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Self-focusing and spatial ringing of intense cw light propagating through a strong absorbing medium

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This paper deals with the strongly nonlinear regime of the atom and field interaction which arises for both high cw input intensities and large absorption. The intensity of the light inside the medium is then highly nonlinear with respect to the incident one, the order of magnitude of the nonlinearity being approximately αl, for a cell of length l and an absorption length α^{-1}. The transverse effects related to such a regime are analytically treated. They are shown to induce self-focusing, blooming, and also spatial ringing. These features strongly depend on the distance of the screen from the cell, on the detuning between the incident wavelength and that of the atomic transition, and on the input intensity. For self-focusing, analytical expressions of the focal point, of the waist, and of the maximum intensity are given. This model for transverse effects in strong nonlinear propagation of pulses would contribute to explain recent anomalous observations.

I. INTRODUCTION

Recent experiments of propagation of pulses generated by dye lasers through dense absorbing gaseous media have displayed many spatial effects which are not yet fully explained.[1–3] The interaction of the light with the material modifies so strongly the spatial shape of the incident beam that the patterns of a visible transmitted beam display blooming spots, surrounding rings, large and/or small random speckles .... These effects are seen to strongly depend on the intensity of the driving pulse, on the detuning between the incident pulsation ω ($\omega=2\pi c/\lambda$) and the atomic one ω_a, $\delta=\omega_a-\omega$, and also on the atomic density.

These transverse deshapings of the pulse must be related to self-focusing or defocusing effects because of the very high intensities of the incident beam, of magnitude of order a few $\mathrm{mW/cm^2}$.

The concept of self-focusing[4–11] implies simultaneously the narrowing of the section of the optical beam and the enhancement of the on-axis intensity. The narrowing of the beam section is a consequence of a relevant nonlinear index of refraction, that requires in the case of a cw light, or a steady-state model, the off-resonance condition (condition I). The enhancement of the on-axis intensity, which is understandable for amplifying material requires two conditions for absorbing media: (a) a dilute material in order that the absorption may be neglected (condition II) and (b) a converging input pulse (condition III).[9] The standard theory for the steady-state self-focusing assumes that the three above conditions are met. Furthermore it assumes a cubic medium, or in other words, it takes only the lowest nonlinearity of the index of refraction into account. This later assumption implies that the input intensity $I(0)$ normalized to the saturation intensity $(1+\delta^2T_2^2)/\beta$ is smaller than unity (The parameter β is equal to $|\vec{\mu}|^2T_1T_2/\hbar^2$, where $\vec{\mu}$ is the dipole moment for an atom, and $T_{1,2}$ are well-known relaxaton times of the Bloch equations; *the intensity is defined as the squared modulus of the electric field.)* This condition requires together with the condition (II) for transparency the inequalities

$$\frac{\alpha l}{1+\delta^2T_2^2} << \frac{\beta I(0)}{1+\delta^2T_2^2} << 1\ , \tag{1.1}$$

to be fulfilled (α^{-1} is the on-resonance absorption length of the Beer's law and l is the cell length).

The usual treatment of self-defocusing in the steady-state model or the adiabatic following model[12] also assumes the inequalities (1.1) to be satisfied.

Self-focusing in absorbing media was predicted and analyzed by Wright and co-workers[13,14] in the case of propagation of very short pulses, (i.e., in transient regime), within the conditions for self-induced transparency[15] (SIT). This effect which proved that self-focusing, may be realized even *for large absorption* ($\alpha l >> 1$), and *on resonance* was experimentally verified by Gibbs and co-workers.[16]

The off-resonance condition (I) can be satisfied without detuning since the time variation of the field phase plays the role of an instantaneous detuning. The enhancement of the on-axis intensity which is found in spite of large αl results from a competition between the absorption and the transverse effects. These latter proceed from the introduction of the transverse Laplacian $\nabla_T^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ in the reduced Maxwell equation that becomes relevant only if the incident wave is not a plane wave.

The transverse effects were generally neglected in transient propagation effects, either self-induced transparency or superfluorescence. The work of Mattar[14] displayed important transverse effects in SIT which were recognized in the aforesaid experiment.[16] More recently, Mattar[17] has shown the primary role of the transverse effects, expecially in order to understand the absence of temporal ringings in some cases of superfluorescence, as it was portended by two of us.[18]

Strong transverse effects would be also expected in the case of an intense cw light propagating through a dense absorbing medium. Such effects would explain some of the anomalous features recently reported.[1-3]

The object of this paper is to deal with the strong nonlinear regime, which appears when the inequalities

$$\frac{\alpha l}{1+\delta^2 T_2^2} >> \frac{\beta I(0)}{1+\delta^2 T_2^2} >> 1 , \tag{1.2}$$

are fulfilled, and to propose an analytical treatment of self-focusing or defocusing for this case [Eq. (1.2)] where the standard theory[4-12] breaks down. Let us summarize the condition of our treatment as follows:

(a) An absorbing medium containing homogeneously broadened two-level atoms nearly resonant with

(b) a monochromatic incident light of Gaussian cross section and plane phase front;

(c) the slowly varying envelope and the paraxial ray approximations leading both to the reduced Maxwell equation;

(d) strongly nonlinear regime [cf. Eq. (1.2)].

With the above conditions let us divide the cell in two parts:

(1) In the first part ($0 \leq z \leq z_{NL}$) the diffraction is negligible (cf. Appendix A where sufficient condition is given), therefore the intensity obeys the implicit equation firstly given by Icsevgi and Lamb for an amplifying medium. In the vicinity of the abscissa z_{NL} defined in Sec. II in case of inequality (1.2), the intensity $I(z,r)$ is strongly nonlinear with respect to $I(o,r)$. It follows a narrowing of the cross section only caused by the nonlinearities of the intensity.

(2) In the second part ($z_{NL} < z$) we treat the diffraction effects in the linear Maxwell equation approximation because $I(z,0) << 1 << \alpha z$ in this domain (Sec. III). Let us point out that as soon as the electric field $\epsilon(z)$ is linear with respect to $\epsilon(z_{NL})$, it is strongly nonlinear with respect to ϵ_0, consequently we call it the "strong nonlinear regime." The general solution for the field amplitude is shown to be an infinite expansion in Kogelnik[19] functions $\psi_n(r,z)$ with waist $w_0/\sqrt{2n+1}$, where w_0 is the waist of the input beam ($s = \pi w_0^2$). For negative detuning *self-focusing is predicted,* the focus, the waist, and an estimation of the on-axis maximum intensity are given. After the focal point, the beam exhibits one or several spatial ringings before it definitely blooms. For positive or zero detuning the beam defocuses from the origin z_{NL}. Also ringings appear on a large range of propagation length, for $\delta > 0$. The patterns of these oscillations are studied as a function of $\beta I(0)$, δ, and z. The intensity patterns reveal also a small ring surrounding the central spot for vanishing detuning.

It would be desirable that a systematical experimental study of these effects may be attempted in order to check the present theoretical work.

I. MODEL

A. Basic equations

Within the semiclassical approach of the interaction atoms and field,[20] the transverse electric field $\vec{E}(x,y,z)$ obeys the Maxwell equation with sources,

$$\Delta\vec{E}(x,y,z)-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=\frac{4\pi}{c^2}\frac{\partial^2\vec{P}(x,y,z)}{\partial t^2} . \quad (2.1)$$

The polarization $\vec{P}(x,y,z)$ is defined for an homogeneous medium as

$$\vec{P}(x,y,z)=N\vec{\mu}\int d\vec{r}\sum_j \delta(\vec{r}-\vec{r}_j)R_j^- + \text{H.c.} , \quad (2.2)$$

where R_j^- is the lowering operator for atom j from its upper level $|+\rangle_j$ to its lower level $|-\rangle_j$, and N is the atomic density. The vector $\vec{\mu}$ is the atomic dipole moment, which is parallel to the electric field in the case of an atomic gas.

With an input electric field,

$$\vec{E}(x,y,z)=\tfrac{1}{2}\vec{x}\epsilon(x,y,z)e^{-i\omega(t-z/c)}+\text{c.c.} ,$$

propagating in the forward direction $\vec{0z}$, the complex amplitude $\epsilon(x,y,z)$ is found to obey a reduced Maxwell equation,[4]

$$\left[\frac{\partial}{\partial z}+\frac{c}{2i\omega}\nabla_T^2\right]\epsilon(x,y,z) =2i\pi\frac{\omega}{c}N|\vec{\mu}|\mathscr{P}(x,y,z) , \quad (2.3)$$

where the second derivative $\partial^2/\partial z^2$ has been neglected. The notation ∇_T^2 means

$$\nabla_T^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} , \quad (2.4)$$

and $\mathscr{P}(x,y,z)$ represents the complex amplitude of the positive frequency part of the polarization.

