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Casimir forces without the vacuum radiation field

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Although the electromagnetic Casimir forces are customarily explained in terms of the vacuum radiation field, they may also be understood in terms of radiation reaction, without explicit reference to the vacuum field. In particular, the Casimir forces may be obtained if one recognizes that the radiation reaction field of a dipole on itself depends on where the dipole is located, for this field is determined by the modal characteristics of the surrounding space. Only in infinite free space does the field of radiation reaction have a spatially invariant meaning. The fact that the Casimir forces may be understood in terms of either the vacuum electromagnetic field or the field of radiation reaction is an example of the general fluctuation-dissipation relation between the two concepts. These ideas are illustrated with derivations of several "vacuum-field effects," including the Casimir-Polder attraction of an atom to a conducting wall and the van der Waals force between two atoms.

I. INTRODUCTION

The quantum theory of radiation predicts a zero-point energy $\frac{1}{2}\hbar\omega$ per normal mode of the field. This is a consequence of the fact that each mode is canonically equivalent to a harmonic oscillator. Summation over all modes leads to an infinite zero-point field energy, a result occasionally considered embarrassing.[1] Consideration of the zero-point electromagnetic energy may be sidestepped, general relativity notwithstanding, on the grounds that a constant additive term in the Hamiltonian has no effect on commutators and equations of motion. At the same time, it is generally believed that there are indeed real physical effects attributable to the zero-point field. Such effects include the Casimir-Polder attraction of a neutral polarizable particle to a conducting wall, the Casimir attraction between two parallel conducting plates, and the van der Waals interaction between polarizable particles.[2]

In 1948 Welton[3] showed that the (nonrelativistic) radiative level shift obtained by Bethe[4] could be attributed to the effect on the atomic electron of the zero-point field. Welton's elegant derivation seems to have influenced many physicists to believe that the ubiquitous phenomenon of spontaneous emission is also a consequence of the quantum-mechanical zero-point field.[5] An atom in a pure stationary state has no dipole moment, i.e., the expectation value of the dipole-moment operator vanishes, so that the process of spontaneous emission cannot be simply understood on the basis of classical radiation theory. But a fluctuating zero-point field, present even when there are no other sources of radiation but the excited atom, can perturb the electron motion and induce a transition to a state of lower energy. The coupling of the electron to the zero-point field thus provides an intuitive explanation of spontaneous emission. But this interpretation is hardly satisfying. In particular, it offers no good explanation of the fact that atoms do not absorb energy from the zero-point field.

In classical electrodynamics the radiation by an accelerating charge is associated with the force of radiation reaction. It seems that the early workers in the quantum theory considered spontaneous emission to be a result of this "radiation force."[6] Of course, radiation reaction also appears in quantum electrodynamics. As in classical theory, the radiation reaction field is just the solution of the inhomogeneous Maxwell equation for the electric field, evaluated at the source. But in quantum theory there is also the zero-point field, i.e., the solution of the homogeneous Maxwell equation, acting on the source. What does the quantum theory of radiation have to say about spontaneous emission, and in particular about the radiative corrections to the energy levels?

Ackerhalt *et al.*[7] have given a Heisenberg-picture treatment of spontaneous emission in which the usual procedure of normal ordering is adopted. They show that spontaneous emission, and the associated level width and (nonrelativistic) shift, can be understood solely on the basis of radiation reaction. As a result of normal ordering, there are no explicit contributions from the zero-point field. In particular, it was interesting that they could obtain essentially the same result as Welton for the level shift, even though Welton dealt directly with the zero-point field. The two approaches were tied together by Senitzky[8] and Milonni *et al.*,[9] who performed Heisenberg-picture calculations in which operator orderings other than normal were used. If annihilation and creation operators were symmetrically ordered, for in-

stance, the calculation showed that the level shift and width could be attributed entirely to the zero-point field, just as Welton had shown. The final expressions for the radiative corrections are independent of the ordering chosen, since atom and field operators commute. Only the interpretation one is most likely to draw from the calculation depends on the way the field operators are ordered.

Jaynes[10] has suggested that this apparent interchangeability of the effects of radiation reaction and the zero-point field in spontaneous emission is but one example of a general fluctuation-dissipation theorem.[11] He suggests that the effects of radiation reaction are the same *as if* a zero-point field were acting on the atom, and that effects like the Casimir-Polder force might also be attributed to radiation reaction. This paper provides support for that suggestion.

Physical consequences of the zero-point field arise from a *change* in the modal structure of the field over its vacuum form. Thus the Lamb shift[12] may be viewed as the change in zero-point field energy resulting from the presence of the atom.[13,14] In a similar fashion the force between two parallel conducting plates may be obtained by calculating the change in zero-point field energy due to the presence of the plates.[2,15] In this paper such effects are approached using the notion that *the radiation reaction field likewise depends upon the modes of the field.*

It does not seem to be generally appreciated that the radiation reaction field must vary with the environment of the emitter. A trivial example, however, can be given to show that this must be the case: The rate of radiation from two identical dipoles oscillating in phase, and separated by a distance small compared with the wavelength, is twice the sum of the single-dipole rates. The radiation reaction field on each dipole is correspondingly twice as large as that it experiences in free space. Each dipole is acted upon by its own free-space radiation reaction field plus the radiation reaction field of the other dipole.

In Sec. II the field is quantized in terms of general mode functions. Within this "macroscopic" approach a modification of the Jordan-Pauli commutation relation is necessary. We obtain an expression for the radiation reaction fields of a free electron and a bound electron, and use the results to obtain corresponding expressions for the energy shifts resulting from the radiation reaction. The results are then used in Sec. III to obtain the Casimir-Polder interaction between a ground-state atom and a perfectly conducting wall. We also consider briefly the radiative level shifts in "murium."[16] In Sec. IV we consider the van der Waals interaction between two ground-state atoms.

