Analysis of K-shell ionization accompanying nuclear scattering

K. W. McVoy* and H. A. Weidenmüller
Max-Planck-Institut für Kernphysik, Heidelberg, West Germany
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Blair *et al.* have given a semiclassical form for the scattering amplitude describing a proton scattered by both a K-shell electron and by the nuclear plus Coulomb force of the target nucleus. This form is at variance with a recent experimental result of Duinker *et al.* We present both simple physical arguments and a detailed theoretical analysis which establish the conditions of validity of Blair's formula. We pay particular attention to the case of resonant nuclear scattering. We find that in the case of the experiment of Duinker *et al.*, the Blair formula is expected to hold.

I. INTRODUCTION AND OVERVIEW

Duinker *et al.*[1] have recently performed a $(p,p'\gamma_K)$ experiment in which a proton scattered elastically through 125° by the ^{12}C nucleus also creates a hole in the atomic K shell, whose subsequent decay produces an x ray. The x ray is measured in coincidence with the scattered proton. The ratio $R(E_p)$ of $p-\gamma$ coincidences to proton singles was investigated as a function of proton bombarding energy E_p across an s-wave resonance at 461 keV of width $\Gamma_N=38$ keV in ^{13}N. It was found that $R(E_p)$ exhibits a resonance enhancement of some 50% above its off-resonance background value. This finding is at variance with a theoretical expression given by Blair *et al.*[2] and, in extended form, in Ref. 3. The amplitude $T_\epsilon(\vec{k},\vec{k}\,';\vec{k}_e)$ for the scattering of a proton by both the nucleus and an electron of the target atom is written approximately as the sum of products,

$$T_\epsilon(\vec{k},\vec{k}\,';\vec{k}_e)\cong f_{\epsilon_N}(\theta_p)T^e(\vec{k}\,',\vec{k}_e)+\tilde{T}^e(\vec{k},\vec{k}_e)f^N_{\epsilon'}(\theta_p)\,, \qquad (1.1)$$

where $\hbar\vec{k}$, $\hbar\vec{k}\,'$, and $\hbar\vec{k}_e$ are the momenta of incident and outgoing proton and ejected electrons, respectively, in the c.m. (center of momenta) system, ϵ is the incident c.m. energy, ϵ' the proton c.m. energy after electron emission, θ_p the angle between $\vec{k}$ and $\vec{k}\,'$, and $f^N_\epsilon(\theta_p)$ the proton's nuclear plus Coulomb elastic scattering amplitude. Equation (1.1) has the intuitively appealing property that it gives T_ϵ as the sum of two contributions which describe electron emission after and before nuclear plus Coulomb scattering, respectively. As we show below, the two factors in each term are essentially the on-shell amplitudes for the proton to scatter separately from the nucleus and from the electron. In the experiment of Duinker *et al.*,[1] the mean energy loss to the electron is estimated to be about 1 keV, i.e., much smaller than the width (38 keV) of the 461-keV resonance in ^{13}N. It is then justified to put $f^N_\epsilon(\theta_p)=f^N_{\epsilon'}(\theta_p)$ so that

$$R(\epsilon)=R(\tfrac{12}{13}E_p)=\frac{\int|T_\epsilon(\vec{k},\vec{k}\,';\vec{k}_e)|^2d^3k_e}{|f^N_\epsilon(\theta_p)|^2} \qquad (1.2)$$

should, in this approximation, be constant across the resonance.

This striking disagreement between experiment and theory has prompted us to take an independent look at the theoretical description of the problem, in an attempt to determine under which physical conditions the form (1.1) for the amplitude given in Refs. 2 and 3 is valid. We do not restrict ourselves to elastic proton-nucleus scattering but include inelastic nuclear processes. We first examine the conditions for Eq. (1.1) within the DWBA (distorted-wave Born approximation) description, in which the nuclear plus Coulomb scattering of the proton by the target nucleus provides the distorted waves which are employed to calculate the proton-electron scattering. We subsequently show that the same conditions suffice to justify the form of the result (1.1) within an adiabatic Born-Oppenheimer approximation in which it is not the proton-electron interaction which is assumed to be small but rather the ratio of the proton velocity to the bound-electron's orbital velocity. We pay particular attention to the fact that the proton-nucleus scattering displays a resonance.

Within these two theoretical approaches to the problem, we find that the form (1.1) for the amplitude is not universally valid, but that it is accurate within a well-defined set of conditions on the parameters of the scattering problem. These conditions are the following.

(i) It is most essential that the proton-electron scattering (ionization) should essentially occur at distances from the target nucleus which greatly exceed the proton's impact parameter b. More precisely, ionization should essentially occur at a distance R_K which satisfies[4]

$$R_K >> lb \;, \tag{1.3}$$

where $l(b)$ is the maximum angular momentum (the maximum impact parameter) which contributes to the nuclear plus Coulomb scattering of the proton by the nucleus. Since, in semiclassical terms, b and $l=bk$ change with the proton scattering angle (for pure Coulomb scattering, e.g., according to $l=\eta\cot\theta_p/2$, where η is the proton-target Sommerfeld parameter), the condition (1.3) will be violated at small scattering angles, where the impact parameter for Coulomb scattering from the nucleus can exceed the electronic K-shell radius R_K. As is shown below, for the experiment of Duinker *et al.*, Eq. (1.3) is satisfied for angles θ_p greater than 30° or so; we recall that the experiment was performed at 125°.

(ii) The momentum transfer,

$$\hbar q=\hbar(k-k') \;, \tag{1.4}$$

to the electron must be a small fraction of the incident proton's momentum k. Since classically q cannot exceed the value $2(m_e/m_p)k$, where m_e and m_p are the mass of electron and proton, respectively, this condition is surely well satisfied.

(iii) The angular momentum transfer $\hbar q R_K$ to the electron must not exceed, roughly speaking, a few units of $\hbar$. In the Duinker *et al.* experiment, $\hbar q\cdot R_K$ is about $1\hbar$.

(iv) Since Duinker *et al.* observe a *resonance enhancement* of the ratio $R(E_p)$, it is vital to establish the condition on the resonance part of the proton scattering wave function that leads to Eq. (1.1). The nuclear resonance can decay into the proton continuum through the Coulomb interaction between proton and bound electron, ejecting the latter in the process. This can be viewed as internal conversion where one of the nuclear states is a scattering state. This process does not lead to the form (1.1) for the total scattering amplitude. We estimate the relative contribution of this process to F_ϵ to be of the order $(E_{\rm res}/\Gamma_N)^{1/2}\cdot(R_N/R_K)$, where $E_{\rm res}$ is the resonance energy, R_N the nuclear radius. The factor $E_{\rm res}/\Gamma_N \cong 10$ describes the resonance enhancement of the process. The factor R_N/R_K accounts for the fact that in a nuclear resonance, the proton is confined to a sphere of volume R_N while in the DWBA matrix element involving two scattering states of the proton, the integration over the proton coordinate extends essentially over a sphere of radius R_K.

To summarize in physical terms: (a) The contribution from internal conversion—the genuine resonance contribution—is relatively suppressed because the ratio $R_K/R_N >> 1$. (b) The large ratio of proton mass to electron mass guarantees that proton scattering on an electron at rest is very forward peaked (for the proton), and transfers very little momentum to the electron. (c) This forward peaking eliminates multiple scattering of the proton, of the nucleus-electron-nucleus type, which would require the proton to backscatter from the electron. (d) For large enough proton scattering angles, the nuclear plus Coulomb scattering of the proton occurs in a volume which is negligibly small compared to the volume of the atomic K shell. Then, proton-nucleus and proton-electron scattering become mutually independent processes. These conditions suffice to justify the form (1.1) given by Blair *et al.*, and make it difficult to understand the experimental result of Ref. 1.

The paper is organized as follows. The essential physical steps leading to the conditions (i)–(iv) are outlined in the following sections. In view of the discrepancy between theory and experiment, we also wish to give a detailed and precise account of our arguments. In order not to destroy the simplicity of the basic reasoning, part of this account is given in several appendices. We use the DWBA form for T_ϵ throughout and show in Sec. V that the conclusions remain unchanged when the adiabatic form is used instead.