Neglecting the derivative term $\partial^2\epsilon/\partial z^2$ in (2.1) requires the paraxial approximation to be valid. This approximation was discussed by Lax and coworkers[21] in relation with the role of the transverse effects and the Coulomb gauge framework. It is found to be valid if the ratio between the wavelength λ and the beam waist w_0 is much smaller than unity. In the present paper, the condition for neglecting $\partial^2\epsilon/\partial z^2$ will be assumed to be met.

The first term in the left-hand side of Eq. (2.3) describes the variation of the field amplitude with penetration in the medium. The left-hand side of Eq. (2.3) reduces to it in the plane-wave approximation, $\epsilon(x,y,z)=\epsilon(z)$. The second term will give rise to the diffraction of the light. It is responsible for self-focusing and defocusing.

The scalar $\mathscr{P}(x,y,z)$ obeys the Bloch equations[20] deduced from the Schrödinger equation

$$\frac{\partial\mathscr{P}}{\partial t}=-\left[i\delta+\frac{1}{T_2}\right]\mathscr{P}-i\frac{|\vec{\mu}|}{\hbar}\epsilon W ,$$

$$\frac{\partial W}{\partial t}=-\frac{1}{T_1}(W+\tfrac{1}{2})-\frac{i|\vec{\mu}|}{2\hbar}(\epsilon\mathscr{P}^*-\text{c.c.}) , \quad (2.5)$$

where $W(x,y,z)$ measures the population difference between the upper and the lower levels for an atom located at (x,y,z). The detuning δ is the difference between the atomic pulsation ω_a and ω

$$\delta=\omega_a-\omega . \quad (2.6)$$

The relaxation times T_1 and T_2 are the radiative lifetime and the homogeneous lifetime, respectively,

$$\frac{1}{T_1}=\frac{4}{3}\frac{\omega^3}{c^2\hbar}\mu^2 ,$$

$$\frac{1}{T_2}=\frac{1}{T_2'}+\frac{1}{2T_1} . \quad (2.7)$$

The relaxation time T_2' results from collisions which conserve the atomic energy.

We will consider incident pulses with coherence times much larger than T_1 and T_2 in order to neglect transient effects. Then, the steady-state conditions for the atoms

$$\frac{\partial}{\partial t}\mathscr{P}(x,y,z)=0 ,$$

$$\frac{\partial}{\partial t}W(x,y,z)=0 , \quad (2.8)$$

leads to a closed equation for the field amplitude

$$\left[\frac{\partial}{\partial z}+\frac{1}{2ik}\nabla_t^2\right]\epsilon(x,y,z)=-\frac{\alpha}{2}\frac{(1-i\delta T_2)}{1+\delta^2T_2^2}\times\frac{\epsilon(x,y,z)}{1+\frac{\beta}{1+\delta^2T_2^2}|\epsilon(x,y,z)|^2} . \quad (2.9)$$

The parameter β was defined in Sec. I. The absorption length α^{-1} is defined by

$$\alpha=\frac{4\pi\omega|\mu|^2N}{c\hbar}T_2\equiv\frac{T_2}{\tau_R l} , \quad (2.10)$$

where τ_R is the superradiant time introduced in the theory of superradiance.[22]

B. Plane-wave approximation

As a first step, Eq. (2.9) can be solved within the plane-wave approximation, $\epsilon(x,y,z) \equiv \epsilon(z)$. Then the solution $\epsilon(z)$ of

$$\frac{\partial}{\partial z}\epsilon = -\frac{\alpha}{2}\frac{1}{1+i\delta T_2}\frac{\epsilon}{1+\frac{\beta}{1+\delta^2 T_2^2}|\epsilon|^2}, \tag{2.11}$$

obeys the well-known equation[23]

$$\epsilon(z) = \epsilon(0)\exp\left[-\frac{1}{1+i\delta T_2}\tfrac{1}{2}[\alpha z - \beta|\epsilon(0)|^2 + \beta|\epsilon(z)|^2]\right]. \tag{2.12}$$

Then the normalized intensity

$$\mathcal{J}(z) = \frac{\beta}{1+\delta^2 T_2^2}|\epsilon(z)|^2, \tag{2.13}$$

obeys the implicit equation

$$\mathcal{J}(z) = \mathcal{J}_0 e^{-\tilde{\alpha} z + \mathcal{J}_0 - \mathcal{J}(z)}, \tag{2.14}$$

with the notations $\mathcal{J}_0 \equiv \mathcal{J}(0)$, and $\tilde{\alpha} = \alpha(1+\delta^2 T_2^2)^{-1}$.

Figures 1(a) and 1(b) exhibit the behavior of $\mathcal{J}(z)$ as a function of $\mathcal{J}_0$ for two penetrations $\tilde{\alpha} z = 10$ and $\tilde{\alpha} z = 30$, respectively (full lines).

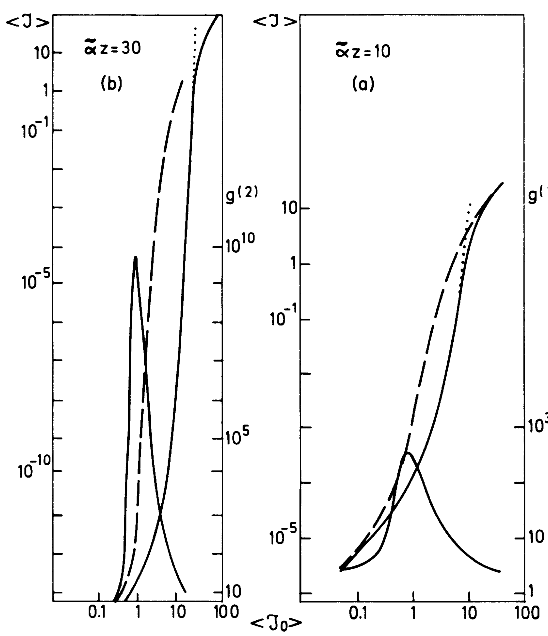

FIG. 1. $\langle \mathcal{J} \rangle$ as given by Eq. (2.14) as a function of $\langle \mathcal{J}_0 \rangle$. The full line corresponds to an input coherent field. The dotted line exhibits the departure of the approximated law [Eq. (2.16)] from the exact one [Eq. (2.14)]. The broken line corresponds to an input chaotic field. The scale of the full line $g^{(2)}$ [Eq. (4.3)] as a function of $\langle \mathcal{J}_0 \rangle$ is given on the right side of the vertical axis. In (a) $\tilde{\alpha} z = 10$ and (b) $\tilde{\alpha} z = 30$.

For $\mathcal{J}_0$ much smaller or much larger than $\tilde{\alpha} z$, $\mathcal{J}(z)$ varies practically linearly with $\mathcal{J}_0$, either

$$\mathcal{J}(z) = \mathcal{J}_0 e^{-\tilde{\alpha} z}, \quad \mathcal{J}_0 << \tilde{\alpha} z \tag{2.15a}$$

for the Beer's law regime, or

$$\mathcal{J}(z) = \mathcal{J}_0 - \tilde{\alpha} z, \quad \mathcal{J}_0 >> \tilde{\alpha} z \tag{2.15b}$$

for a quasitransparent medium. Between these two limits the Figs. 1 exhibit a nonlinear variation of $\mathcal{J}(z)$ with respect to $\mathcal{J}_0$. For example, for penetration $\tilde{\alpha} z = 30$, $\mathcal{J}(z)$ decreases from 1 to 2×10^{-8} as $\mathcal{J}_0$ decreases from 28 to 10. This very strong nonlinear regime for $\mathcal{J}(z)$ can be well described by the law

$$\mathcal{J}(z) = \mathcal{J}_0 e^{-\tilde{\alpha} z + \mathcal{J}_0}, \tag{2.16}$$

valid for $\mathcal{J}(z) < 1$. The expression (2.16) is plotted with dotted lines in Figs. 1(a) and 1(b). More precisely, the strong nonlinear law (2.16) can be shown to be valid for any

$$\mathcal{J}_0 < \tilde{\alpha} z - (\ln \tilde{\alpha} z + 1), \tag{2.17}$$

for which $\mathcal{J}(z) < 1/e << \tilde{\alpha} z - \mathcal{J}_0$ in the case $\tilde{\alpha} z > 1$. We will assume that the nonlinear regime begins at penetration z_{NL} for which $\mathcal{J}(z)$ is equal to e^{-1}. Then, the origin for nonlinearities is related to $\mathcal{J}_0$ and $\tilde{\alpha}$ by

$$z_{NL} = \tilde{\alpha}^{-1}(\mathcal{J}_0 + 1 + \ln \mathcal{J}_0), \tag{2.18}$$

when using Eq. (2.14) together with $\mathcal{J}(z_{NL}) = e^{-1}$.