II. QUANTIZATION OF THE FIELD AND RADIATION REACTION

In the absence of any sources it is convenient to use the gauge in which the scalar potential vanishes and the vector potential is transverse. The Maxwell equations are satisfied if the vector potential obeys the wave equation $\Box\vec{A}(\vec{x},t)=0$. We assume the existence of a complete set of transverse mode functions. In the Fourier series

$$\vec{A}(\vec{x},t)=\sum_\alpha[A_\alpha\vec{F}_\alpha(\vec{x})e^{-i\omega_\alpha t}+A_\alpha^*\vec{F}_\alpha^*(\vec{x})e^{i\omega_\alpha t}]\,, \quad (2.1)$$

where the A_α are constants, the mode functions $\vec{F}_\alpha(\vec{x})$ must satisfy the Helmholtz equation

$$\nabla^2\vec{F}_\alpha(\vec{x})+k_\alpha^2\vec{F}(\vec{x})=0,\quad k_\alpha^2=\omega_\alpha^2/c^2\,, \quad (2.2)$$

in addition to the condition of transversality,

$$\vec{\nabla}\cdot\vec{F}_\alpha(\vec{x})=0\,. \quad (2.3)$$

The (transverse) electric and magnetic field vectors are given by

$$\vec{E}(\vec{x},t)=-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

$$=i\sum_\alpha k_\alpha[A_\alpha(t)\vec{F}_\alpha(\vec{x})-A_\alpha^*(t)\vec{F}_\alpha^*(\vec{x})]\,, \quad (2.4)$$

and

$$\vec{B}(\vec{x},t)=\vec{\nabla}\times\vec{A}$$

$$=\sum_\alpha[A_\alpha(t)\vec{\nabla}\times\vec{F}_\alpha(\vec{x})+A_\alpha^*(t)\vec{\nabla}\times\vec{F}_\alpha^*(\vec{x})]\,, \quad (2.5)$$

where $A_\alpha(t)=A_\alpha\exp(-i\omega_\alpha t)$. In addition to Eqs. (2.2) and (2.3), the mode functions satisfy the appropriate boundary conditions on $\vec{E}$ and $\vec{B}$. They are assumed to form an orthonormal set:

$$\int d^3x\,\vec{F}_\alpha^*(\vec{x})\cdot\vec{F}_\beta(\vec{x})=\delta_{\alpha\beta}\,. \quad (2.6)$$

When the field is quantized $A_\alpha(t)$ becomes simply $(2\pi\hbar c^2/\omega_\alpha)^{1/2}a_\alpha(t)$, where $a_\alpha(t)$ is the annihilation operator for mode α, i.e., $[a_\alpha,a_\beta^\dagger]=\delta_{\alpha\beta}$. Then

$$\vec{A}(\vec{x},t)=\sum_\alpha\left(\frac{2\pi\hbar c^2}{\omega_\alpha}\right)^{1/2}[a_\alpha(t)\vec{F}_\alpha(\vec{x})+a_\alpha^\dagger(t)\vec{F}_\alpha^*(\vec{x})]\,, \quad (2.7)$$

and $\vec{E}$ and $\vec{B}$ follow from (2.4) and (2.5). It is important to note that the particular form taken by the mode functions has no bearing whatsoever on the standard quantization procedure. All of the quantum mechanics of the field are contained, as it were, in their time dependence. The spatial dependence of the field arises from the mode functions $\vec{F}_\alpha(\vec{x})$, which are computed using *classical* electromagnetic theory.

A. The Hamiltonian

The Hamiltonian for the system comprising the electromagnetic field and a spinless, nonrelativistic electron bound by some potential function V is

$$H=\frac{1}{2m}\left(\vec{\mathrm{p}}-\frac{e}{c}\vec{\mathrm{A}}\right)^2+V(\vec{\mathrm{r}})+\frac{1}{8\pi}\int d^3x\left[\vec{\mathrm{E}}^2(\vec{\mathrm{x}},t)+\vec{\mathrm{B}}^2(\vec{\mathrm{x}},t)\right], \tag{2.8}$$

where m and e ($=-|e|$) are the electron mass and charge. The energy associated with the longitudinal part of the electric field arises from instantaneous Coulomb interactions not explicitly included in Eq. (2.8).[17] In terms of the mode functions $\vec{\mathrm{F}}_\alpha(\vec{\mathrm{x}})$ and the photon annihilation and creation operators, the Hamiltonian (2.8) takes the form

$$H=\frac{p^2}{2m}+V(\vec{\mathrm{r}})+\sum_\alpha\hbar\omega_\alpha a_\alpha^\dagger a_\alpha-\frac{e}{mc}\sum_\alpha\left(\frac{2\pi\hbar c^2}{\omega_\alpha}\right)^{1/2}\left[a_\alpha\vec{\mathrm{F}}_\alpha(\vec{\mathrm{r}})\cdot\vec{\mathrm{p}}+a_\alpha^\dagger\vec{\mathrm{F}}_\alpha^*(\vec{\mathrm{r}})\cdot\vec{\mathrm{p}}\right]$$
$$+\frac{e^2}{2mc^2}\sum_\alpha\sum_\beta(2\pi\hbar c)(\omega_\alpha\omega_\beta)^{1/2}\left[a_\alpha a_\beta\vec{\mathrm{F}}_\alpha(\vec{\mathrm{r}})\cdot\vec{\mathrm{F}}_\beta(\vec{\mathrm{r}})+a_\alpha^\dagger a_\beta^\dagger\vec{\mathrm{F}}_\alpha^*(\vec{\mathrm{r}})\cdot\vec{\mathrm{F}}_\beta^*(\vec{\mathrm{r}})+a_\alpha^\dagger a_\beta\vec{\mathrm{F}}_\alpha^*(\vec{\mathrm{r}})\cdot\vec{\mathrm{F}}_\beta(\vec{\mathrm{r}})\right.$$
$$\left.+a_\alpha a_\beta^\dagger\vec{\mathrm{F}}_\alpha(\vec{\mathrm{r}})\cdot\vec{\mathrm{F}}_\beta^*(\vec{\mathrm{r}})\right], \tag{2.9}$$

where the additive constant corresponding to the zero-point field energy has been dropped.