II. DWBA TREATMENT OF PROTON-ELECTRON SCATTERING

As stated in Sec. I, we first consider the range of validity of the factorized form (1.1) of T_ϵ within a DWBA formulation, in which the weak ($\frac{1}{137}$) proton-electron interaction is considered as a first-order perturbation on the dominant proton-nucleus scattering. Since our primary purpose is to present an independent search for possible deviations from the factorized form of T_ϵ given by Blair *et al.*, we have taken some pains to construct a proton-nucleus scattering wave function which explicitly contains the effects of a resonance (which does not normally occur in distorted-wave calculations). For generality, we have considered both a one-body ("shape") and a many-body ("compound nucleus") resonant state, but in order not to interrupt the main argument, we present the detailed discussion of this wave function in Appendix A. As mentioned in Sec. I, we consider the general case of inelastic proton-nucleus scattering. To focus on the essentials in the main text, we deal with this general case in Appendix B, and concentrate here on the one-body aspects, treating the resonance as a shape resonance, and the proton-nucleus scattering as an elastic process. In this case, the proton-nucleus scattering functions $\varphi_\epsilon^{(\pm)}(\vec{k},\vec{r}_p)$ depend on the single variable r_p (proton's positron) and the incident (final) momentum $\hbar\vec{k}(\hbar\vec{k}')$, respectively. Let $\Xi_0(r_e)$ be the (normalized) K-shell bound-state wave function of the electron in the target nucleus, and $\Xi_E^{(-)}(\vec{k}_e,\vec{r}_e)$ a scattering wave function of the electron in the Coulomb field of the target nucleus with asymptotic momentum $\hbar\vec{k}_e$ subject to an incoming wave boundary condition, and normalized to a delta function in energy.

The DWBA form for T_ϵ is given by

$$T_\epsilon(\vec{k},\vec{k}';\vec{k}_e)=\left\langle \varphi_\epsilon^{(-)}(\vec{k}',\vec{r}_p)\Xi_E^{(-)}(\vec{k}_e,\vec{r}_e)\left|\frac{e^2}{|\vec{r}_p-\vec{r}_e|}\right|\varphi_\epsilon^{(+)}(\vec{k},\vec{r}_p)\Xi_0(r_e)\right\rangle . \tag{2.1}$$

We have used the electron-proton interaction as a perturbation. This interaction must be supplemented by a recoil term (see Appendix A); such a term does not alter our conclusions and is therefore omitted here. In the published literature, one sometimes uses electronic wave functions for the united atom; in this case, the perturbation takes a different form.[3] However, our arguments carry through for this case just as well; see our discussion of the adiabatic approximation in Sec. V.

According to Appendix A, the functions $\varphi_\epsilon^{(\pm)}$ have the form (we suppress the proton spin which is irrelevant)

$$\begin{aligned}\varphi_\epsilon^{(\pm)}(\vec{k},\vec{r}_p)=&[(2m_p)/(\pi k\hbar^2)]^{1/2}\sum_{l,m=0}^{\infty} i^l Y_l^{m*}(\hat{k})Y_l^m(\hat{r}_p)\exp[\pm i\delta_l(\epsilon)]\\ &\times\left[r_p^{-1}F_\epsilon^l(r_p)+\delta_{l,0}\int d\epsilon'(\epsilon^\pm-\epsilon')r_p^{-1}F_{\epsilon'}^0(r_p)V_N^{\epsilon'}(\epsilon-\mathcal{E}_N^\pm)^{-1}V_N^\epsilon\right]\\ &+(4\pi r_p)^{-1}F_0(r_p)\exp[\pm i\delta_0(\epsilon)](\epsilon-\mathcal{E}_N^\pm)^{-1}V_N^\epsilon .\end{aligned} \tag{2.2}$$

Here, m_p is the reduced mass of the proton, $\delta_l(\epsilon)$ $(l>0)$ are the nuclear-plus-Coulomb scattering phase shifts of the proton (we assume for simplicity that the Coulomb potential of the target nucleus is screened and omit the logarithmic phase shift of an infinitely extended Coulomb potential), and $F_\epsilon^l(r_p)$ $(l>0)$ are the real radial scattering wave functions of the proton which are regular at the origin and behave for large distances as

$$F_\epsilon^l(r_p)\underset{r_p\to\infty}{\longrightarrow}\sin(kr_p-\tfrac{1}{2}l\pi+\delta_l) . \tag{2.3}$$

It is the purpose of Eq. (2.2) to exhibit the explicit influence of the resonance which to be definite we have assumed to occur in the s wave. According to Appendix A, $\delta_0(\epsilon)$ and $F_\epsilon^0(r_p)$ are smooth background phase shift and smooth background real regular radial scattering function, respectively, for s waves, the latter again with asymptotic behavior (2.3), while the effect of the resonance resides in the last two terms of Eq. (2.2). The real radial wave function $F_0(r_p)$ is normalized according to

$$\int_0^\infty dr_p F_0^2(r_p)=1 ; \tag{2.4}$$

it is the "wave function of the resonance" and differs from the functions $F_\epsilon^0(r_p)$ in decaying exponentially (or in some other fashion) outside the target nucleus, where the functions F_ϵ^0 have oscillatory behavior. The real quantity V_N^ϵ is the matrix element which couples the resonance to the proton continuum. It is responsible for the width

$$\Gamma_N=2\pi(V_N^\epsilon)^2 \tag{2.5}$$

of the resonance. The complex energy $\mathcal{E}_N^\pm=E_N\pm i\frac{1}{2}\Gamma_N$ is given in terms of the width and the real resonance energy E_N.

Of the three additive terms appearing in Eq.

(2.2), the first $[F_\epsilon^l(r_p)]$ describes the nonresonant background. The other two are resonant, but differ in that the first [containing $F_{\epsilon'}^0(r_p)$] is long ranged in r_p, and the last $[F_0(r_p)]$ is, as mentioned, of bound-state form and so localized in or near the nucleus. To simplify the notation we designate these three terms, in the above order, as 1, 2, and 3. Substituting Eq. (2.2) into (2.3) then produces 9 terms $\langle i \mid T \mid j \rangle \equiv T_{ij}$, with $i,j=1,2,3$. The contributions T_{ij} have a simple physical interpretation along the lines of multiple-scattering theory and a great formal similarity to the work of Ref. 5. T_{11} is obviously the sum of the nonresonant contributions; it will receive detailed attention in Sec. IV. The term T_{12} can be read as formation and subsequent decay of the nuclear resonance, propagation of the proton in an intermediate state (integration over the intermediate energy ϵ', followed by ejection of the electron. The term T_{21} has the time-reversed interpretation. The term T_{13} signifies formation of the nuclear resonance, followed by a process which is a generalization of, but similar to, internal conversion: The Coulomb interaction between proton and electron leads to the simultaneous emission of the electron and decay of the resonance. (In internal conversion, one considers transitions between two nuclear levels, whereas here we deal with the transition from a nuclear level to the proton continuum.) The interpretation of the other terms T_{ij} can be given in a similar way.

III. NEGLECT OF "INTERNAL CONVERSION" PROCESSES

We show that all contributions to T containing the resonance wave function $F_0(r_p)$, i.e., all terms T_{ij} with i or j equal to 3, are negligibly small. We simplify the argument by considering only atomic monopole transitions (the extension to higher multipoles is straightforward). We compare the s-wave contribution to T_{13} with the s-wave contribution to T_{11} and show that the former is very small compared to the latter.

The two contributions are denoted by T_{13}^0 and T_{11}^0, respectively, and are given by

$$T_{13}^0=[(2m_p)/\pi\hbar^2k)]^{1/2}(4\pi)^{-2}\exp(2i\delta_0)V_N^\epsilon(\epsilon-\mathscr{E}_N^+)^{-1} \times\int_0^\infty dr_pF_0(r_p)F_{\epsilon'}^0(r_p)g_0(k_e,r_p)\ , \tag{3.1}$$

$$T_{11}^0=[(2m_p)/(\pi\hbar^2k)](4\pi)^{-2}\exp(2i\delta_0) \times\int_0^\infty dr_pF_\epsilon^0(r_p)F_{\epsilon'}^0(r_p)g_0(k_e,r_p)\ . \tag{3.2}$$