Before we conclude this paragraph, let us study the order of the nonlinearity of $\mathcal{J}$ with respect to $\mathcal{J}_0$: firstly we prove that the slope of $\mathcal{J}(\mathcal{J}_0)$ in Eq. (2.16) is an increasing function of $\mathcal{J}_0$ and $-\tilde{\alpha} z$. We have

$$\ln \mathcal{J} = \ln \mathcal{J}_0 - \tilde{\alpha} z + \mathcal{J}_0; \tag{2.19}$$

therefore the slope of the curve $\ln\mathscr{I}$ as a function of $\ln\mathscr{I}_0$ is

$$\frac{d(\ln\mathscr{I})}{d(\ln\mathscr{I}_0)}=1+e^{(\ln\mathscr{I}_0)}, \tag{2.20a}$$

which grows monotonically in the domain defined by Eq. (2.17). The maximum of the slope corresponds to $(\mathscr{I}_0)_{\max}=\tilde{\alpha}z-(\ln\tilde{\alpha}z+1)$; it is

$$\left[\frac{d(\ln\mathscr{I})}{d(\ln\mathscr{I}_0)}\right]_{\max}=\tilde{\alpha}z-\ln\tilde{\alpha}z . \tag{2.20b}$$

For $(\mathscr{I}_0)_{\max}$, $\mathscr{I}(\mathscr{I}_0)$ can be written as

$$\mathscr{I}(z)\propto(\mathscr{I}_0)^n , \tag{2.21}$$

where

$$n=\tilde{\alpha}z-\ln\tilde{\alpha}z , \tag{2.22}$$

gives the order of nonlinearity of the intensity.

In conclusion the behavior of $\mathscr{I}(z)$ as z increases from $z=0$ can be described as follows when the inequalities

$$\tilde{\alpha}l>>\mathscr{I}_0>>1 \tag{2.23}$$

are fulfilled.

As z is much smaller than $\tilde{\alpha}^{-1}\mathscr{I}_0$, the medium is quasitransparent; no modification of the input beam is expected. As z increases, weak nonlinearities appear with still appreciable values of $\mathscr{I}(z)$ larger than unity. The strong nonlinearities, that we are dealing with, will next appear for $z\sim z_{NL}$. In Sec. III, when we will treat the role of the diffraction effects on the strong nonlinear regime (valid for $z\geq z_{NL}$), we will neglect them in the regime of weak nonlinearities ($z<z_{NL}$). This approximation will be discussed in the Appendix.

C. Introduction of a Gaussian input profile

Let us assume that the transverse profile of the input beam is Gaussian

$$\epsilon_0(r)=\epsilon_0 e^{-r^2/w_0^2} . \tag{2.24}$$

As long as the diffraction term is neglected in the Maxwell equation, the solutions for $\epsilon(z,r)$ and $I(z,r)$ are

$$\epsilon(z,r)=\epsilon_0(r)\exp\left[-\frac{1}{2(1+i\delta T_2)}[\alpha z-\beta I_0(r)+\beta I(z,r)]\right], \tag{2.25}$$

and

$$I(z,r)=I_0(r)\exp\left[\frac{1}{(1+\delta^2T_2^2)}[\alpha z-\beta I_0(r)+\beta I(z,r)]\right], \tag{2.26}$$

respectively, and the nonlinear solution for the intensity, which concerns us, becomes

$$I(z,r)=I_0(r)\exp\left[-\frac{1}{1+\delta^2T_2^2}[\alpha z-\beta I_0(r)]\right], \tag{2.27}$$

valid for any $z\geq z_{NL}$. The solutions (2.25) and (2.26) mean that the electric field amplitude on a cylindrical surface of radius r obeys the same propagation law as for a plane-wave input pulse of intensity $I_0e^{-2r^2/w_0^2}$.

The law (2.27) exhibits the deformation of the input Gaussian shape due to strong nonlinearities when the diffraction is neglected. The half-width w of the profile (2.27) defined with the help of the relation

$$I(z,w)=\frac{1}{e}I(z,0) , \tag{2.28}$$

is given by

$$\frac{w^2}{w_0^2}=-\ln\left[1+\frac{1}{\mathscr{I}_0}\left[1-\frac{w^2}{w_0^2}\right]\right] , \tag{2.29}$$

when using Eqs. (2.27), (2.28), and definition (2.13). This expression becomes

$$w\simeq w_0\left[\frac{1+\delta^2T_2^2}{2\beta I_0}\right]^{1/2} , \tag{2.30}$$

in the limit of interest, $\beta I_0>>1+\delta^2T_2^2$. This narrowing of the beam results from the nonlinear absorption regime, the smaller the input intensity the more absorbed the light.

The diffraction term of the Maxwell equation leads to the coupling between the annular rings.[14] The transverse effects can be predicted in an elegant manner with the help of the transverse component of the energy current flow.[13,14] It is defined by the relation

$$J_T(z,r)=i[\epsilon(z,r)\nabla_T\epsilon^*(z,r)-\text{c.c.}]\ , \tag{2.31a}$$

or equivalently

$$J_T(z,r)=2\,|\,\epsilon(z,r)\,|^2\frac{\partial S(z,r)}{\partial r}\ , \tag{2.31b}$$

with $\epsilon=|\,\epsilon\,|\,e^{iS}$. Now it is known that $\partial S/\partial r$ is directly proportional to the variation $\partial\vec{r}_T/\partial z$ of the transverse component $\vec{r}_T$ of a ray normal to the surface of constant phase.[21] Then if J_T is negative, it indicates a focusing of the beam while if J_T is positive, spreading of the beam is expected. Transient self-focusing and self-defocusing have yet been related to the variations of the transverse energy current flow by Mattar *et al.*, especially for superfluoresence.[17]

At the origin z_{NL}, the calculation of $J_T(z_{NL},r)$ leads to

$$J_T(z_{NL},r)=\frac{2r}{w_0^2}\delta T_2\frac{\beta I_0}{1+\delta^2T_2^2}I(z_{NL},r)\ , \tag{2.32}$$

when using the definition (2.31) together with Eq. (2.27). Then if the transverse effects are taken into account from z_{NL}, the sign of $J_T(z_{NL},r)$ predicts either narrowing (with eventually enhancement of the on-axis intensity) for $\delta<0$ or spreading for $\delta>0$, in agreement with the well-known results of the standard self-focusing theory.[11]

D. When are the transverse effects important in the strong nonlinear regime?

A powerful way to get an insight into the diffraction effects is to apply the Huyghens's principle inside the cell. The amplitude of the positive frequency part of field scattered by the atoms at location $\vec{x}$ is proportional to

$$\epsilon_{\text{scatt}}(\vec{x})=-4i\pi Nk^3\,|\,\vec{\mu}\,|\int_v d^3x'\frac{\exp[ik(z-z')-ik\,|\,\vec{x}-\vec{x}'\,|\,]}{k\,|\,\vec{x}-\vec{x}'\,|}\mathscr{P}(\vec{x}')\ , \tag{2.33}$$

where $\mathscr{P}(\vec{x})$ is the amplitude of the field radiated by a microscopic source located at $\vec{x}$. The diffraction kernel $e^{-ik|\vec{x}-\vec{x}'|}/k\,|\,\vec{x}-\vec{x}'\,|$ can be well approximated by the integral[24]

$$\int_0^1 du\,J_0[k(1-u^2)^{1/2}X(r,r')]e^{iku\,|z-z'|}\ ;$$

in the paraxial approximation (the quantity $X(r,r')$ gives the length of the projection of the vector $\vec{x}-\vec{x}'$ on a plane perpendicular to the 0z axis). Then if the cylindrical coordinates ($r\cos\varphi$, $r\sin\varphi$, z) are used and if the integration over φ is performed, Eq. (2.33) becomes

$$\epsilon_{\text{scatt}}(\vec{x})=-4i\pi N\,|\,\vec{\mu}\,|\,k^3\int_{z_{NL}}^{l}dz'\int_0^\infty dr'r'\mathscr{P}(z',r')e^{ik(z-z')}\times\int_0^1 du\,e^{-iku\,|z-z'|}J_0[kr(1-u^2)^{1/2}]J_0[kr'(1-u^2)^{1/2}]\ . \tag{2.34}$$

Its part which propagates in the forward direction is

$$\epsilon_{\text{scatt}}^{\text{forw}}(\vec{x})=-4i\pi N\,|\,\vec{\mu}\,|\,k^3\int_{z_{NL}}^{z}dz'\int_0^\infty r'dr'\mathscr{P}(z'r')\times\int_0^1 du\,e^{ik(1-u)(z-z')}J_0[kr(1-u^2)^{1/2}]J_0[kr'(1-u^2)^{1/2}]\ . \tag{2.35}$$

The expression (2.35) can be calculated by introducing in its right-hand member the solution $\mathscr{P}(z',r')$ of the reduced Maxwell equation in which the diffraction has been neglected,

$$\frac{\partial}{\partial z}\epsilon=2i\pi\frac{|\,\vec{\mu}\,|}{c}\omega N\mathscr{P}\ .$$

Then by an obvious argument of self-consistency these diffraction effects will be really negligible, only if the right-hand member reduces to the difference $\epsilon(z,r)-\epsilon(z_{NL},r)$.