B. Heisenberg equations of motion

From the Hamiltonian it follows easily that the electron coordinate operator obeys the Lorentz-Heaviside force law

$$\frac{md^2\vec{\mathrm{r}}}{dt^2}=-\vec{\nabla}V(\vec{\mathrm{r}})+e\vec{\mathrm{E}}(\vec{\mathrm{r}})+\frac{e}{c}\frac{d\vec{\mathrm{r}}}{dt}\times\vec{\mathrm{B}}(\vec{\mathrm{r}}). \tag{2.10}$$

From the Heisenberg equation of motion for a_α,

$$\frac{da_\alpha}{dt}=-i\omega_\alpha a_\alpha+\frac{ie}{\hbar c}\left(\frac{2\pi\hbar c^2}{\omega_\alpha}\right)^{1/2}\vec{\mathrm{F}}_\alpha^*(\vec{\mathrm{r}})\cdot\frac{d\vec{\mathrm{r}}}{dt}, \tag{2.11}$$

together with Eqs. (2.2) and (2.7), it follows that

$$\nabla^2A_i(\vec{\mathrm{x}},t)-\frac{1}{c^2}A_i(\vec{\mathrm{x}},t)=-\frac{4\pi e}{c}\delta_{ij}^\perp(\vec{\mathrm{x}}-\vec{\mathrm{r}})\frac{dr_j}{dt}, \tag{2.12}$$

where $\delta^\perp$ is the transverse delta function. In deriving this result we have used the conditions

$$\sum_\alpha F_{\alpha i}^*(\vec{\mathrm{x}})F_{\alpha j}(\vec{\mathrm{x}}')=\delta_{ij}^\perp(\vec{\mathrm{x}}-\vec{\mathrm{x}}') \tag{2.13}$$

and

$$\mathrm{Im}\sum_\alpha\omega_\alpha^{-1}F_{\alpha i}^*(\vec{\mathrm{x}})F_{\alpha j}(\vec{\mathrm{x}}')=0. \tag{2.14}$$

The first is just the assumption that the mode functions from a complete set for vector functions of zero divergence. The second condition may be proved using the Helmholtz equation and (2.13). Equations (2.10) and (2.12) are well known, of course, and they are obviously valid regardless of what are the mode functions $\vec{\mathrm{F}}_\alpha(\vec{\mathrm{x}})$ that are most conveniently employed in a given situation.

It is interesting to consider the field commutation relations. It is found, for instance, that

$$[A_i(\vec{\mathrm{x}},t),A_j(\vec{\mathrm{x}}',t')]=4\pi i\hbar c^2\,\mathrm{Im}\sum_\alpha\omega_\alpha^{-1}F_{\alpha i}(\vec{\mathrm{x}})F_{\alpha j}^*(\vec{\mathrm{x}}')\times e^{-i\omega_\alpha(t-t')}. \tag{2.15}$$

In the case of "free space" we follow the familiar approach of taking the mode functions to be plane waves satisfying periodic boundary conditions on a large cube:

$$\vec{\mathrm{F}}_\alpha(\vec{\mathrm{x}})\to\vec{\mathrm{F}}_{\vec{\mathrm{k}}\lambda}(\vec{\mathrm{x}})=L^{-3/2}\hat{\epsilon}_{\vec{\mathrm{k}}\lambda}e^{i\vec{\mathrm{k}}\cdot\vec{\mathrm{x}}}, \tag{2.16}$$

where L is the side of the cube and the unit vector $\hat{\epsilon}_{\vec{\mathrm{k}}\lambda}$ ($\lambda=1,2$) specifies the polarization of the wave, $\vec{\mathrm{k}}\cdot\hat{\epsilon}_{\vec{\mathrm{k}}\lambda}=0$. The wave vectors $\vec{\mathrm{k}}=(2\pi/L)(n_x,n_y,n_z)$, where $n_i=0,\pm1,\pm2,\ldots$. Then (2.15) becomes

$$[A_i(\vec{\mathrm{x}},t)A_j(\vec{\mathrm{x}}',t')]=\frac{4\pi i\hbar c^2}{L^3}\sum_{\vec{\mathrm{k}},\lambda}\omega_k^{-1}\epsilon_{\vec{\mathrm{k}}\lambda i}\epsilon_{\vec{\mathrm{k}}\lambda j}\sin(\vec{\mathrm{k}}\cdot\vec{\mathrm{r}}-\omega_k\tau), \tag{2.17}$$

where $\vec{\mathrm{r}}=\vec{\mathrm{x}}-\vec{\mathrm{x}}'$, $\tau=t-t'$. In the mode continuum limit $L\to\infty$,

$$[A_i(\vec{\mathrm{x}},t),A_j(\vec{\mathrm{x}}',t')]=-\frac{i\hbar c}{2\pi^2}\int\frac{d^3k}{k}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\times e^{i\vec{\mathrm{k}}\cdot\vec{\mathrm{r}}}\sin(kc\tau). \tag{2.18}$$

Well-known relations such as

$$[A_i(\vec{\mathrm{x}},t),E_j(\vec{\mathrm{x}}',t)]=-4\pi i\hbar c\,\delta_{ij}^\perp(\vec{\mathrm{x}}-\vec{\mathrm{x}}'), \tag{2.19}$$

and the Jordan-Pauli commutator[18]

$$[E_i(\vec{\mathrm{x}},t),E_j(\vec{\mathrm{x}}',t')]=-4\pi i\hbar c\left(\frac{\delta_{ij}}{c^2}\frac{\partial^2}{\partial t\,\partial t'}-\frac{\partial^2}{\partial x_i\partial x_j'}\right)\times D(|\vec{\mathrm{r}}|,\tau), \tag{2.20}$$

where

$$D(r,t)=\frac{1}{4\pi r}[\delta(r-ct)-\delta(r+ct)], \tag{2.21}$$

follow from (2.18)