For simplicity of notation, we have put $\delta_0(\epsilon)=\delta_0(\epsilon')$. This obviously does not affect the magnitudes of T_{13}^0 and of T_{11}^0. The function $g_0(k_e,r_p)$ is the form factor for the atomic monopole transition, obtained by integrating the product of $\Xi_0,\Xi_\epsilon^{(-)}$ and the interaction in expression (2.1) over the variable $\vec{r}_e$. This form factor is still a function of the electron momentum $\hbar k_e$. For dipole transitions, the momentum dependence of $g_1(r_p)$ is displayed in Fig. 3 of Ref. 6. As the energy of the ejected electron changes by a factor 100, the characteristic features of the dipole form factor—position and width of the broad maximum occuring at about $r_p=R_K$, the K-shell radius—can be seen not to change by more than a factor of 2 or 3. Keeping in mind that we aim at giving an order-of-magnitude estimate of T_{11} and T_{13}, and that the characteristic energy dependence of the monopole form factor is expected to be similar to that of the dipole term, we approximate $g_0(k_e,r_p)$ by an energy-independent function. The form of this function is suggested by Fig. 2 of Ref. 6 which shows $g_0(k_e,r_p)$ for nuclear charge 20 to be a monotonically decreasing function of r_p, with a half-maximum value taken roughly at $r_p=R_K$, the atomic K-shell radius. In our case,

$$R_K=\tfrac{1}{6}a_B\cong\tfrac{1}{12}\text{Å}\ , \tag{3.3}$$

where a_B is the Bohr radius. We accordingly parametrize $g_0(k_e,r_p)$ in the form

$$g_0(k_e,r_p)\cong\exp(-r_p/R_K)\ . \tag{3.4}$$

We emphasize that (3.4) does not constitute a precise approximation to $g_0(r_p)$ (we have paid no attention to normalization); it suffices, however, to show that $\mid T_{13}\mid << \mid T_{11}\mid$. We approximate the bound s-wave function $F_0(r_p)$ of the proton in a similar way, writing

$$F_0(r_p)\cong(2/R_N)^{1/2}\exp(-r_p/R_N)\ , \tag{3.5}$$

which is consistent with Eq. (2.4). To evaluate the integral in Eq. (3.2), we note that the range of integration extends over about 10 000 fm, while the classical turning point of s-wave protons at 460 keV lies roughly at 20 fm. It is therefore justified to use for F_ϵ^0 the asymptotic form (2.3). Using this form, we find two contributions, one containing the factor $\exp[i(k+k')r_p]$, the other, the factor $\exp[i(k-k')r_p]$. since $k\cong0.15\ \text{fm}^{-1}$, the first contribution contains a function which oscillates rapidly over the range R_K of integration, and thus is negligible in comparison with the second, in which the momentum transfer $\hbar q$ defined in Eq.

(1.4) appears. This momentum transfer is typically of the order $\hbar(R_K)^{-1}$. With these approximations (which are further treated in Sec. IV A), we find

$$T^0_{11} \approx -[(2m_p)/(\pi\hbar^2 k)](4\pi)^{-2} \times \exp(2i\delta_0)\tfrac{1}{2}R_K(1+q^2R_K^2)^{-1} . \quad (3.6)$$

The magnitude of T_{13} given by Eq. (3.1) is overestimated if we replace $F^0_\epsilon(r_p)$ (which oscillates, and is suppressed near the origin because of the Coulomb penetration factor) by unity. this yields, with $R_K >> R_N$,

$$T^0_{13} \le 2[(2m_p)/(\pi\hbar^2 k)]^{1/2}\cdot(4\pi)^{-2} \times (V^\epsilon_N/\Gamma_N)(2R_N)^{1/2} . \quad (3.7)$$

As a result, we find, using Eq. (2.5), that

$$|T^0_{13}/T^0_{11}| \le 4(E_{\rm res}/\Gamma_N)^{1/2}(R_N/R_K)^{1/2} \times (kR_K)^{-1/2}(1+q^2R_K^2) . \quad (3.8)$$

Since k^{-1} is of the order of R_N, this agrees with the qualitative estimate given in the Sec. I. Quantitatively, we see that with $R_K \cong 8000$ fm, $R_N \cong 4$ fm, $k \cong 0.15$ fm^{-1}, $E_{\rm res}/\Gamma_N \cong 10$, $R_K q \lesssim 3$, the ratio $|T^0_{13}/T^0_{11}| < 0.0005$. Even though this estimate is based on qualitative features of g_0 and F_0 it shows that the contribution of T^0_{13} can never explain the finding of Ref. 1. It is conceivable, of course, that at an energy where $(T_{11}+T_{12}+T_{21})$ nearly cancel, T_{13} gives a contribution of a few per cent, but this is not enough.

The same arguments can be used to show that $|T^0_{23}/T^0_{21}| << 1$. Finally, comparing T^0_{33} with T^0_{11}, we find analogously that the ratio T^0_{33}/T^0_{11} is roughly given by $(E_{res}/\Gamma_N)\cdot(R_K k)^{-1}(1+q^2R_K^2)$ so that T^0_{33}, too, is negligible.

These findings justify the neglect of the component proportional to $F_0(r_p)$ in the wave function (2.2).

IV. THE RANGE OF VALIDITY OF THE FACTORIZED FORM (1.1) OF $T_\epsilon(\vec{k},\vec{k}\,';\vec{k}_e)$

Even within the DWBA and after neglect of the terms involving $F_0(r_p)$, we do not obtain the factorized form (1.1) in general. The reason is that in the angular range where Eq. (1.1) is valid, the factors themselves are in essence the on-shell amplitudes describing the scattering of the proton independently by the target nucleus and by the electron; this form can only be expected to hold when these scatterings occur at very different points in space. The latter condition requires, in turn, the nuclear plus Coulomb scattering by the target to occur well inside the electronic K-shell radius R_K. For Coulomb trajectories, this will only be true if the scattering angle θ_p is sufficiently large. Mathematically we understand this to demand that the integral over r_p in the ionization matrix element (2.1) come predominantly from r_p values at which the distorted waves of the proton are accurately described by their asymptotic (large r_p) form, i.e., well outside the range of Coulomb and centrifugal effects from the target nucleus.

To bring out the basic simplicity of our argument most clearly, we proceed in two steps. We first show that using the asymptotic form of the wave functions (2.2), one does obtain the factorized form (1.1). We subsequently establish the range of validity of this formula.

A. Asymptotic evaluation of the DWBA matrix element

For any value of r_p, and under neglect of $F_0(r_p)$, the functions $\phi_\epsilon(\pm)$ given by Eq. (2.2) can be recast into the conventional form

$$\phi^{(\pm)}_\epsilon = [(2m_p k)/(\pi\hbar^2)]^{1/2}(4\pi)^{-1}\exp(i\vec{k}\cdot\vec{r}_p) + \phi^{(\pm)}_{\epsilon,\rm scatt} , \quad (4.1)$$

with the resonant ($F^0_{\epsilon'}$) contributions contained in $\phi^{(\pm)}_{\epsilon,\rm scatt.}$. Asymptotically in r_p ($r_p \to \infty$), this term has the form

$$\phi^{(\pm)}_{\epsilon,\rm scatt} \to [(2m_p)/(\pi k\hbar^2)]^{1/2} r_p^{-1}\exp(\pm ikr_p) , \sum_{l,m=0}^{l_{\max}} i^{l\mp l} Y^{m*}_l(\hat{k})Y^m_l(\hat{r}_p)(2i)^{\pm 1}\begin{bmatrix} S_e - 1 \\ S^*_e - 1 \end{bmatrix} . \quad (4.2)$$

We have used the definitions

$$S_l = \exp(2i\delta_e) \quad l > 0 , \qquad S_0 = \exp(2i\delta_0)\cdot[(\epsilon - \mathscr{E}^-_N)/(\epsilon - \mathscr{E}^+_N)] . \quad (4.3)$$

The derivation of the $l > 0$ part of Eq. (4.2) is straightforward.[7] For $l = 0$, we use the fact that the ϵ' integration on the right-hand side of Eq. (2.2) can be expressed in terms of the proton's s-wave radial Green's functions; using the space representation, this Green's function, too, can be evaluated asymptotically in standard fashion.[7] We note that in the asymptotic region, the resonance manifests itself only in the energy dependence of $S_0(\epsilon)$.

The symbol $l_{\max}$ on the right-hand side of Eq. (4.2) indicates that only a finite number of partial waves contributes to $\varphi^{(+)}_{\epsilon,\text{scatt}}$ if the scattering angle θ_p is held fixed. The dependence of $l_{\max}$ on θ_p and the ensuing restriction on the values of r_p for which the asymptotic form (4.2) can be used, are treated in Sec. IV B.

We introduce the nuclear plus Coulomb scattering amplitude for the proton,

$$f^N_\epsilon(\hat{k},\hat{r}_p)=(k)^{-1}\sum_{l,m=0}^{l_{\max}}(2i)^{-1}[S_l(\epsilon)-1]\times Y_l^{m*}(\hat{k})Y_l^m(\hat{r}_p). \quad (4.4)$$

With the help of this function, we rewrite Eq. (4.2) as

$$\varphi^{(\pm)}_{\epsilon,\text{scatt}}\underset{r_p\to\infty}{\longrightarrow}[(2m_pk)/(\pi\hbar^2)]^{1/2}\times r_p^{-1}\exp(\pm ikr_p)\begin{bmatrix}f^N_\epsilon(\hat{k},\hat{r}_p)\\ f^{N*}_\epsilon(-\hat{k},\hat{r}_p)\end{bmatrix}. \quad (4.5)$$

We note that $\varphi^{(\pm)}_{\epsilon,\text{scatt}}$ is proportional to the nuclear plus Coulomb scattering amplitude of the proton in the asymptotic region; it is this fact which renders $R(E_p)$ independent of the resonance.