Let us perform this calculation in the strong nonlinear regime. A rough approximation of the field amplitude consists in assuming that its transverse shape is a Gaussian of width $w_0/\sqrt{\mathscr{I}_0}$ as deduced from Eq. (2.29),

$$\epsilon(z,r) \simeq \epsilon_0 e^{-(1/2)\tilde{\alpha}z+(1/2)\mathscr{I}_0 - r^2\mathscr{I}_0/w_0^2} . \qquad (2.36)$$

Then the integration over the area in Eq. (2.30) can be simply performed and will lead to an easier estimation of the role of the diffraction. If we put Eq. (2.36) into Eq. (2.35), after taking into account the relation between $\mathscr{P}(z,r)$ and $\epsilon(z,r)$ we get

$$\epsilon_{\text{scatt}}^{\text{forw}}(z,r) = -4i\pi \frac{|\vec{\mu}|^2}{\hbar} T_2 k^3 \epsilon_0 e^{+(1/2)\mathscr{I}_0} \int_{z_{NL}}^{z} dz' \, e^{-(1/2)\tilde{\alpha}(z-z')} \times \frac{w_0^2}{2\mathscr{I}_0} \int du \, J_0[kr(1-u^2)^{1/2}] \times \exp\left[ik(1-u)(z-z') - \frac{k^2 w_0^2 (1-u^2)}{4\mathscr{I}_0} \right] . \qquad (2.37)$$

The change of variables

$$\frac{k^2 w_0^2}{4\mathscr{I}_0}(1-u) = v^2 , \qquad (2.38)$$

with the present condition $kw_0/\sqrt{\mathscr{I}_0} >> 1$ in order to satisfy the paraxial approximation, allows us to write

$$\frac{w_0^2}{2\mathscr{I}_0} \int_0^1 du \, J_0[kr(1-u^2)^{1/2}] e^{ik(1-u)(z-z')} e^{(k^2 w_0^2/4\mathscr{I}_0)(1-u^2)}$$
$$\simeq \frac{4}{k^2} \int_0^{kw_0/\sqrt{\mathscr{I}_0} \to \infty} dv \, v J_0 \left[2r \frac{\sqrt{2\mathscr{I}_0} v}{w_0} \right] e^{-2v^2[1-2i(z-z')\mathscr{I}_0/kw_0^2]}$$
$$= \frac{2}{k^2} \frac{1}{\left[1 - \dfrac{2i(z-z')\mathscr{I}_0}{kw_0^2} \right]} \exp\left[-r^2 \frac{\mathscr{I}_0}{w_0^2} \frac{1}{\left[1 - \dfrac{2i(z-z')\mathscr{I}_0}{kw_0^2} \right]} \right] . \qquad (2.39)$$

Therefore the integral (2.39) can be considered as independent of $z - z'$ for any z and z' in the cell, only if the quantity $2l\mathscr{I}_0/kw_0^2$ is much smaller than unity. This implies that the Fresnel number $\mathscr{N} = \pi w_0^2/\lambda l$ has to be much larger than the normalized intensity $\mathscr{I}_0$,

$$\mathscr{N} >> \frac{|\vec{\mu}|^2}{\hbar^2} \frac{T_1 T_2 |\epsilon_0|^2}{1+\delta^2 T_2^2} . \qquad (2.40)$$

If this latter inequality is satisfied, then Eq. (2.39) reduces to $2k^{-2} e^{-r^2\mathscr{I}_0/w_0^2}$ and Eq. (2.34) becomes

$$\epsilon_{\text{scatt}}^{\text{forw}}(\vec{x}) = \epsilon(z,r) - \epsilon(z_{NL},r) , \qquad (2.41)$$

as expected when the diffraction effects are negligible.

In conclusion, the diffraction term is found to be irrelevant only if the Fresnel number for the cell obeys the inequality (2.40) which generalizes to the nonlinear propagation case the customary condition $\mathscr{N} >> 1$ for vanishing transverse effects. Let us point out that the condition (2.40) can be understood like $\tilde{\mathscr{N}} >> 1$, where $\tilde{\mathscr{N}}$ is the Fresnel number associated with the narrowed radius $w_0\sqrt{\mathscr{I}_0}$.

No interactions between the annular rings with intensity $I(z,r)$ as given by Eq. (2.26) will be expected for large Fresnel number, and the patterns of $I(l,r)$ displayed on a screen located far from the source will be the Fourier transform of a small spot of radius $\propto I_0^{-1/2}$, i.e., a spread spot of radius $\propto I_0^{1/2}$.

III. ANALYTICAL TREATMENT OF THE DIFFRACTION FOR STRONG NONLINEAR REGIME

A. Model

We are strictly dealing with the strong nonlinear variation of the propagating beam with respect to the input intensity.

Two assumptions are made:

(a) the strong nonlinear regime begins at penetration z_{NL}, as given by Eq. (2.17),

(b) the diffraction effects were negligible before the field reaches z_{NL}. This assumption is discussed in the Appendix. A sufficient condition is given by $\mathscr{N} >> (\mathscr{I}_0)^3/\alpha l$.

Then we have to solve the reduced Maxwell equation

$$\frac{\partial}{\partial z}\epsilon+\frac{1}{2ik}\nabla_T^2\epsilon=-\frac{\alpha}{2}\frac{1-i\delta T_2}{1+\delta^2T_2^2}\frac{\epsilon}{1+\tilde{\beta}|\epsilon|^2}, \tag{3.1a}$$

where $\tilde{\beta}=\beta[1+\delta^2T_2^2]^{-1}$, for any $z\geq z_{NL}$, with the initial condition

$$\epsilon(z_{NL},r)=\epsilon_0 e^{-r^2/w_0^2}\times e^{-(1/2)[\tilde{\alpha}z-\beta I_0\exp(-2r^2/w_0^2)](1-i\delta T_2)}. \tag{3.1b}$$

In Eq. (3.1b) we recognize the linear refractive index increment (n_0-1)

$$(n_0-1)=\frac{1}{2}\frac{\tilde{\alpha}\delta T_2}{k}=+\frac{1}{2}\frac{\alpha}{k}\frac{\delta T_2}{1+\delta^2T_2^2}, \tag{3.2a}$$

deduced from the identity

$$\tfrac{1}{2}\tilde{\alpha}\delta T_2 z=(n_0-1)kz . \tag{3.2b}$$

Therefore we can define the "diffraction" function

$$\tilde{\epsilon}(z,r)=\epsilon(z,r)e^{+(1/2)\tilde{\alpha}z(1-i\delta T_2)}, \tag{3.3}$$

which obeys the differential equation

$$\left[\frac{1}{2ik}\nabla_T^2+\frac{\partial}{\partial z}\right]\tilde{\epsilon}(z,r)=-\frac{\tilde{\alpha}}{2}(1-i\delta T_2)\tilde{\epsilon}(z,r)\times\left[\frac{1}{1+\tilde{\beta}e^{-\alpha z}|\tilde{\epsilon}(z,r)|^2}-1\right]. \tag{3.4}$$

Before we develop a treatment of focusing effects in the present case of strong nonlinearities, let us recall the approach in the low-intensity and small-absorption limit [Eq. (1.1)]. In this latter case the focusing (defocusing) is treated from the input $z=0$ with $\tilde{\epsilon}(z=0,r)=\epsilon_0(r)$ and the term in large parentheses in the right-hand member of Eq. (3.4) is generally approximated by the lowest nonlinear term $-\tilde{\beta}|\epsilon_0(r)|^2$, that leads to the well-known cubic refractive index.