However, *these familiar commutation relations hold only for plane-wave mode functions*. It may be seen directly from Eq. (2.15) that field commutation relations depend on the form of the mode functions. Every set of mode functions satisfies the Helmholtz equation, but the solutions to the Helmholtz equation that one employs for a given physical situation are determined by the electromagnetic boundary conditions. In other words, the field commutation relations are determined not only by the commutation rules for a_α and $a_\alpha^\dagger$, but also by boundary conditions. Suppose, for instance, that we quantize the field in the half-space bounded by a perfectly conducting plate in the plane $z=0$. In this case the x and y components of the electric field must vanish at $z=0$, so we choose our mode functions accordingly. For example, the x components of the $\vec{F}_\alpha(\vec{x})$ may be written as

$$F_{\alpha x}(\vec{x}) = \left(\frac{2}{V}\right)^{1/2} \epsilon_{\vec{k}\lambda x} e^{i\vec{k}_\parallel \cdot \vec{x}} \sin(k_z z)\,, \tag{2.22}$$

where V is the quantization volume and $\vec{k}_\parallel$ denotes the component of $\vec{k}$ that is parallel to the plate. Equation (2.15) now takes the following form for the x component of the vector potential.

$$[A_x(\vec{x},t), A_x(\vec{x}',t')] = \frac{8\pi i\hbar c^2}{V} \operatorname{Im} \sum_{\vec{k}} \omega_k^{-1} \left(1 - \frac{k_x^2}{k^2}\right) e^{i\vec{k}_\parallel \cdot (\vec{x}-\vec{x}')} \sin(k_z z) \sin(k_z z') e^{-i\omega_k \tau}\,, \tag{2.23}$$

where we have used the equation $\sum_{\lambda=1,2} \epsilon^2_{\vec{k}\lambda x} = 1 - k_x^2/k^2$. From (2.23), with the quantization volume $V \to \infty$, we obtain

$$[A_x(\vec{x},t), E_x(\vec{x}',t)] = -\,4\pi i\hbar c[\delta_{xx}^{\perp}(\vec{r}_1) - \delta_{xx}^{\perp}(\vec{r}_2)]\,, \tag{2.24}$$

where $\vec{r}_1 = \vec{x} - \vec{x}' = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$ and $\vec{r}_2 = (x-x')\hat{x} + (y-y')\hat{y} + (z+z')\hat{z}$. Similarly,

$$[E_x(\vec{x},t), E_x(\vec{x}',t')] = -\,4\pi i\hbar c\left(\frac{1}{c^2}\frac{\partial^2}{\partial t \partial t'} - \frac{\partial^2}{\partial x \partial x'}\right) \times [D(|\vec{r}_1|,\tau) - D(|\vec{r}_2|,\tau)] \tag{2.25}$$

in place of the Jordan-Pauli relation (2.20). Equations (2.24) and (2.25) reduce to their free-space counterparts in the limiting case in which the two points $\vec{x}$ and $\vec{x}'$ are infinitely far from the conducting plate.

It may appear surprising that such ubiquitous commutation relations as (2.19) and (2.20) require any modification at all. Indeed one can adduce arguments, from a somewhat deeper level of analysis, that these commutation relations are always valid. A brief digression therefore seems worthwhile.

It is well known from the work of Bohr and Rosenfeld[19] that the commutation relation (2.20) implies the simultaneous measureability of the electric field at two space-time points, provided that these points cannot be connected by a light signal [i.e., provided that $D(r,\tau) = 0$]. With this in mind the modification of this relation as in Eq. (2.25), for instance, seems quite reasonable, for whereas $|\vec{r}_1|/c$ is the time taken by a light ray to go directly from $\vec{x}$ to $\vec{x}'$, the time $|\vec{r}_2|/c$ is that taken to go from $\vec{x}$ to $\vec{x}'$ with the intermediate reflection of the light from the conductor at $z=0$. In other words, $\vec{x}$ and $\vec{x}'$ can be connected by a light signal not only directly but also indirectly via the "mirror." Thus there are basically two ways in which two space-time points in the half-space may be connected by a light signal, and Eq. (2.25) accounts for both possibilities. A measurement of the field at $(\vec{x},t)$ can influence a measurement at $(\vec{x}',t')$ not only if $|\vec{r}_1| = c(t'-t)$, but also if $|\vec{r}_2| = c(t'-t)$.

On the other hand, the use of mode functions like (2.22) implies a macroscopic, and indeed classically deterministic, treatment of the conductor. Ideally one would begin with a multiparticle generalization of the Hamiltonian (2.8), with the field quantized in terms of "free-space" modes for which relations such as (2.19) and (2.20) hold. Provided the particles constituting the conductor act collectively in an essentially classical manner, their net effect should be equivalent to the quantization of the field in terms of the classical modes of the right half-space. In fact this is basically the only justification for the *expedient* of quantizing the field in terms of mode functions satisfying macroscopic boundary conditions.

C. Radiation reaction field. Nonrelativistic free electron

From Eq. (2.11) we obtain

$$a_\alpha(t) = a_\alpha(0)e^{-i\omega_\alpha t} + \frac{ie}{\hbar c}\left(\frac{2\pi\hbar c^2}{\omega_\alpha}\right)^{1/2} \times \int_0^t dt_1 \vec{F}_\alpha^*[\vec{r}(t_1)] \cdot \frac{d\vec{r}(t_1)}{dt} e^{i\omega_\alpha(t_1 - t)}\,. \tag{2.26}$$