Inserting the relations (4.1) and (4.5) into the DWBA matrix element (2.1), we obtain four contributions, involving as integrands the factors $\exp[i(\vec{k}-\vec{k}\,')\vec{r}_p]$, $\exp(ikr_p-\vec{k}\,'\cdot\vec{r}_p)\exp(i\vec{k}\cdot\vec{r}_p+ik'r_p)$, and $\exp[i(k+k')r_p]$, respectively.

To evaluate these four contributions further, we expand the form factor $G(\vec{k}_e,\vec{r}_p)$ in Eq. (2.1), i.e., the result of the integration over $\vec{r}_e$, in multipoles,

$$G(\vec{k}_e,\vec{r}_p)=\sum_{L,M=0}^{L_{\max}}Y_L^{*M}(\hat{k}_e)Y_L^M(\hat{r}_p)g_L(k_e,r_p) \quad (4.6)$$

and make use of the well-known fact that only a few multipoles ($L_{\max}\lesssim 2$) contribute to K-shell ionization.[6] This is intuitively clear because $qR_K\cong 1$. [In writing Eq. (4.6), we have also used the fact that G is independent of the direction of the spin of the electron.]

The first of the four above-mentioned contributions involves the integral

$$I_1=\int d^3r_p\exp[i(\vec{k}-\vec{k}\,')\cdot\vec{r}_p]G(\vec{k}_e,\vec{r}_p)\ . \quad (4.7)$$

To show that this is negligible for scattering angles $\theta_p\gtrsim 6°$ or so, we use the monopole term in Eq. (4.6) (the argument applies similarly to $L>0$), with $g_0(k_e,r_p)\cong\exp(-r_p/R_K)$. Then,

$$I_1\cong 8\pi R_K^3\cdot(1+|\vec{k}-\vec{k}\,'|^2R_K^2)^{-2}\ . \quad (4.8)$$

For scattering angles large enough that $|\vec{k}-\vec{k}\,'|\approx 2k$, and for $kR_K>>1$, $I_1\approx(k^4R_K)^{-1}<<I_2$ or I_3 as estimated below. This is intuitively obvious: The small momentum transfer to the electron makes it impossible for the proton to scatter into sizeable angles, unless this is accomplished by nuclear plus Coulomb scattering on the target. The neglect of I_1 is ultimately justified by the smallness of m_e/m_p.

The fourth contribution involves the integral

$$I_4=(4\pi)^2\int d^3r_pG(\vec{k}_e,\vec{r}_p)r_p^{-2}\exp[i(k+k')r_p]\times f^N_\epsilon(\hat{k},\hat{r}_p)\cdot f^N_{\epsilon'}(-\hat{k}',\hat{r}_p)\ . \quad (4.9)$$

(The units are the same as for I_1.) Again, we observe that the radial integral over the monopole term (which we take once more as representative) yields a factor

$$\int_0^\infty dr_pg_0(k_e,r_p)\exp[i(k+k')r_p]\cong R_K(1-i\cdot 2kR_K)^{-1}\ .$$

We see that I_4 has the order of magnitude (recall that $kR_K>>1$)

$$|I_4|\cong(4\pi)^2R_K(2kR_K)^{-1}k^{-2}\ ; \quad (4.10)$$

we have used that $k^2\int|f^N_\epsilon|^2d\Omega_{r_p}\simeq 1$ for the low l values ($l\leq l_{\max}$) here considered, and have for simplicity also approximated

$$k\cdot k'\cdot\int d\Omega_{r_p}f^N_\epsilon(\hat{k},\hat{r}_p)f^N_{\epsilon'}(-\hat{k}',\hat{r}_p)$$

by unity. This is clearly an overestimate. Again, I_4 is negligible compared with I_2 and I_3 which are estimated below. We mention that I_4 describes the process of nuclear-atom-nuclear scattering by the proton mentioned in Sec. I. This process involves a backscattering into the nucleus by the electron of the nuclear-scattered proton; such backscattering requires a large momentum transfer to the electron. This causes the factor $\exp[i(k+k')r_p]$ to appear; the ultimate reason for the neglect of I_4 is thus the same as for I_1 and rests in the inequality $m_e/m_p<<1$.

We turn to the evaluation of the remaining two contributions. They involve the two integrals

$$I_2=(4\pi)\cdot\int d^3r_p G(\vec{k}_e,\vec{r}_p)\times\exp(ikr_p-i\vec{k}'\cdot\vec{r}_p)r_p^{-1}f_\epsilon^N(\hat{k},\hat{r}_p)\,,$$
$$I_3=(4\pi)\cdot\int d^3r_p G(\vec{k}_e,\vec{r}_p)\times\exp(ik'r_p+i\vec{k}\cdot\vec{r}_p)r_p^{-1}f_\epsilon^N(-\hat{k}',\hat{r}_p)\,. \quad (4.11)$$

Since f_ϵ contains a summation over l restricted by $l_{\max}$, and since G contains angular momenta L not exceeding $L_{\max}\approx 2$ [see Eqs. (4.4) and (4.6)], we see that in the angular momentum expansion of the plane-wave factors appearing in Eqs. (4.11), only terms with $l\le l_{\max}+L_{\max}$ contribute. For such a limited range of l values, an asymptotic expansion of the Bessel functions can be used[7]; it has the *same range of validity* as Eq. (4.2). We find[7] (see Appendix C)

$$\exp(i\vec{k}\cdot\vec{r}_p)\to[(2\pi)/(ikr_p)][\exp(ikr_p)\delta(\Omega_{\vec{k}}-\Omega_{\vec{r}_p})-\exp(-ikr_p)\delta(\Omega_{\vec{k}}+\Omega_{\vec{r}_p})]\,. \quad (4.12)$$

The delta functions in Eq. (4.12) are simply representations of the finite sums

$$\sum_{l,m=0}^{l_{\max}+L_{\max}} Y_e^{m*}(\hat{k})Y_e^m(\hat{r}_p)\,;$$

for the restricted set of angular momenta here considered, they are identical to these sums.

Inserting Eq. (4.12) into Eq. (4.11), we find that both I_2 and I_3 consist of two terms, involving the factors $\exp[i(k+k')r_p]$ and $\exp[i(k-k')r_p]$, respectively. Approximating $G(\vec{k}_e,\vec{r}_p)$ by $\exp[-r_p/R_K]$ as before, the first is of order $|1-i(k-k')R_K|/|1-i(k+k')R_k|$ relative to the second, and so negligible in the limit $(k+k')R_K>>1>>(k-k')R_K$. This finally yields

$$I_2=(4\pi)(2\pi i)(k')^{-1}f_\epsilon^N(\hat{k},\hat{k}')\int_0^\infty dr_p G(\vec{k}_e,\vec{r}_p)\Big|_{\hat{r}_p=\hat{k}'}\exp[i(k-k')r_p]\,,$$
$$I_3=(4\pi)(2\pi i)(k)^{-1}f_\epsilon^N(\hat{k}',\hat{k})\int_0^\infty dr_p G(\hat{k}_e,\vec{r}_p)\Big|_{\hat{r}_p=-\hat{k}}\exp[i(k'-k)r_p]\,, \quad (4.13)$$

where we have $f_\epsilon(\hat{k},\hat{k}')=f_\epsilon(\theta_p)=f_\epsilon(\vec{k}',\vec{k})=f_\epsilon(-\vec{k}',-\vec{k})$. Estimating f_ϵ roughly again by k^{-1} and evaluating the integrals in Eq. (4.13) in the monopole approximation, $I_2\approx I_3\approx 8\pi^2R_Kk^{-2}(1-iqR_K)^{-1}\approx 8\pi^2R_Kk^{-2}$, so that $I_4/I_2\sim(kR_K)^{-1}\sim 10^{-3}$ and $I_1/I_2\sim(16\pi k^2R_K^2)^{-1}<10^{-7}$.