Here the nonlinearities appear in a somewhat different way: Because of the strong absorption ($\alpha z_{NL}>>1$), the nonlinearities in the right-hand member of Eq. (3.4) are assumed to be negligible with respect to the strong nonlinearities yet included in the initial condition

$$\tilde{\epsilon}(z_{NL},r)=\epsilon_0(r)e^{(1/2)\tilde{\beta}(1-i\delta T_2)I_0(r)} . \tag{3.5}$$

Therefore our treatment of focusing (defocusing) effects amounts to the study of the diffraction of light with the particular shape (3.5) of the cross section resulting from the nonlinearities of the medium.

The introduction of the term $e^{-\alpha z}|\tilde{\epsilon}(z,r)|^2$ needs a numerical integration of Eq. (3.4). Its role will be discussed elsewhere.

The treatment that we propose is really straightforward. The initial condition for $\epsilon(z_{NL},r)$ can be expanded as an infinite series in powers of I_0

$$\tilde{\epsilon}(z_{NL},r)=\epsilon_0\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{\beta I_0 e^{i\varphi}}{2(1+\delta^2T_2^2)^{1/2}}\right]^n e^{-(2n+1)r^2/w_0^2}, \tag{3.6}$$

with

$$\varphi=-\arctan\delta T_2 \tag{3.7}$$

or, in other words, as an infinite series of Gaussian functions

$$\psi_n(z_{NL},r)=e^{-(2n+1)r^2/w_0^2} . \tag{3.8}$$

Because of the above discussion about Eq. (3.4), it follows that $e^{-(2n+1)r^2/w_0^2}$ is the initial value of the function $\psi_n(z,r)$, solution of

$$\frac{\partial}{\partial z}\psi_n(z,r)+\frac{1}{2ik}\nabla_T^2\psi_n(z,r)=0 , \tag{3.9}$$

for any $z\geq z_{NL}$. The solution of Eq. (3.9) together with Eq. (3.8) was first discussed by Kogelnik.[19] They are

$$\psi_n(z,r)=\frac{w_{0n}}{w_n(z)}e^{-i\Phi_n(z)-r^2/w_n^2+ir^2k/2R_n(z)}, \tag{3.10}$$

with

$$w_{0n}=\frac{w_0}{\sqrt{2n+1}},$$

$$w_n^2(z)=w_{0n}^2\left[1+\left[\frac{2n+1}{\mathcal{N}(z-z_{NL})}\right]^2\right],$$

$$R_n(z)=(z-z_{NL})\left[1+\left[\frac{\mathcal{N}(z-z_{NL})}{2n+1}\right]^2\right],$$

$$\mathcal{N}(z-z_{NL})=\frac{\pi w_0^2}{\lambda(z-z_{NL})}, \tag{3.11}$$

$$\Phi_n(z)=\arctan\left[\frac{2n+1}{\mathcal{N}(z-z_{NL})}\right].$$

Finally the solution for $\epsilon(z,r)$ is

$$\epsilon(z,r)=\epsilon_0 e^{-(\tilde{\alpha}/2)z(1-i\delta T_2)} \times \sum_{n=0}^{\infty}\left[\frac{\beta I_0 e^{i\varphi}}{2(1+\delta^2T_2^2)^{1/2}}\right]^n \frac{1}{n!}\psi_n(z,r)\ , \tag{3.12}$$

with $\psi_n(z,r)$ given by Eq. (3.10). The solution (3.12) is valid inside the cell, $z\leq l$. At the exit of the material, the field amplitude retains the expansion (3.12), but the exponential factor $e^{-\tilde{\alpha}z/2}$ has to be replaced by $e^{-(1/2)\tilde{\alpha}l}$. Therefore the expansion (3.12) gives rise directly to the patterns of the transmitted intensity on a screen located at any $z>l$.

Owing to the series expansion of $\epsilon(z,r)$ oscillations of the transmitted intensity $I(z,r)$ may be expected.

The standard theory of the self-focusing often refers to two integrals of motion.[10] So let us give their expressions in the case of a strong nonlinear regime. The function $\tilde{\epsilon}(z,r)$ obeys the two integrals of motion,

$$\int_0^\infty r\,dr\,|\tilde{\epsilon}(z,r)|^2=|\tilde{\epsilon}(z_{NL},r)|^2\ , \tag{3.13}$$

$$\int r\,dr\left[\left|\frac{\partial}{\partial r}\tilde{\epsilon}(z,r)\right|^2 + 2ik\left[\tilde{\epsilon}^*\frac{\partial}{\partial z}\tilde{\epsilon}-\text{c.c}\right]\right]=0\ , \tag{3.14}$$

which are easily deduced from the reduced Maxwell equation. The first constant of motion which conserves energy gives

$$\int_0^\infty r\,dr\,|\tilde{\epsilon}(z,r)|^2=\tfrac{1}{2}w_0^2\frac{I_0}{\mathcal{I}_0}(e^{\mathcal{I}_0}-1)\ , \tag{3.15}$$

when using (3.1b) together with (3.13). Note that the left-hand side of (3.15) is exactly equal to the product of the on-axis intensity

$$\tilde{I}(z_{NL},0)=I_0e^{\mathcal{I}_0}\ , \tag{3.16}$$

by its area $\pi w_0^2/2\mathcal{I}_0$ [cf. Eq. (2.30)].

The identity (3.14) is obviously satisfied by any Kogelnik function (3.11). Unfortunately, due to the expansion (3.7), the constant of the motion (3.14) does not lead to any useful information.

B. Results

The intensity $\tilde{I}(z,r)$

$$\tilde{I}(z,r)=I_0\sum_{p,m}\frac{1}{p!m!}\psi_m\psi_p^* e^{-i(m-p)\arctan\delta T_2}\left[\frac{\beta I_0}{2(1+\delta^2T_2^2)^{1/2}}\right]^{m+p}\ , \tag{3.17}$$

has been calculated as a function of the propagation variable z for various normalized input intensity $\mathcal{I}_0$ and for various positive or negative values of δT_2.

1. *Self-focusing*

The transverse energy current J_T, calculated at the origin z_{NL}, predicts focusing for a negative detuning ($\omega_a<\omega$). Then the value

$$\tilde{I}(z_{NL},0)=I_0\exp\left[\frac{\beta I_0}{1+\delta^2T_2^2}\right]\ , \tag{3.18a}$$

and the width of the intensity defined at half-height of the peak is expected to decrease from its initial value

$$\bar{r}_0=w_0\left[\frac{\ln 2}{2}\frac{1+\delta^2T_2^2}{\beta I_0}\right]^{1/2}\ . \tag{3.18b}$$

The double summation (3.17) has been numerically performed for values of βI_0 extending from 40 to 450, and values of $|\delta|T_2$ of magnitude of order a few units. It displays self-focusing of the intensity with respect to the initial conditions (3.18). However, the net enhancement of the on-axis intensity has to be related to the input intensity I_0 and depends strongly on the attenuation factor $\exp(-\alpha l/1+\delta^2T_2^2)$.

Our systematical calculations of $\tilde{I}(z,r)$ exhibit the following features.

The focal point z_f measured from z_{NL} is given by the relation

$$\frac{\pi w_0^2}{\lambda z_f} = \frac{I_0}{|\delta| T_2} , \qquad (3.19a)$$

or

$$\frac{z_f}{l} = \mathcal{N} \frac{|\delta| T_2}{\beta I_0} , \qquad (3.19b)$$

where $(1+\delta^2 T_2^2)^{1/2}$ has been approximated by $|\delta| T_2$, valid for $|\delta| T_2 \gtrsim 2$.

The waist $\tilde{r}_{\min}$ defined at half-height of the peak intensity obeys approximately the law

$$\tilde{r}_{\min} = w_0 \left[\frac{\ln 2}{2} \frac{|\delta| T_2}{\beta I_0} \right]^{1/2} . \qquad (3.20)$$

Let us point out that the focal point z_f and $\sqrt{2}\tilde{r}_{\min}$, which is the radius associated with the field amplitude define a local Fresnel number

$$\frac{2\pi \tilde{r}^2_{\min}}{\lambda z_f} ,$$

of magnitude of order unity for which the integral (2.35) is maximum.