The first term on the right-hand side is the source-free or "vacuum" term, whereas the second is due to the electron. When the second term is used in Eq. (2.7) it gives the vector potential field of the point electron. In the nonrelativistic approximation the time dependence in the mode

function may be ignored in Eq. (2.26). This corresponds to the neglect of recoil or, what is essentially the same thing, the neglect of Doppler shifts. In this approximation we obtain from Eq. (2.7) the following source vector potential at the position of the electron:

$$\vec{A}_{RR}(\vec{r},t)=2\pi iec\sum_{\alpha}\omega_{\alpha}^{-1}\vec{F}_{\alpha}(\vec{r})\cdot\vec{F}_{\alpha}^{*}(\vec{r}) \times\int_{0}^{t}dt_{1}\frac{d\vec{r}(t_{1})}{dt}e^{i\omega_{\alpha}(t_{1}-t)}+\text{H.c.} \quad (2.27)$$

In the case of free space the mode functions are given by (2.16) and we obtain after some algebra the result

$$\vec{A}_{RR}(\vec{r},t)=-\frac{4e}{3\pi c^{2}}\int_{0}^{t}dt_{1}\frac{d^{2}\vec{r}(t)}{dt^{2}}\frac{\partial}{\partial t_{1}}\left(\frac{\sin\Omega(t_{1}-t}{t_{1}-t}\right) =\frac{4e\Omega}{3\pi c^{2}}\frac{d\vec{r}(t)}{dt}-\frac{2e}{3c^{2}}\frac{d\vec{r}(t)}{dt^{2}}, \quad (2.28)$$

where Ω is a high-frequency cutoff. Thus we obtain a familiar expression for the electric field of radiation reaction for a nonrelativistic, point electron

$$\vec{E}_{RR}(t)=\frac{2e}{3c^{3}}\frac{d^{3}\vec{r}(t)}{dt^{3}}-\frac{\delta m}{e}\frac{d^{2}\vec{r}(t)}{dt^{2}}, \quad (2.29)$$

where $\delta m=4e^{2}\Omega/3\pi c^{3}$ is the electromagnetic mass, which is linearly divergent in the nonrelativistic theory without cutoff.

The second-order, nonrelativistic result for the electron self-energy due to the $\vec{A}_{\text{vac}}\cdot\vec{p}$ interaction, where $\vec{A}_{\text{vac}}$ is the vector potential associated with the vacuum field, is

$$\Delta E=-\left(\frac{2e^{2}\Omega}{3m^{2}c^{3}}\right)p^{2}. \quad (2.30)$$

This result may also be regarded as the change in the energy of the electron arising from the difference between its bare mass m_0 and its renormalized mass, $m=m_0+\delta m$:

$$\frac{p^{2}}{2m}-\frac{p^{2}}{2m_{0}}\cong-\frac{\delta m}{m}\left(\frac{p^{2}}{2m}\right)=\Delta E. \quad (2.31)$$

Whereas expression (2.30) is derived by considering the interaction of the electron with the vacuum field, the identical expression (2.31) is obtained from the point of view of radiation reaction. The equivalence of the two interpretations of ΔE in this simple example serves to prelude the main point of this paper.

Equation (2.27) indicates that the reaction field of the electron, and therefore its electromagnetic mass, depends upon the field mode structure at the position of the electron. The electromagnetic mass of the electron should therefore depend upon its location, a possibility which obviously does not exist in free space. The modification of electromagnetic mass due to the presence of conducting plates has in fact been considered both theoretically[20] and experimentally,[21] although the effect has not been directly demonstrated. The important point for our purposes here is that the proposed modification of δm may be obtained either by using the radiation reaction field (2.27) *or* by calculating the two-photon-vertex contribution of the vacuum field to ΔE. In other words, the equivalence of the two interpretations of ΔE for the case of free space holds in the general case as well.

D. Radiation reaction field. Nonrelativistic bound electron

We now consider an electron bound to a nucleus. In this case we shall be interested in radiative level shifts ΔE_j of the eigenstates $|j\rangle$ of the Hamiltonian $p^2/2m+V(r)$. We regard such shifts as arising from the radiation reaction field of the electron. Although this interpretation has a more rigorous justification in a detailed analysis,[7,22] a simpler, intuitive approach seems preferable here.

In classical electromagnetism, an energy $-\frac{1}{2}\vec{d}\cdot\vec{E}$ is associated with a polarizable particle in an electric field, where $\vec{d}$ is the induced dipole moment. We therefore assume that the energy shift in state $|j\rangle$, arising from the interaction of the bound electron with an electric field $\vec{E}(t)$, is

$$\Delta E_{j}=-\tfrac{1}{2}\langle j,F|\,e\vec{r}(t)\cdot\vec{E}(t)\,|j,F\rangle, \quad (2.32)$$

where $\vec{r}(t)$ and $\vec{E}(t)$ are Heisenberg-picture operators, with the electron coordinate $\vec{r}(t)$ measured from the nucleus, and the state of the field is labeled by F. In fact, this result may be obtained by a transformation on the Hamiltonian, as shown by Craig and Power.[23] It is valid in the approximation that $\vec{r}$ depends linearly on $\vec{E}$ via the polarizability, so that our analysis based on (2.32) will be valid to order e^2.

Suppose there are no external fields, i.e., fields from other sources in the universe. Then $\vec{E}(t)$ is simply the vacuum or source-free field $\vec{E}_0(t)$ plus the radiation reaction field $\vec{E}_{RR}(t)$ of the electron on itself. $\vec{E}_0(t)$ is $\vec{E}_0^{(+)}(t)+\vec{E}_0^{(-)}(t)$, where the positive-frequency part

$$\vec{E}_{0}^{(+)}(t)=i\sum_{\alpha}(2\pi\hbar\omega_{\alpha})^{1/2}\vec{F}_{\alpha}(\vec{x})a_{\alpha}(0)e^{-i\omega_{\alpha}t} \quad (2.33)$$

and the negative-frequency part $\vec{E}_0^{(-)}(t)=[\vec{E}_0^{(+)}(t)]^{\dagger}$. In writing (2.33) we assume that electron excursions from the nucleus are sufficiently small that the mode functions $\vec{F}_{\alpha}(\vec{r})$ may be evaluated at the coordinate $\vec{x}$ of the nucleus; this coordinate $\vec{x}$ will in effect define the position of the "atom." Simi-