Collecting our results, we find that $T_\epsilon(\vec{k},\vec{k}';\vec{k}_e)$ has indeed the form of Eq. (1.1). The two amplitudes appearing in this equation are given by

$$T_\epsilon^e(\vec{k}',\vec{k}_e)=[(im_p)/(\pi\hbar^2)]\int_0^\infty dr_p G(\vec{k}_e,\vec{r}_p)\Big|_{\hat{r}_p=\hat{k}'}\exp[i(k-k')r_p]\,,$$
$$\tilde{T}_\epsilon^e(\vec{k},\vec{k}_e)=[(im_p)/(\pi\hbar^2)]\int_0^\infty dr_p G(\vec{k}_e,\vec{r}_p)\Big|_{\hat{r}_p=-\hat{k}}\exp[i(k'-k)r_p]\,. \quad (4.14)$$

We have put $(k'/k)\cong 1$ which is consistent with our other approximations. Equations (1.1) and (4.14) are consistent with the results of Refs. 2 and 3. Equations (1.1) and (4.14) were obtained under neglect of terms of the order m_e/m_p, and with the help of the asymptotic form (4.5) [which because of $L_{\max}\lesssim 2$ ensures the validity of Eq. (4.12)]. We now discuss the validity of this form.

B. Range of validity of the asymptotic form (4.5)

In keeping with the entire main text of the manuscript, we establish this range for elastic nuclear plus Coulomb scattering of the proton. For inelastic nuclear scattering, fewer l waves contribute, and the condition is seen to be fulfilled under weaker conditions than in the elastic case. Hence, the most conservative estimate of the r_p range where the asymptotic representation (4.5) is valid, is given by the asymptotic condition[7] for elastic Coulomb scattering wave functions

$$kr_p>>l(l+1)+\eta^2\,, \quad (4.15)$$

where η is the proton-nucleus Sommerfeld parameter. In the limit $l>>\eta$, this is the condition mentioned in Sec. I, where we have replaced r_p by R_K, the K-shell radius as an estimate of the region in which the proton form factor $G(\vec{k}_e,\vec{r}_p)$ is concentrated.

The condition (4.15) imposes a lower limit on r_p for each nuclear partial wave, which in physical terms demands that the centrifugal and Coulomb potentials be much smaller than the incident (or outgoing) energy throughout this r_p range. Thus we require that (4.15) be met for the largest l value contributing to the nuclear plus Coulomb scattering at the scattering angle θ_p. Since this l value is obviously determined by the Coulomb (without nuclear) potential, the simplest estimate is given, on semiclassical grounds, by

$$l(\theta_p)=\eta\cot(\theta_p/2)\ . \tag{4.16}$$

For the Duinker experiment[1] and $\theta_p=125°$, this gives $l(\theta_p)\approx 1$. To allow for wave effects, we can use the stationary-phase approximation in the manner described in Sec. 5 1.1 of Ref. 8 to estimate the spread of l values around this central value. Unfortunately, this stationary-phase approximation works only if the central value given by Eq. (4.16) is about 10 or larger, as otherwise the representation of the Legendre polynomials in terms of asymptotic formulas is not accurate enough. For $l(\theta_p)=10$, the spread due to wave effects can then be estimated to be five or six, and we expect a similar number to apply for the central value $l(\theta_p)=1$. If only five or six (or even eight) l values contribute significantly to scattering into $\theta_p=125°$, then the condition (4.15) is met for $k=0.15$ fm, $R_K\cong 8000$ fm, and the validity of the Blair formula is established.

An independent check on the validity of our reasoning can be read off from Figs. 4 to 6 of Ref. 6. There, a numerical evaluation of the DWBA matrix element for Coulomb scattered proton wave functions is compared with a semiclassical evaluation using classical Coulomb trajectories for the proton. The figures show that for small values of the atomic number Z of the target nucleus and large scattering angles ($\theta_p \gtrsim 90°$ or so), the agreement between the two calculations is excellent (better than 1 part in 10^4). We therefore feel very confident in the use of the asymptotic form (4.5). We emphasize, however, that the use of this approximation is restricted to sufficiently large scattering angles. This is not because of the nuclear or the nuclear resonance scattering, both of which occur in partial waves of low l so that the use of asymptotic forms presents no problem. Rather, it is due to the Coulomb field. For scattering angles $\theta_p<50°$, the classical l value increases steeply with $\widetilde{\theta}_p$, and the use of the asymptotic form (4.5) ceases to be correct.

V. SUMMARY AND CONCLUSIONS

Starting from a many-body treatment of the nuclear resonance (given in Appendix A) and using a DWBA form for the K-shell ionization amplitude, we have shown that the T matrix can be written in the form (1.1). This result is based on the neglect of terms which are small of the order m_e/m_p, and on a truncated summation over angular momenta in the proton scattering wave function which is legitimate for sufficiently large proton scattering angles ($\theta \gtrsim 50°$ in the case of the reaction of Ref. 1). This truncation allows us to use the asymptotic form of the proton scattering wave function, and this directly implies the form (1.1) for T.

Our result (1.1) is not restricted to a DWBA form of the K-shell ionization amplitude. It holds equally if one includes the recoil term, or uses adiabatic wave functions for the electrons (defined as functions of r_p such that for each fixed r_p, they solve the electron's Dirac equation) and calculates the K-shell ionization amplitude to first order in the deviation from adiabaticity. The validity of this statement follows from the observation that in deriving the form (1.1), we have only used two features of the electronic transition matrix elements: (i) the inequalitites $m_p >> m_e$ and $R_K >> R_N$ which remain valid for adiabatic wave functions; (ii) the form (4.6) which only uses rotational invariance. This shows that the factorization (1.1) holds under very general conditions, i.e., as long as a first-order treatment of *some* kind of interaction between proton and electron is justified.

We have shown that for the experiment reported in Ref. 1, the form (1.1) implies that the ratio $R(E_p)$ of coincidence over singles is independent of the nuclear scattering process, including the presence of a nuclear resonance, if the nuclear scattering amplitude $f_\epsilon^N(\theta_p)$ changes sufficiently little with energy ϵ within an energy interval defined by the energy transfer to the K-shell electron. This is the case for the scattering of protons on ^{12}C, and leads to the conclusion that the result of Ref. 1 is at variance with scattering theory.

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APPENDIX A: s-WAVE RESONANT NUCLEAR SCATTERING AS A MANY-BODY PROCESS

In this appendix, we derive the wave function describing s-wave protons scattered elastically by ^{12}C, including the ^{13}N resonance. In order to establish a somewhat general theoretical framework which encompasses any type of nuclear resonance, we consider two cases. The first considers the resonance as being due to a bound state embedded in the elastic proton continuum. The theoretical description uses the shell-model approach to nuclear reactions.[9] In the second case, we consider an s-wave shape resonance, which we describe in the framework of the approach developed by Wang and Shakin.[10] In the first case, we deal with a genuine many-body wave function, while in the second, it suffices to consider a one-body potential scattering problem. We confine ourselves throughout to s-wave scattering. By inclusion of the proper angular momentum factors, this restriction can easily be lifted if application to other cases is envisaged.

1. The resonance is caused by a bound state embedded in the continuum

We describe the resonance in a truncated space of functions which we construct as follows.[9] Let $\{\chi_\epsilon^0\}$ be a set of antisymmetric wave functions describing the nucleus ^{12}C in its ground state, and a proton with angular momentum zero in a scattering state with asymptotic c.m. energy ϵ. The states $|\chi_\epsilon^0\rangle$ contain the real s-wave radial wave functions $F_\epsilon^0(r_p)$ of the proton which are regular at the origin and which are solutions of a one-body Schrödinger equation containing a screened Coulomb and a nuclear Woods-Saxon potential. The parameters of the Woods-Saxon potential can be chosen in such a way that the elastic s-wave scattering phase shift $\delta_0(\epsilon)$, defined by

$$F_\epsilon^0(r_p)\rightarrow\sin[kr_p+\delta_0(\epsilon)] \tag{A1}$$

and $\epsilon=\hbar^2k^2/(2m_p)$, where m_p is the proton reduced mass, correctly reproduces the observed *nonresonant* cross section. The functions χ_ϵ^0 are normalized to a delta function in energy

$$\langle\chi_{\epsilon'}^0|\chi_\epsilon^0\rangle=\delta(\epsilon-\epsilon') \; . \tag{A2}$$

For later use, we also define

$$\chi_\epsilon^{0(\pm)}=\exp[\pm i\delta_0(\epsilon)]\chi_\epsilon^0 \; . \tag{A3}$$