The law for the conservation of energy requires

$$\int dr\, r\tilde{I}(z,r) = \frac{1}{\ln 2} \tilde{r}_0^2 \tilde{I}(z_{NL},0) , \qquad (3.21)$$

when using Eq. (3.15) together with definitions (3.18). If the beam would retain the same shape as the initial one $\tilde{I}(z_{NL},r)$ the on-axis intensity of the focus, $\tilde{I}_{\max}$, would be given by

$$\tilde{I}_{\max} = \frac{r_0^2}{r^2_{\min}} \tilde{I}(z_{NL},0) = |\delta| T_2 \tilde{I}(z_{NL},0) . \qquad (3.22)$$

In fact the direct calculation of Eq. (3.17) shows that $\tilde{I}_{\max}$ does not obey the law (3.22). The exact value of $\tilde{I}_{\max}$ is approximately twice the expression (3.22). Nevertheless, the expression (3.22) can be used to prescribe a condition for the enhancement of the intensity with respect to the input. The on-axis intensity at its waist will be enhanced with respect to I_0, if the inequality

$$\tilde{I}_{\max} e^{-\alpha l/\delta^2 T_2^2} > I_0 , \qquad (3.23)$$

is met, or by using the lower value (3.22) for $\tilde{I}_{\max}$,

$$|\delta| T_2 e^{-(\alpha l - \beta I_0)/\delta^2 T_2^2} > 1 . \qquad (3.24)$$

This latter inequality has to be consistent with our framework condition

$$\frac{\beta I(z_{NL},0)}{\delta^2 T_2^2} < 1 , \qquad (3.25)$$

which is fulfilled for

$$\frac{\beta I_0}{\delta^2 T_2^2} < e^{(\alpha l - \beta I_0)/\delta^2 T_2{}^2} . \qquad (3.26)$$

Both conditions (3.24) and (3.26) lead to

$$\frac{\ln \beta I_0}{\delta^2 T_2^2} < \frac{\alpha l - \beta I_0}{\delta^2 T_2^2} < \ln |\delta| T_2 , \qquad (3.27)$$

which can be satisfied for large I_0 and $|\delta| T_2$. Note that inequality (3.27) together with the condition for strong absorption means also

$$1 < \frac{\beta I_0}{\delta^2 T_2^2} << |\delta| T_2 . \qquad (3.28)$$

This result is very interesting since self-focusing is shown to be expected even for a large-absorption regime. Moreover, we are able to predict the *location of the focus* given by $z_f + z_{NL}$ from the input in the cell

$$\frac{z_f + z_{NL}}{l} \simeq \frac{I_0}{\alpha l} + \mathcal{N} \frac{|\delta| T_2}{\beta I_0} , \qquad (3.29)$$

the waist of the beam, $\sqrt{(\ln 2/2)(|\delta| T_2/\beta I_0)}$, and we can also give a good estimation of the on-axis intensity at the focus.

Figure 2 displays the shape of the intensity $I(z_f,r)$ at the focus together with the behavior of the transverse energy current J_T. This latter reaches its minimum approximately at $r = \tilde{r}_{\min}$.

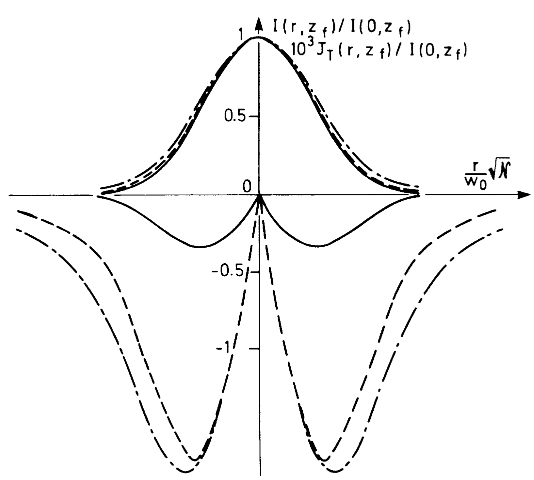

FIG. 2. Shape of the intensity at its focus z_f for three different detunings as a function of the normalized coordinates $(r/w_0)\sqrt{\mathcal{N}}$. At each pulse corresponds a negative transverse energy current J_T, normalized to the maximum intensity: full lines for $\delta T_2 = -3$, broken lines for $\delta T_2 = -6$, and dots and dashes for $\delta T_2 = -9$.

The coincidence of the three curves for $-\delta T_2=3,6,9$ illustrates Eqs. (3.19) and (3.20).

Figures 3 show the deformation of the pulse from z_{NL} to $z\rightarrow\infty$ as a function of the normalized coordinate

$$Z=(z-z_{NL})\frac{\lambda}{\pi w_0^2}=\frac{1}{\mathcal{N}(z-z_{NL})}\ , \tag{3.30}$$

for $\beta I_0=200$ and $\delta T_2=-6$. Figure 3(a) exhibits the behavior of $I(z,r)$ before and after the focus ($z_f=3\times10^{-2}$). The formation of the spatial ringings which occur after the focus is detailed on Fig. 3(b) with $0.08<Z<0.2$. After the beam goes by its focus, it will definitely diverge. This property is shown on Fig. 3(c) where the normalized intensity $I(z,r)/I_{\max}(z)$ is plotted as a function of $\ln[1+(r/w_0]$ and for $10^{-1}<z<10^3$. The oscillations which appear after the focus still exist for large Z but they are more and more attenuated for increasing Z.

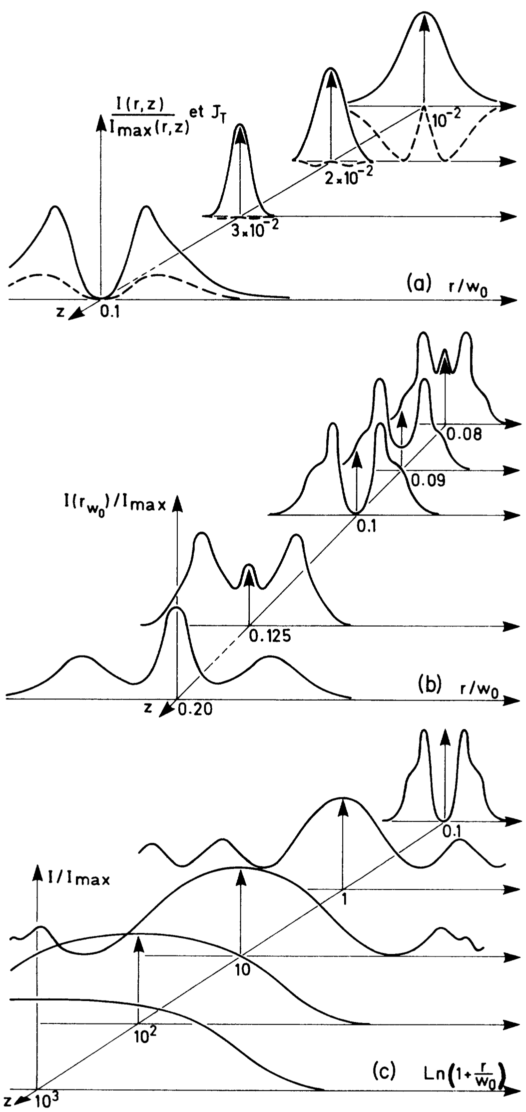

FIG. 3. Transverse shape of the intensity as a function of the normalized penetration $Z=z\lambda/\pi w_0^2$ for $\delta T_2=-6$ and $\beta I_0=200$.

2. *Defocusing*

For positive detuning, the beam defocuses as it propagates from its origin z_{NL}. When the propagation length $(z-z_{NL})$ approaches $\pi w_0^2/\lambda$, then the intensity patterns exhibit one or several rings.

Figure 4 displays the transverse shapes of $\tilde{I}(z,r)$ for fixed detuning, $\delta T_2=5$, and $Z=10$, as a function of the input intensity βI_0. These figures demonstrate that the patterns, especially the number of the oscillations, vary with βI_0. The number of the rings is found to be approximately

$$\frac{1}{3\pi}\frac{\beta I_0}{(1+\delta^2T_2^2)^{1/2}}\ .$$

The half-width of the blooming, $\bar{r}_b$, defined as the position of the maximum of the outside ring *plus* its half-width is an increasing function of βI_0. The variation of $\bar{r}_b$ is plotted in Fig. 5 as a function of βI_0 extending from 20 to 240 for two penetrations, $Z=1$ and $Z=10$. Up to now we have not found an analytical expression to fit the curves. We can only bound $\bar{r}_b$ as follows.

