving delta function in Eq. (36) and provides the correct quantum cutoff for the k integration. As a result, Eq. (37) reproduces the correct weak-coupling answer without the introduction of a pseudopotential. Unfortunately, such a computational strategem is necessary for more strongly coupled plasmas since the most common techniques for finding equilibrium correlation functions, such as the solution of the HNC equation employed here, are based upon classical mechanics and cannot handle the attractive Coulomb potential. If a completely quantum-mechanical method for calculating these structure factors ever becomes practical, it could be employed in Eq. (37).

The approach described here differs from that adopted by Hansen and his co-workers.[15,16] They model the electron-ion plasma as a classical system of pseudoparticles which interact through the Coulomb potential at large distances, but the potential is modified at distances less than the thermal de Broglie wavelength in a way that simulates quantum diffraction effects. The fact that columns B and D in Tables I and II are in essential agreement indicates that the behavior of the classical pseudoparticles imitates that of the true, quantum system fairly well, at least in the regions of temperature and density examined so far. However, a major drawback of both theories is that, while they allow for strong correlations, they fail to include the effects of close, strong collisions such as those described in the weak-coupling theory of Williams and DeWitt.[4] Such effects can be very important[5,17] and should be investigated further.

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