To introduce the ^{13}N resonance as a bound state embedded in the continuum, let φ_N^0 be an antisymmetric normalized A-body wave function, orthogonal to the $\{\chi_\epsilon^0\}$, with spin $\frac{1}{2}$ and positive parity which describes the ^{13}N resonance in the frame of a shell-model approach as a bound state, so that

$$\langle\chi_\epsilon^0|\varphi_N^0\rangle=0, \;\; \langle\varphi_N^0|\varphi_N^0\rangle=1 \; , \tag{A4}$$

and that the expectation value E_0 of the nuclear Hamiltonian H_N,

$$E_0=\langle\varphi_N^0|H_N|\varphi_N^0\rangle \tag{A5}$$

approximates the ^{13}N resonance energy. (Note that our energy scale is normalized in such a way that ^{12}C in its ground state plus a proton at rest at infinity have zero energy.) We consider the nuclear Hamiltonian in the space of functions $\{\chi_\epsilon^0,\varphi_N^0\}$ and write it as

$$H_N=\int d\epsilon\,|\chi_\epsilon^0\rangle\epsilon\langle\chi_\epsilon^0|+|\varphi_N^0\rangle E_0\langle\varphi_N^0| + \int d\epsilon\, V_N^\epsilon(|\chi_\epsilon^0\rangle\langle\varphi_N^0|+|\varphi_N^0\rangle\langle\chi_\epsilon^0|) \; . \tag{A6}$$

The last term in Eq. (A6) describes the coupling between the bound states $|\varphi_N^0\rangle$ and the scattering states $|\chi_\epsilon^0\rangle$ and causes $|\varphi_N^0\rangle$ to become a resonance. The coupling matrix elements

$$V_N^\epsilon=\langle\chi_\epsilon^0|H_N|\varphi_N^0\rangle \tag{A7}$$

can be and have been chosen real.

In the space of functions $\{\chi_\epsilon^0,\varphi_N^0\}$, we write the total s-wave scattering function Ψ_E^0 in the form

$$\Psi_E^0=\int d\epsilon\, a_E(\epsilon)|\chi_\epsilon^0\rangle+\mathscr{C}_E|\varphi_N^0\rangle \; . \tag{A8}$$

Substituting Eq. (2.8) into the Schrödinger equation $H_N|\Psi_E^0\rangle=E|\Psi_E^0\rangle$, we find after a straightforward calculation for $|\Psi_E^0\rangle$ the explicit form[9]

$$|\Psi_E^{0(\pm)}\rangle=\exp[\pm i\delta_0(E)]\cdot\left[|\chi_E^0\rangle+|\varphi_N^0\rangle(E-\mathscr{E}_N^\pm)^{-1}V_N^E+\left[\int d\epsilon(E^\pm-\epsilon)^{-1}|\chi_\epsilon^0\rangle V_N^\epsilon\right]\cdot(E-\mathscr{E}_N^\pm)^{-1}V_N^E\right] \; . \tag{A9}$$

Here, $E^{\pm}$ denotes the way in which the singularity $E=\epsilon$ is avoided, in the usual manner. The complex (energy-dependent) resonance energy $\mathscr{E}_N^{\pm}$ is defined by

$$\mathscr{E}_N^{\pm}=E_0+\Delta_N(E)\mp\frac{i}{2}\Gamma_N(E)=E_{\rm res}\mp\frac{i}{2}\Gamma_N(E)\,, \tag{A10}$$

where

$$\Delta_N(E)\mp\frac{i}{2}\Gamma_N(E)=\int d\epsilon(E^{\pm}-\epsilon)^{-1}(V_N^{\epsilon})^2\,. \tag{A11}$$

The last two terms in Eq. (A9) give the resonance contribution to the nuclear s-wave scattering.

Taking the asymptotic behavior of $|\Psi_E^{0(\pm)}\rangle$, we find the s-wave part $S_0(E)$ of the nuclear scattering matrix,

$$S_0(E)=\exp[2i\delta_0(E)]\frac{E-\mathscr{E}_N^{-}}{E-\mathscr{E}_N^{+}}\,. \tag{A12}$$

The construction of $|\Psi_E^0\rangle$ in Eq. (A9), and of $S_0(E)$ in Eq. (2.12), is formal in the sense that we have not calculated explicitly the quantities $\{\chi_\epsilon,\varphi_N^0,\xi_N,V_N^\epsilon\}$. Such constructions, based on a nuclear shell model, can be given.[9] For our purpose, i.e., the calculation of the ratio $R(E_p)$, are, however, not needed.

2. The resonance is a shape resonance in the nuclear potential

By a procedure due to Wang and Shakin,[10] this can be cast into a form that is identical to that of Appendix A 1. We briefly recapitulate their procedure here with the aim of making our presentation self-contained. Let $\{\varphi_\epsilon^0\}$ be a set of antisymmetric wave functions describing the nucleus ^{12}C in its ground state, and a proton with angular momentum zero in a scattering state with asymptotic c.m. energy ϵ. The states $|\varphi_\epsilon^0\rangle$ contain, in full analogy to Appendix A 1, the real s-wave radial wave functions $F_\epsilon^0(r_p)$ of the proton with asymptotic behavior (A 1) where now, however, $\delta_0(\epsilon)$ displays a single-particle resonance. In other words, $\delta_0(\epsilon)$ increases (nearly) by π over an energy interval of length $\Gamma_{\rm sp}$ centered at $\epsilon=E_{\rm sp}$. We define a normalized wave function $\varphi_{\rm sp}^0$ which describes the resonance, and associated projection operators $Q=|\varphi_{\rm sp}^0\rangle\langle\varphi_{\rm sp}^0|$ and

$$P=\int d\epsilon\,|\varphi_\epsilon^0\rangle\langle\varphi_\epsilon^0|-Q.$$

For our purposes, the detailed choice of $\varphi_{\rm sp}^0$ is largely arbitrary except that for $r_p\lesssim$ the radius of the Coulomb barrier, $|\varphi_{\rm sp}^0\rangle$ should essentially coincide (except for a normalization constant) with $|\varphi_\epsilon^0\rangle$ taken at $\epsilon=E_{\rm sp}$. One possibility is to choose $|\varphi_{\rm sp}^0\rangle=\int d\epsilon\,a(\epsilon)\,|\varphi_\epsilon^0\rangle$ with $a(\epsilon)$ real, sharply centered at $\epsilon=E_{\rm sp}$, and $\int|a(\epsilon)|^2d\epsilon=1$, but other choices are possible. The function $\varphi_{\rm sp}^0$ plays the role of φ_N^0 of Appendix A 1 in the present context. Since the set of functions $\{|\varphi_\epsilon^0\rangle,|\varphi_{\rm sp}^0\rangle\}$ is overcomplete we introduce modified scattering functions χ_ϵ^0 which are orthogonal to $\varphi_{\rm sp}^0$ and do not contain the single-particle resonance. This can be done[10] by defining

$$(\epsilon-PH_NP)\,|\chi_\epsilon^0\rangle=0\,. \tag{A13}$$

Here, $H_N=\int d\epsilon\,|\varphi_\epsilon^0\rangle\epsilon\langle\varphi_\epsilon^0|$ is the nuclear Hamiltonian truncated to the set of proton scattering states. Using the requirement $\langle\chi_\epsilon^0|\varphi_{\rm sp}^0\rangle=0$, the definition of P given above, and the abbreviation $G_\epsilon=({\rm Pr}/\epsilon-H_N)$ where Pr stands for principal-value integral, we find for $|\chi_\epsilon^0\rangle$ the explicit form

$$|\chi_\epsilon^0\rangle=(1-\langle\varphi_{\rm sp}^0|G_\epsilon|\varphi_{\rm sp}^0\rangle^{-1}G_\epsilon Q)\,|\varphi_\epsilon^0\rangle\,. \tag{A14}$$

These functions obey Eq. (A13) and are normalized to a delta function. In the space of states $\{|\chi_\epsilon^0\rangle,|\varphi_{\rm sp}^0\rangle\}$ the nuclear Hamiltonian takes the form (A6), with the index N replaced everywhere by sp, and with $E_0=\langle\varphi_{\rm sp}^0|H_N|\varphi_{\rm sp}^0\rangle$ and $V_{\rm sp}^\epsilon=\langle\chi_\epsilon^0|H_N|\varphi_{\rm sp}^0\rangle$. The further treatment outlined in Appendix A 1 now applies. We shall therefore not deal separately with the case of a single-particle resonance in the sequel, but use the formulas of Appendix A 1 throughout as constituting the most general case.