In the limit $Z>>1$ the expression (3.12) can be

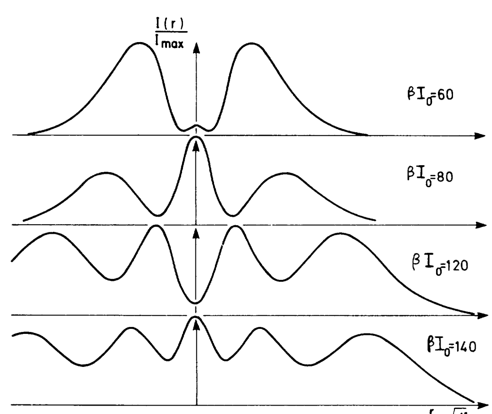

FIG. 4. Transverse shape of the intensity at $Z=10$, for $\delta T_2=+5$ and with $\beta I_0=60$, 80, 120, and 140.

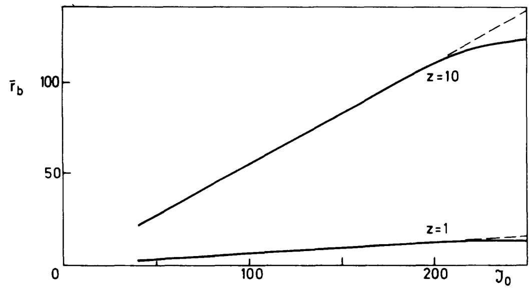

FIG. 5. Blooming radius r_b as a function of βI_0, for two penetrations, $Z=1$ and 10, and for $\delta T_2=+5$.

approximated by

$$\tilde{\epsilon}(z,r)=\mathcal{N}\epsilon_0\sum_{n=0}^{\infty}\left[\frac{\beta I_0 e^{i\varphi}}{2(1+\delta^2T_2^2)^{1/2}}\right]^n\frac{1}{n!(2n+1)}\times e^{-(r^2/w_0^2)[\mathcal{N}^2/(2n+1)]}\ . \quad (3.31)$$

It is easy to show that the weight of the exponential of waist $w_0\sqrt{2n+1}/\mathcal{N}$ is maximum for

$$n=\frac{\beta I_0}{2(1+\delta^2T_2^2)^{1/2}}-1\ . \quad (3.32)$$

For large values of this index n, given by Eq. (3.32), the maximum weight can be approximated by e^n when using the Stirling formula for $n!$. Therefore the field amplitude becomes negligible for any r such that

$$\exp\left[n-\frac{r^2}{w_0^2}\frac{\mathcal{N}^2}{2n+1}\right]=\exp\left[\frac{\beta I_0}{2(1+\delta^2T_2^2)^{1/2}}-\frac{r^2}{w_0^2}\frac{\mathcal{N}^2}{(2n+1)}\right]<<1\ , \quad (3.33)$$

i.e., for any

$$r\gtrsim w_0\frac{\beta I_0}{(1+\delta^2T_2^2)^{1/2}}\frac{1}{\sqrt{2}\mathcal{N}}\ . \quad (3.34)$$

So, we can predict that the blooming radius $\bar{r}_b$ is smaller than the right-hand member of the inequality (3.34), for any penetration $Z>1$.

Figure 6 displays the role of the detuning for $\beta I_0=60$ and $Z=10$. The radius r_b is found to vary like $(\delta T_2)^{-1}$ for the considered range of the variable.

The deformation of the pulse shape along the propagation is shown on Fig. 7, with $\beta I_0=60$ and

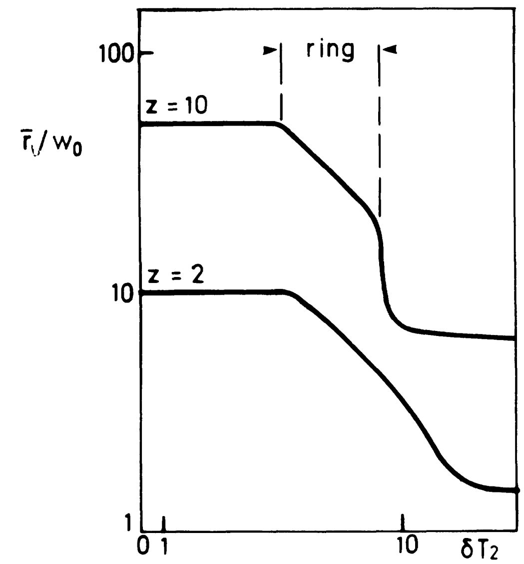

FIG. 6. r_b as a function of δT_2 for $\beta I_0=60$ and $Z=10$.

$\delta T_2=5$. The transverse intensity is plotted as a function of Z with $0.1<Z<1$ to visualize the formation of a ring. For large Z the ring disappears and it remains a large spot of light. The variation of the blooming radius $\bar{r}_b$ is given in the last figure (Fig. 8). For $Z<1$, $\bar{r}_b$ is proportional to $Z^{1/2}$, i.e., $\mathcal{N}^{-1/2}$ and for $Z>1$, it varies like Z, i.e., $\mathcal{N}^{-1}$ in agreement with Eq. (3.34).

IV. PERSPECTIVE

If the observation of spatial ringing has been often mentioned,[1–4] its systematic study has not

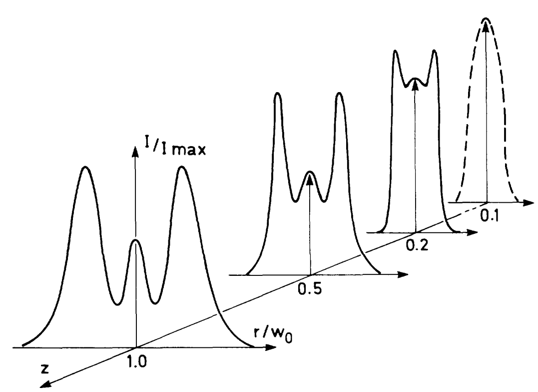

FIG. 7. Transverse shape of the intensity as a function of Z for $\delta T_2=+5$ and $\beta I_0=60$.

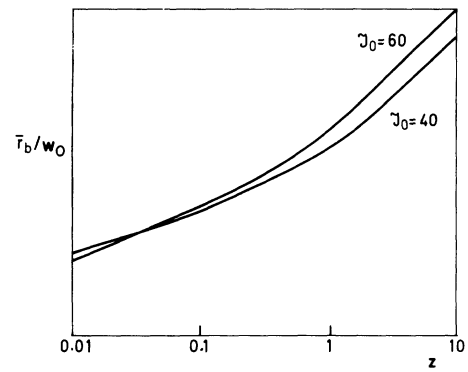

FIG. 8. r_b as a function of Z for $\delta T_2 = +5$ and $\beta I_0 = 40$ and 60.

yet been performed except in the case of the linear conical shell first reported by Skinner and Kleiber.[25]

However, some experimental results[26] indicate that the diameter of the ring displayed on a screen located far from the source (~ 2 m) decreases with the detuning and increases with both I_0 and $\tilde{\alpha} l$.

The variation of the ring diameter with respect to the atomic concentration does not explicitly appear in our model. However, it can be understood as follows: The nonlinear intensity I varies approximately like $(I_0)^{\tilde{\alpha} l}$ [cf. Eqs. (2.21) and (2.22)] and $I(r)$ like $I_0^{\tilde{\alpha} l} e^{-\tilde{\alpha} l r^2/w_0^2}$. Therefore, if the transverse effects are neglected in the cell as a first approximation, the Fourier transform of $I(r)$ displayed on a screen has a width proportional to $\tilde{\alpha}^{1/2}$, i.e., $N^{1/2}$. So we can predict that any ring, if it exists, will be located inside this width.

The great interest of self-focusing is so known that it is unnecessary to dwell upon the interest of cw light self-focusing in strong absorbing cell.