larly we obtain from (2.26) the radiation reaction field $\vec{E}_{RR}(t) = \vec{E}^{(+)}_{RR}(t) + \vec{E}^{(-)}_{RR}(t)$, where

$$\vec{E}^{(+)}_{RR}(t) = -2\pi e \sum_\alpha \vec{F}_\alpha(\vec{x}) \cdot \vec{F}^*_\alpha(\vec{x}) \times \int_0^t dt_1 \frac{d\vec{r}(t_1)}{dt} e^{i\omega_\alpha(t_1 - t)}, \tag{2.34}$$

and $\vec{E}^{(-)}_{RR}(t) = [\vec{E}^{(+)}_{RR}(t)]^\dagger$. Both $\vec{E}^{(+)}_0(t)$ and $\vec{E}^{(+)}_{RR}(t)$ are functions of $\vec{x}$.

As is well known, a normal ordering of field operators is convenient when the initial field state is the vacuum ($F = F_{\text{vac}}$), for then $\vec{E}^{(+)}_0(\vec{x}, t) \mid F_{\text{vac}}\rangle = 0$. We therefore normally order the field operators in (2.32):

$$\begin{aligned}\Delta E_j(\vec{x}) &= -\tfrac{1}{2}e\langle j, F_{\text{vac}} \mid [\vec{r}(t)\cdot\vec{E}^{(+)}_0(t) + \vec{E}^{(-)}_0(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{E}^{(+)}_{RR}(t) + \vec{E}^{(-)}_{RR}(t)\cdot\vec{r}(t)] \mid j, F_{\text{vac}}\rangle \\ &= -\tfrac{1}{2}e\langle j, F_{\text{vac}} \mid \vec{r}(t)\cdot\vec{E}^{(+)}_{RR}(t) + \vec{E}^{(-)}_{RR}(t)\cdot\vec{r}(t) \mid j, F_{\text{vac}}\rangle \\ &= -\text{Re}\langle j, F_{\text{vac}} \mid e\vec{r}(t)\cdot\vec{E}^{(+)}_{RR}(t) \mid j, F_{\text{vac}}\rangle .\end{aligned} \tag{2.35}$$

This expression is equivalent to (2.32) because

$$[\vec{r}(t), \vec{E}^{(\pm)}(t)] = [\vec{r}(t), \vec{E}^{(\pm)}_0(t) + \vec{E}^{(\pm)}_{RR}(t)] = 0. \tag{2.36}$$

Henceforth we shall drop the label vac, so that $|j\rangle$ will be used for $|j, F_{\text{vac}}\rangle$. Combining Eqs. (2.35) and (2.34), therefore, we have

$$\Delta E_j = 2\pi^2 e\, \text{Re} \sum_\alpha F_{\alpha M}(\vec{x}) F^*_{\alpha N}(\vec{x}) \times \int_0^t dt_1 \left\langle j \left| r_M(t) \frac{dr_N(t_1)}{dt} \right| j \right\rangle e^{i\omega_\alpha(t_1 - t)}, \tag{2.37}$$

where M, N (=1, 2, 3) label the components of $\vec{r}$ and we follow the summation convention for repeated indices.

Denoting by $U(t)$ the time evolution operator associated with the complete Hamiltonian, we have

$$\left\langle j \left| r_M(t) \frac{dr_N(t_1)}{dt} \right| j \right\rangle = \frac{\partial}{\partial t_1} \sum_i \langle j \mid U^\dagger(t) r_M(0) U(t) \mid i\rangle \times \langle i \mid U^\dagger(t_1) r_N(0) U(t_1) \mid j\rangle , \tag{2.38}$$

where the summation is over a complete set of states. We now make the approximation that quantities like $\langle j \mid U^\dagger(t) r_M(0) U(t) \mid i\rangle$ evolve in time nearly according to their evolution in the case of the unperturbed atom:

$$\langle j \mid U^\dagger(t) r_M(0) U(t) \mid i\rangle \cong e^{i\omega_{ji}t}\langle j \mid r_M(0) \mid i\rangle = r_{Mji} e^{i\omega_{ji}t}, \tag{2.39}$$

where $\hbar\omega_{ji} = E_j - E_i$. Within this approximation Eq. (2.37) becomes

$$\begin{aligned}\Delta E_j &= 2\pi e^2 \sum_i \sum_\alpha F_{\alpha M}(\vec{x}) F^*_{\alpha N}(\vec{x}) \omega_{ji} r_{Mji} r_{Nij} \int_0^t dt_1 \sin(\omega_\alpha - \omega_{ji})(t_1 - t) \\ &= -2\pi e^2 \sum_i \sum_\alpha \frac{\omega_{ji}}{\omega_\alpha - \omega_{ji}} \mid \vec{F}_\alpha(\vec{x}) \cdot \vec{r}_{ji} \mid^2 ,\end{aligned} \tag{2.40}$$

where we have used the relation

$$\int_0^t dt_1 \sin(\omega_\alpha - \omega_{ji})(t_1 - t) \cong -\mathcal{P}\left(\frac{1}{\omega_\alpha - \omega_{ji}}\right) \tag{2.41}$$

for $t \gg |\omega_{ji}|^{-1}$, so that integrals over ω_α arising from (2.40) are to be understood as Cauchy principal parts. Equation (2.40) gives the level shift arising from the coupling of the electron to its own radiation reaction field.

Let us again consider the simplest case, that in which the atom is in free space, for which the field modes are given by (2.16). From Eq. (2.40) we obtain

$$\Delta E_j = -\frac{2e^2}{3\pi c^2} \sum_i \omega_{ji} \mid \vec{r}_{ji} \mid^2 \int_0^\Omega \frac{d\omega\, \omega^2}{\omega - \omega_{ji}} \tag{2.42}$$

in the mode continuum limit

$$\sum_{\vec{k},\lambda} \to \frac{L^3}{8\pi^3} \int d^3k \sum_\lambda .$$

This is the level shift due to radiation reaction in free space.