3. Inclusion of higher partial waves. The total proton scattering wave function

Using the results of Appendixes A 1 and A 2, and the assumption that in the energy interval of interest, resonances in partial waves with angular momentum >0 do not play any role, we can easily write down the total proton scattering wave functions. Let $\{\chi_\epsilon^{lm}\}$ denote a set of antisymmetric wave functions describing the nucleus ^{12}C in its ground state, and a proton with angular momentum $\hbar l$, z component $\hbar m$, and asymptotic c.m. energy ϵ in a scattering state. The states $|\chi_\epsilon^{lm}\rangle$ contain the spherical harmonic $Y_l^m(\hat r_p)$ and real l-wave radial wave functions $F_\epsilon^l(r_p)$ of the proton which are regular at the origin and behave asymptotically like

$$F_\epsilon^l(r_p) \underset{r_p\to\infty}{\longrightarrow} \sin[kr-\tfrac{1}{2}l\pi+\delta_l(\epsilon)]\ . \qquad (A15)$$

The states $|\chi_\epsilon^{lm}\rangle$ are normalized to a delta function in energy. We define

$$\Psi_\epsilon^{lm(\pm)}=\exp[\pm i\delta_l(\epsilon)]\,|\chi_\epsilon^{lm}\rangle,\ \ l>0 \qquad (A16)$$

and write the total scattering wave function in the form[7]

$$\Psi_\epsilon^{(\pm)}(\vec{k};\vec{r}_1,\ldots,\vec{r}_A)=\sum_{l,m} i^l Y_l^{m*}(\hat{k})\,|\Psi_\epsilon^{lm(\pm)}\rangle\ . \qquad (A17)$$

They consist asymptotically ($r_p\to\infty$) of an antisymmetric product of ^{12}C in its ground state and a plane proton wave with momentum $\hbar\vec{k}$ plus outoing (+) or incoming (−) spherical waves. We emphasize again that we use a screened Coulomb potential the effect of which is included in the δ_l. The state $|\Psi_E^{0(\pm)}\rangle$ was defined in Appendix A 1. Since by definition

$$\langle\Psi_E^{lm(+)}|\Psi_{E'}^{l'm'(+)}\rangle=\delta_{ll'}\delta_{mm'}\delta(E-E')\ , \qquad (A18)$$

we also have

$$\langle\Psi_E^{(+)}(\vec{k})\,|\,\Psi_{E'}^{(+)}(\vec{k}\,')\rangle=\delta(E-E')\delta(\Omega_{\vec{k}}-\Omega_{\vec{k}'})\ . \qquad (A19)$$

The proton scattering amplitude is given by[8]

$$f_\epsilon^N(\theta_p)=k^{-1}\left[(2i)^{-1}(S_0-1)+\sum_{l>0}^{\infty}(2l+1)P_l(\cos\theta_p)\times\exp[i\delta_l(\epsilon)]\sin\delta_l(\epsilon)\right], \qquad (A20)$$

where θ_p is the scattering angle, and S_0 was defined in Eq. (A12).

4. Evaluation of the DWBA matrix element

To take account of the indistinguishability of the scattered proton and the protons in the target nucleus, we write the additional Coulomb interaction between electron and scattered proton in the form

$$\Delta V^c=\sum_{j=1}^{7}\frac{e^2}{|\vec{r}_e-\vec{r}_j|}-\left\langle {}^{12}C\left|\sum_{j=1}^{6}\frac{e^2}{|\vec{r}_e-\vec{r}_j|}\right|{}^{12}C\right\rangle, \qquad (A21)$$

where $|{}^{12}C\rangle$ is the wave function of ^{12}C in its ground state.

Let $\vec{k}(\vec{k}\,')$ be the momentum vector of the incident (outgoing) proton, $\vec{k}_e$ that of the ejected electron. In Born approximation, the transition amplitude is proportional to

$$T_\epsilon(\vec{k},\vec{k}\,';\vec{k}_e)=\langle\Psi_\epsilon^{(-)}(\vec{k}\,')\Xi_E^{(-)}(\vec{k}_e)\,|\,\Delta V^c\,|\,\Psi_\epsilon^{(+)}(\vec{k})\Xi_0\rangle\ . \qquad (A22)$$

In comparing our form for T with the semiclassical result given for instance in Ref. 6, we note that the semiclassical treatment also contains a recoil term proportional to $\vec{r}_e\cdot\overset{\leftrightarrow}{R}_c$ where $\vec{R}_c$ is the position coordinate of the ^{12}C nucleus. The quantum analog of this recoil term is given by (Ref. 6) $(6e^2/R^3)(m_e/m_c)\cdot(\vec{r}_e\cdot\vec{R})$, where R is the distance between the proton and the c.m. of the electron-^{12}C system, m_e the mass of the electron, m_c the mass of the ^{12}C nucleus. We add this term to ΔV^c and call the resulting operator ΔV. It is ΔV which must appear in the matrix element (A22). The recoil term is known to give sizable contributions to electron ejection by dipole emission. In our further derivation, we often simplify the writing by taking into account only the Coulomb terms. The recoil term should always be added, but this will be seen not to affect our general line of reasoning.

The expression (A22) still contains an integration over the 12 nucleons which constitute the target ^{12}C. To reduce T to an expression involving only the coordinates of the electron and the scattered proton, we observe that the scattering wave functions $|\chi_\epsilon^l\rangle$ introduced in Appendixes A 1–A 3 are antisymmetrized products of the wave function of ^{12}C in its ground state, and the proton scattering wave function

$$r_p^{-1}NF_\epsilon^l(r_p)Y_l^m(\hat{r}_p)\ , \qquad (A23)$$

where the normalization factor N must be chosen such as to satisfy Eq. (2.2) and has the value[7]

$$N=[(2m_p)/(\pi\hbar^2k)]^{1/2}\ . \qquad (A24)$$

Using standard techniques, it is easy to show that in all cases where the expression (A22) involves the states $|\chi_\epsilon^l\rangle$ both in the bra and in the ket, the integration over the target nucleons can be carried out with the result that $|\chi_\epsilon^l\rangle$ is replaced by the expression (A23), and ΔV is replaced by $e^2/|\vec{r}_p-\vec{r}_e|$. The only term in $\Psi_\epsilon^{(\pm)}$ which requires special attention is the term containing the resonance wave function φ_N^0, see Eq. (A9). Using a fractional parentage expansion, we write φ_N^0 in the form

$$|\varphi_N^0\rangle=\alpha\,|{}^{12}\mathrm{C}\rangle\,|r_p^{-1}F_0(r_p)\rangle+\cdots\,, \qquad \text{(A25)}$$

where $1/r_pF_0(r_p)$ is a bound-state wave function for a proton with angular momentum zero, normalized to unity, α a fractional parentage coefficient with $|\alpha|\leq 1$, and where the dots indicate terms involving excited states of ^{12}C. In cases where the expression (A22) involves the state $|\varphi_N^0\rangle$ in the bra and the $|\chi_\epsilon^l\rangle$ in the ket or vice versa, we use the expansion (A25) to reduce the matrix element to a form involving only the function $(1/r_p)F_0(r_p)$, the expression (A23) with ΔV replaced by $e^2/|\vec{r}_p-\vec{r}_e|$, and with a multiplicative factor given by $\alpha\sqrt{7^{-1}}$. (The factor $\sqrt{7}$ arises from the antisymmetrization.) Finally, in the case where the expression (A23) involves the state $|\varphi_N^0\rangle$ on both sides of ΔV, we use the assumption that the Coulomb field generated by all but the last of the protons is the same as the Coulomb field of ^{12}C. This leaves us with a matrix element involving the interaction $e^2/|\vec{r}_p-\vec{r}_e|$, and the proton density matrix $\rho(r_p)$ defined by

$$\rho(r_p)=(\varphi_N^0\,|\,\varphi_N^0)\,, \qquad \text{(A26)}$$

where the parentheses denote an integration over all coordinates but that of the last proton.

We shall be led to the conclusion that the contributions to T coming from $F_0(r_p)$ and from $\rho(r_p)$ are negligible. We shall derive this result under the simplifying assumptions that $\alpha\sqrt{7^{-1}}=1$, and that $r_p^2\rho(r_p)=[F_0(r_p)]^2$. The reader is invited to check that our conclusion does not depend on this simplification. Using it, we can, however, write the T-matrix element in the simple form given in Sec. II.

APPENDIX B: GENERALIZATION TO PROTON INELASTIC NUCLEAR SCATTERING

Having presented a very explicit and detailed discussion of the case of elastic nuclear scattering in Appendix A, we feel justified in only sketching the derivation that generalizes Eq. (1.1) to inelastic nuclear scattering, and in giving the final formula that replaces Eq. (1.1).