Furthermore, let us point out a particular consequence of the self-focusing effect in a strong nonlinear medium when the input intensity is a random function. It appears that a chaotic field previously self-focused as described in this paper may be more efficient than a chaotic field for multilevel excitation for an atom (or a molecule). The statistical properties of the pulse are modified due to the propagation inside the nonlinear medium. Especially, the n-order moment of the intensity, which is initially

$$\langle I_0^n \rangle = \frac{1}{\langle I_0 \rangle} \int_0^\infty dI_0 I_0^n e^{-I_0/\langle I_0 \rangle} = n! \langle I_0 \rangle^n , \quad (4.1)$$

for a chaotic field, is after propagation

$$\langle I^n(z) \rangle = \frac{1}{\langle I_0 \rangle} \int_0^\infty dI_0 [I(z,I_0)]^n e^{-I_0/\langle I_0 \rangle} . \quad (4.2)$$

It is obvious that the transmitted field keeps the statistical properties of the input field only in the case of a *linear cell.* Then the normalized moments $g^{(n)}$

$$g^{(n)}(z) = \frac{\langle I^n(z) \rangle}{\langle I(z) \rangle^n} \quad (4.3)$$

are unchanged and equal to $n!$ as for the chaotic input field. For a nonlinear cell the situation is quite different. Figures 1 display the variation of the second-order normalized moment of the intensity $g^{(2)}$ as a function of $\langle \mathscr{I}_0 \rangle$ for $\tilde{\alpha} z = 10$ and $\tilde{\alpha} z = 30$. In the two limiting cases of linear regime, either small or large $\langle \mathscr{I}_0 \rangle$, the beam retains its initial statistical properties, and $g^{(2)}$ is of magnitude of order 2. Between these limits, $g^{(2)}$ reaches a large maximum for $\langle \mathscr{I}_0 \rangle$ of magnitude of order unity. The enormous values of $g^{(2)}$ displayed in Fig. 1 are a consequence of the strong nonlinearities ($I \propto I_0^{\tilde{\alpha} z}$).

If the mean transmitted intensity $\langle I(z) \rangle$ is smaller than the input one, any moment $\langle I^n(z) \rangle$ can be shown to be smaller than the incident one. But, if the beam focuses, then the on-axis nth intensity moment may become larger than the incident one. It depends whether the nonlinearities will cancel out or not when averaging over I_0 [see integration (4.2)]. A positive answer can lead to interesting effects. Especially, the probability for an atom to simultaneously absorb n photons will be much larger with an exciting chaotic field previously self-focused and self-bunched than with an unperturbed chaotic field. Propagation effects of the field statistics would be studied in a further publication.

ACKNOWLEDGMENTS

We acknowledge F. Mattar for fruitful and lively discussions during his visit in Orsay.

APPENDIX

We derive the sufficient condition for neglecting the transverse effects on the propagation of a Gaussian cross section electric field in a resonant steady-state medium. The generalization for the off-resonant case is straightforward.

We recall the equation of propagation for the electric field amplitude when the diffraction term is included

$$\left[\frac{\partial}{\partial z}+\frac{1}{2ik}\nabla\mathscr{E}(r,z)\right.$$
$$=-\frac{\alpha}{2}\frac{\mathscr{E}(r,z)}{1+\beta\,|\,\mathscr{E}(r,z)\,|^2}\,. \tag{A1}$$

We scale the axial variable by α^{-1} and the transverse one by w_0. Then the dimensionless equation becomes

$$\left[\frac{\partial}{\partial \bar{z}}+f\nabla_\tau^2\right]\mathscr{E}(\rho,\bar{z})=-\frac{1}{2}\frac{\mathscr{E}(\rho,\bar{z})}{1+\beta\,|\,\mathscr{E}(\rho,\bar{z})\,|^2}\,, \tag{A2}$$

where

$$\rho=\frac{r}{w_0},\quad \bar{z}=\frac{\alpha}{2}z\,, \tag{A3}$$

∇_τ^2 is the transverse Laplacian in ρ, and f gives the magnitude of the diffraction effects on an absorption length

$$f=\frac{1}{k\alpha w_0^2}\,. \tag{A4}$$

Since f is a very small parameter we may introduce the diffraction term in a perturbative way. This approach was used by Mattar when he discussed the coherent resonant self-focusing.[14]

We develop the field amplitude in power of f,

$$\mathscr{E}(\rho,\bar{z})=\mathscr{E}^{(0)}(\rho,\bar{z})+f\mathscr{E}^{(1)}(\rho,\bar{z})+\cdots\,, \tag{A5}$$

where $\mathscr{E}^{(0)}(\rho,\bar{z})$ obeys Eqs. (2.11) and (2.12).

We introduce the series expansion of Eq. (A.1). At resonance $\mathscr{E}^{(0)}(\rho,\bar{z})$ can be chosen real, then $\mathscr{E}^{(1)}(\rho,\bar{z})$ is purely imaginary and obeys the following equation:

$$\frac{\partial}{\partial \bar{z}}\mathscr{E}^{(1)}(\rho,\bar{z})-i\nabla_\tau^2\mathscr{E}^{(0)}(\rho,\bar{z})$$
$$=-\frac{\mathscr{E}^{(1)}(\rho,\bar{z})}{1+\beta\,|\,\mathscr{E}^{(0)}(\rho,\bar{z})\,|^2}\,. \tag{A6}$$

Then the condition

$$|\,f\mathscr{E}^{(1)}(\rho,\bar{z})\,| << |\,\mathscr{E}^{(0)}(\rho,\bar{z}) \tag{A7}$$

will provide a sufficient condition for the transverse effects to be negligible. The solution of Eq. (A6) is

$$\mathscr{E}^{(1)}(\rho,z)=i\mathscr{E}^{(0)}(\rho,\bar{z})\int_0^{\bar{z}}d\bar{z}'\frac{1}{\mathscr{E}^{(0)}(\rho,\bar{z})}$$
$$\times\nabla_\tau^2[\mathscr{E}^{(0)}(\rho,\bar{z})]\,. \tag{A8}$$

The integration of Eq. (A8) can be performed by using the identity deduced easily from Eq. (A1)

$$\frac{\partial}{\partial\rho}[I_0(\rho,\bar{z})]=2\rho[1+\beta I_0(\rho,0)]\frac{\partial I_0}{\partial\bar{z}}(\rho,\bar{z})\,. \tag{A9}$$

Then the final expression of $\mathscr{E}^{(1)}(\rho,\bar{z})$ is given by

$$\mathscr{E}^{(1)}(\rho,\bar{z})=-2i\mathscr{E}^{(0)}(\rho,\bar{z})\left[[1+(1-\rho^2)\mathscr{I}_0(\rho,0)]\bar{Z}-\rho^2[1+\mathscr{I}_0(\rho,0)]^2\left[\bar{Z}+\ln\frac{1+\mathscr{I}_0(\rho,\bar{z})}{1+\mathscr{I}_0(\rho,0)}\right]\right]\,, \tag{A10}$$

with

$$\bar{Z}=\bar{z}+\mathscr{I}_0(\rho,\bar{z})-\mathscr{I}_0(\rho,0)\,. \tag{A11}$$

The magnitude of

$$|\,\mathscr{E}^{(1)}(\rho,\bar{z})\,|\,/\,|\,\mathscr{E}^{(0)}(\rho,\bar{z})\,|$$

is given by the quantity

$$x(\rho)=2\bar{Z}\{\,1+(1-\rho^2)\mathscr{I}_0(\rho,0)$$
$$-\rho^2[1+\mathscr{I}_0(\rho,0)]^2\,\}\,. \tag{A12}$$

For $\rho=0$, we get

$$x(0)=2\bar{Z}\,\mathscr{I}_0(0,0)=2\bar{Z}\mathscr{I}_0\,, \tag{A13}$$

and for $\rho=1$ [which corresponds approximately to the maximum of $x(\rho)$],

$$x(1)=4\bar{Z}\frac{\mathscr{I}_0}{e}\left[2\frac{\mathscr{I}_0}{e}-1\right]\,. \tag{A14}$$

For large input intensities ($\mathscr{I}_0>>e$) we find from Eqs. (A5), (A13), and (A14), that the condition for the transverse effects to be negligible is

$$8f\bar{Z}\frac{\mathscr{I}_0^2}{e^2}<<1\,. \tag{A15}$$

Since $\bar{Z}$ is smaller than $\bar{z}$, a sufficient condition is

$$\frac{\bar{z}}{kw_0^2}\mathscr{I}_0^2 << 1 \,. \qquad \text{(A16)}$$

If the condition (A16) is realized for any $0 \le z \le l$ the transverse effect will begin at the output of the cell. Then the solution given in Sec. III describes the exact solution of the field outside the cell.

The transverse effects will be negligible before $z_{NL} \simeq \beta I_0/\alpha$ if the condition

$$\frac{(\mathscr{I}_0)^3}{\alpha l} << \mathscr{N} \,, \qquad \text{(A17)}$$

deduced from (A16) is verified. In Sec. II, we have shown that the diffraction becomes significant inside the cell if the inequality was realized

$$\mathscr{I}_0 >> \mathscr{N} \,. \qquad \text{(A18)}$$

In conclusion, the conditions for negligible diffraction before z_{NL} and significant one before the end of the cell can be summarized by the inequalities

$$\frac{(\mathscr{I}_0)^3}{\alpha l} << \mathscr{N} < \mathscr{I}_0 \,. \qquad \text{(A19)}$$

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