The limit of a free electron may be obtained by taking the transition frequencies ω_{ji} between electron bound states to be small compared with photon frequencies ω in (2.42):

$$\begin{aligned}\Delta E^{(\Omega)}_j &= -\frac{2e^2}{3\pi c^3} \sum_i \omega_{ji} \mid \vec{r}_{ji} \mid^2 \int_0^\Omega d\omega\, \omega \\ &= \frac{e^2\hbar}{\pi m c^3} \int_0^\Omega d\omega\, \omega .\end{aligned} \tag{2.43}$$

The second line follows when the Thomas-Reiche-Kuhn sum rule,

$$\frac{\hbar}{m} = -\tfrac{2}{3}\sum_i |\vec{r}_{ji}|^2 \omega_{ji} \tag{2.44}$$

is employed. Now $\Delta E_j^{(0)}$ is just the A^2 contribution to the free-electron self-energy:

$$\Delta E_j^{(0)} = \frac{e^2}{2mc^2}\langle F_{\rm vac}|A^2|F_{\rm vac}\rangle . \tag{2.45}$$

We have obtained this well-known result, however, using the notion of radiation reaction, without invoking the vacuum field.

$\Delta E_j^{(0)}$ as given by (2.45) is obviously independent of the electron state $|j\rangle$. We may therefore regard the difference between ΔE_j and $\Delta E_j^{(0)}$ as the true level shift of state $|j\rangle$, i.e.,

$$\Delta E_j' = \Delta E_j - \Delta E(A^2)$$

$$= -\frac{2e^2}{3\pi c^3}\sum_i \omega_{ji}^2 |\vec{r}_{ji}|^2 \int_0^\Omega \frac{d\omega\,\omega}{\omega - \omega_{ji}} . \tag{2.46}$$

This is just the familiar contribution to the radiative level shift from the $\vec{A}\cdot\vec{p}$ interaction in second-order perturbation theory, prior to mass renormalization.[4] This must be the case because we are dealing directly with the coupling between the electron and the (radiation reaction) electric field, which includes both $\vec{A}\cdot\vec{p}$ and A^2 interactions, and in (2.46) we have subtracted away the A^2 contribution. If we again consider a free-electron limit by ignoring ω_{ji} in the integrand of Eq. (2.46), we obtain, therefore, the free-electron energy due to the $\vec{A}\cdot\vec{p}$ interaction

$$\langle j|\Delta E|j\rangle = -\frac{2e^2}{3\pi c^3}\sum_i \omega_{ji}^2 |\vec{r}_{ji}|^2 \int_0^\Omega d\omega$$

$$= \left\langle j\left| -\frac{\delta m}{m}\left(\frac{p^2}{2m}\right)\right| j\right\rangle , \tag{2.47}$$

which, of course, is equivalent to (2.31). The difference between $\Delta E_j'$ and $\langle j|\Delta E|j\rangle$ gives what Bethe[4] computed as the observable radiative level shift.

III. CASIMIR-POLDER ATTRACTION OF AN ATOM BY A PERFECTLY CONDUCTING WALL

As our first example of a position-dependent radiation reaction field level shift, we consider a ground-state atom near a perfectly conducting wall at $z = 0$. The level shift is given by Eq. (2.40), where in the present example the mode functions $\vec{F}_\alpha(\vec{x})$ are those for the half-space $z > 0$ bounded at $z = 0$ by a plane on which the tangential components of $\vec{F}_\alpha(\vec{x})$ must vanish. Consider a plane wave with wave vector $\vec{k} = k_1\hat{x} + k_2\hat{y} + k_3\hat{z} = \vec{k}_\parallel + k_3\hat{z}$. From this wave and the reflected wave with wave vector $\vec{k}^{(r)} = \vec{k}_\parallel - k_3 z$ we can form a mode function

$$\vec{F}_{\vec{k}1}(\vec{x}) = \left(\frac{2}{V}\right)^{1/2}(\hat{k}_\parallel \times \hat{z})\sin(k_3 z)e^{i\vec{k}_\parallel\cdot\vec{x}}, \tag{3.1}$$

which is normalized in the sense of (2.6). V is once again a quantization volume, and the carets denote unit vectors. Similarly we can form a mode

$$\vec{F}_{\vec{k}2}(\vec{x}) \propto (\hat{k}_\parallel\times\hat{z})\times\hat{k}e^{i\vec{k}\cdot\vec{x}} + (\hat{k}_\parallel\times\hat{z})\times\hat{k}^{(r)}e^{i\vec{k}^{(r)}\cdot\vec{x}},$$

which when normalized becomes

$$\vec{F}_{\vec{k}2}(\vec{x}) = \frac{1}{k}\left(\frac{2}{V}\right)^{1/2}[k_\parallel\hat{z}\cos(k_3 z) - k_3\hat{k}_\parallel i\sin(k_3 z)]e^{i\vec{k}_\parallel\cdot\vec{x}}. \tag{3.2}$$

From Eq. (2.40), therefore, we obtain

$$\Delta E_j = -2\pi e^2\sum_i\sum_{\vec{k}}\sum_{s=1,2}\frac{\omega_{ji}}{\omega_k - \omega_{ji}}|\vec{F}_{\vec{k}s}(\vec{x})\cdot\vec{r}_{ji}|^2, \tag{3.3}$$

where $\omega_k^2 = (k_\parallel^2 + k_3^2)c^2$. In the mode continuum limit $V\to\infty$,

$$\Delta E_j(z) = -\frac{e^2}{\pi^2}\sum_i\int_{-\infty}^{\infty}d^2k_\parallel\int_0^\infty dk_3\frac{\omega_{ji}}{\omega_k - \omega_{ji}} \times\left([(\hat{k}_\parallel\times\hat{z})\cdot\vec{r}_{ji}]^2\sin^2k_3z + \frac{k_\parallel^2}{k^2}(\hat{z}\cdot\vec{r}_{ji})^2\cos^2k_3z + \frac{k_3^2}{k^2}(\vec{r}_{ji}\cdot\hat{k}_\parallel)^2\sin^2k_3z\right). \tag{3.4}$$