We consider the case of Δ open channels and a bound state embedded in the continuum which, upon coupling to the channels, turns into a nuclear resonance. We use the formalism of Ref. 9. Our derivation can be extended to include both a number >1 of nuclear resonances, and direct transitions between the channels (which here we omit). We have decided not to include these cases only because this would have made the presentation unwieldy.

We consider a nuclear system (proton plus target nucleus) of fixed spin. Let χ_ϵ^c, $c=1,\ldots,\Delta$, denote antisymmetric products of the proton scattering wave function in channel c and the wave function of the residual nucleus, and Φ the bound state embedded in the continuum. According to Ref. 9, the total scattering wave function $\Psi_\epsilon^{c(+)}$, consisting of an incoming wave in channel c and outgoing waves in all other channels, is given by

$$\Psi_\epsilon^{c(+)}=\sum_{c'=1}^{\Delta}\int_{\epsilon_{c'}}^{\infty}d\epsilon'a_\epsilon^c(\epsilon';c')\chi_{\epsilon'}^{c'}+\mathscr{C}_\epsilon^c\Phi\,, \qquad \text{(B1)}$$

where

$$a_\epsilon^c(\epsilon';c')=\delta_{cc'}\delta(\epsilon-\epsilon')\exp(i\delta_c)+(\epsilon^\dagger-\epsilon')^{-1}V^{c'}(\epsilon')(\epsilon-\mathscr{E}_N)^{-1}\times V^c(\epsilon)\exp(i\delta_c)\,, \qquad \text{(B2)}$$

$$\mathscr{C}_\epsilon^c=\exp(i\delta_c)(\epsilon-\mathscr{E}_N)^{-1}V^c(\epsilon)\,, \qquad \text{(B3)}$$

where $\delta_c(\epsilon)$ is the nonresonant nuclear phase shift in channel c, ϵ the total energy, ϵ_c the threshold energy in channel c, $V^c(\epsilon)=\langle\chi_\epsilon^c\,|\,H\,|\,\Phi\rangle$ the coupling between channels, and the quasibound state, and $\mathscr{E}_N$ the energy of the nuclear resonance, with $\mathrm{Im}\mathscr{E}_N=\frac{1}{2}\Gamma_N=\pi\sum_c(V^c)^2$. The reader will observe that Eqs. (B1)–(B3) are the straighforward generalizations of formulas given in Appendix A to the inelastic case. The final state in the DWBA formula (A22) is described by a wave function $\Psi_\epsilon^{c'(-)}$ with an outgoing wave in the inelastic channel c' and incoming waves in all other channels. The explicit construction of $\Psi_\epsilon^{c'(-)}$ leads to formulas very similar to Eqs. (B1)–(B3). The spin of $\Psi_\epsilon^{c'(-)}$ may differ from the spin of $\Psi_\epsilon^{c(+)}$ because angular momentum is transferred by the DWBA matrix element (A22). Eventually, one has to perform a sum over initial (and final) spins to build up the plane-wave solution; this step is omitted here.

Inserting $\Psi_\epsilon^{c(+)}$ and $\Psi_\epsilon^{c'(-)}$ into the DWBA matrix element (A22), and using the many-body operator (A21) in the matrix element, we find various types of contributions. Some of these involve the state Φ and lead to internal-conversion type of matrix elements; these can again be shown to be negligible. The remaining matrix elements contain the functions χ_ϵ^c and $\chi_\epsilon^{c'}$ in both ket and bra. Since the operator (A21) is a sum of one-body operators for the nucleons, it follows by straightforward arguments that the indices c and c' on the states χ_ϵ^c in bra and ket must coincide to yield a nonvanishing contribution. Intuitively this means that the proton cannot suffer inelastic nuclear scattering through its interaction with the electron.

Omitting Φ, inserting the forms (B1) and (B2) and the corresponding forms for $\Psi_\epsilon^{c'(-)}$ into the DWBA matrix element, choosing a case with $c \neq c'$, and observing that the indices on χ_ϵ^c in bra and ket must coincide, we can reduce the DWBA expression to a one-body matrix element in the nuclear degrees of freedom which is the sum of three terms. Writing Eq. (B2) symbolically in the form $a_\epsilon^c = a_\epsilon^c$ (background) $+ a_\epsilon^c$ (resonance), we find that these three terms contain, respectively, a_ϵ^c (background)$\cdot a_\epsilon^{c'}$ (resonance), a_ϵ^c (resonance)$\cdot a_\epsilon^{c'}$ (background), and a_ϵ^c (resonance)$\cdot a_\epsilon^{c'}$ (resonance). The last of these three contributions is again negligible. This is because it contains for most of the integration range in the DWBA matrix element a product of two outgoing spherical waves $e^{i(k+k')r_p}$. The remaining two terms are similar in form to the ones kept in Appendix A. In these terms, we can again replace the one-body Green's function for the proton appearing in a_ϵ^c (resonance) by its asymptotic form, if the condition (1.3) is met. Subsequently, we use the asymptotic form of the proton radial wave function in a_ϵ^c (background). In this manner, we find finally

$$\begin{aligned} T^{cc'}_{F_i m_i; F_f m_f;\epsilon}(\vec{k},\vec{k}\,';\vec{k}_e) &= f^{cc'}_{F_i m_i; F_f m_f;\epsilon}(\hat{k},\hat{k}')T^e_{\epsilon'}(\vec{k}\,',\vec{k}_e) \\ &+ f^{cc'}_{F_i m_i; F_f m_f;\epsilon'}(\hat{k},\hat{k}')\tilde{T}^e_{\epsilon}(\vec{k},\vec{k}_e)\,. \end{aligned} \tag{B4}$$

For simplicity, we have used a channel-spin representation where F_i, F_f and m_i, m_f denote the initial and final channel spins and their z projections, respectively. The symbol $f^{cc'}_{F_i m_i; F_f m_f;\epsilon}(\vec{k},\vec{k}\,')$ denotes the inelastic nuclear scattering amplitude for scattering $c \rightarrow c'$ with k vectors and m values as indicated. The quantities T and $\tilde{T}$ are defined in Eqs. (4.14). On the right-hand side of the first (second) of Eqs. (4.14), the symbol $k\,(k')$ must be replaced by the magnitude of the proton momentum after (before) nuclear scattering and before (after) electron scattering, respectively.

APPENDIX C: THE ASYMPTOTIC FORMULA (4.12)

We consider integrals of the form

$$\int d^3 r_p e^{i\vec{k}\cdot\vec{r}_p} f(\vec{r}_p)\,,$$

where $f(\vec{r})$ is some function which is orthogonal to all spherical harmonics $Y_l^m(\hat{r})$ with $l > l_0 = l_{\max} + L_{\max}$. We consider $e^{i\vec{k}\vec{r}_p}$ as an integral operator, acting upon the space of such functions. Expanding the plane wave in spherical harmonics, we are clearly allowed to restrict the summation to $l \leq l_0$. Using for the spherical Bessel functions the asymptotic expansion,[7] we find

$$e^{i\vec{k}\cdot\vec{r}_p} \underset{r_p\to\infty,\ l\leq l_0}{\longrightarrow} [(2\pi)/(ikr_p)]\left[\exp(ikr_p)\sum_{l,m=0}^{l_0} Y_l^{*m}(\hat{k})Y_l^m(\hat{r}_p) - \exp(-ikr_p)\sum_{l,m=0}^{l_0}(-)^l Y_l^{m*}(\hat{k})Y_l^m(\hat{r}_p)\right]. \tag{C1}$$

As an integral operator in the space of functions defined above, the first double sum equals $\delta(\Omega_{\vec{k}} - \Omega_{\vec{r}_p})$. Using a parity argument, we find similarly that the second double sum equals $\delta(\Omega_{\vec{k}} + \Omega_{\vec{r}_p})$. This yields the formula (4.12).

*On leave from Department of Physics, University of Wisconsin, Madison, Wisconsin.

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[4]More correctly, Eq. (1.3) should read $R_K > > lb + \eta^2/k$, where η is the proton-target Sommerfield parameter, which is about 1.5 in the experiment of Ref. 1, and therefore small. It is amusing to note that Eq. (1.3), which can also be written in the form $R_K >> kb^2$, can be given a diffraction-pattern interpretation. If the nuclear scattering is thought of as defining an "l window" of size $l = kb$, the portion of the beam passing through this window (i.e., the portion which is scattered by the nucleus) will spread out in a diffraction pattern. Equation (1.3) is just the condition which causes the subsequent electron scattering to occur far enough from the nucleus that it is in the Fraunhofer rather than in the Fresnel region of this pattern.

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