If we average over the orientation of the matrix elements $\vec{r}_{ji}$ of the spherically symmetric system, we have

$$\Delta E_j(z) = \Delta E_j - \frac{e^2}{3\pi}\sum_i|\vec{r}_{ji}|^2\int_0^\infty dk_3k_3^2\ln\left|k_3 - \frac{\omega_{ji}}{c}\right|\cos(2k_3z) + \frac{e^2}{3\pi}\sum_i|\vec{r}_{ji}|^2\int_0^\infty dk_3k_3^2\ln k_3\cos(2k_3z), \tag{3.5}$$

where ΔE_j is the z-independent, free-space level shift (2.42). The complete level shift is obtained by adding to (3.5) the electrostatic shift V_j arising from the instantaneous Coulomb interaction between the dipole at z and its electrostatic image at $-z$. V_j is associated with the longitudinal elec-

tric field, which has not been included explicitly in the Coulomb-gauge formulation of the Hamiltonian (2.8).[17]

Barton has shown how physically meaningful results may be extracted from integrals like those appearing in (3.5).[24] He shows that the last term in (3.5) is in fact just $-V_j$, so that the complete z-dependent level shift is given by the second term.[24,25] For large separations, i.e., for z large compared with the wavelengths of transitions to the ground state $|j\rangle$, Barton[24] shows that this term may be evaluated to give precisely the Casimir-Polder potential,[26]

$$U_j(z) = -3\alpha_j \hbar c/8\pi z^4, \tag{3.6}$$

where

$$\alpha_j = -\frac{2e^2}{3\hbar}\sum_i |\vec{r}_{ji}|^2/\omega_{ji}$$

is the zero-frequency polarizability, which is positive for the ground state.

A. Connection with approaches based on the zero-point electromagnetic field

Derivations of the Casimir-Polder potential based on standard second-order perturbation theory (cf. Refs. 24 and 26) rely formally on nonvanishing interaction matrix elements between the vacuum state and one-photon states. Other derivations invoke the zero-point electromagnetic field in a more explicit manner, such that the change in the zero-point field energy due to the presence of the atom may be regarded as the physical origin of the Casimir-Polder effect.[27,28] We can now show how it is that these derivations give exactly the same result as our radiation-reaction derivation.

For this purpose we return to Eq. (2.40) and write it as

$$\Delta E_j = -\frac{2\pi e^2}{3}\sum_i\sum_\alpha \frac{\omega_\alpha |\vec{r}_{ji}|^2}{\omega_\alpha - \omega_{ji}} |\vec{F}_\alpha(\vec{x})|^2 + \frac{2\pi e^2}{3}\sum_i |\vec{r}_{ji}|^2 \sum_\alpha |\vec{F}_\alpha(\vec{x})|^2, \tag{3.7}$$

where for simplicity we again assume spherical symmetry. Now the z-dependent part of the second term may be shown to correspond to the last term of (3.5), i.e., it is cancelled by the electrostatic shift V_j. The z-independent part contributes to the Lamb shift, which is not of interest here. Thus we drop the second term of (3.7) entirely and write the level shift as

$$\begin{aligned}\delta E_j(\vec{x}) &= \Delta E_j(\vec{x}) + V_j(\vec{x}) \\ &= -\frac{2\pi e^2}{3}\sum_i\sum_\alpha \frac{\omega_\alpha |\vec{r}_{ji}|^2}{\omega_\alpha - \omega_{ji}} |\vec{F}_\alpha(\vec{x})|^2 \\ &= -\frac{e^2}{3\hbar}\sum_i\sum_\alpha \frac{|\vec{r}_{ji}|^2}{\omega_\alpha - \omega_{ji}} |\vec{\mathcal{E}}_\alpha(\vec{x})|^2,\end{aligned} \tag{3.8}$$

where the $\vec{\mathcal{E}}_\alpha(\vec{x})$ are defined by the expression

$$\vec{E}(\vec{x},t) = i\sum_\alpha [\vec{\mathcal{E}}_\alpha(\vec{x}) a_\alpha(t) - \vec{\mathcal{E}}_\alpha^*(\vec{x}) a_\alpha^\dagger(t)] \tag{3.9}$$

for the electric field operator. For essentially long-range interactions we may suppose that only low-frequency contributions are important in (3.8), and therefore that ω_α may effectively be neglected compared with ω_{ji} in the denominator. Thence

$$\begin{aligned}\delta E_j(\vec{x}) &\cong \frac{e^2}{3\hbar}\sum_i \frac{|\vec{r}_{ji}|^2}{\omega_{ji}} \sum_\alpha |\vec{\mathcal{E}}_\alpha(\vec{x})|^2 \\ &= -\tfrac{1}{2}\alpha_j \sum_\alpha |\vec{\mathcal{E}}_\alpha(\vec{x})|^2,\end{aligned} \tag{3.10}$$

where α_j is again the static polarizability. This expression simply relates the energy shift to the work done by the field to induce the dipole moment in the atom. It is, in essence, the starting point of derivations of δE_j based on zero-point electromagnetic energy.[27,28] Within approximation (2.39), one could equally well begin not with (2.35) but with the expression[29]

$$\begin{aligned}\delta E_j(\vec{x}) &= -\tfrac{1}{2}\alpha_j \langle \vec{E}_0^2(\vec{x},t)\rangle_{\text{vac}} \\ &= -\tfrac{1}{2}\alpha_j \sum_\alpha |\vec{\mathcal{E}}_\alpha(\vec{x})|^2 \langle a_\alpha(0) a_\alpha^\dagger(0)\rangle_{\text{vac}} \\ &= -\tfrac{1}{2}\alpha_j \sum_\alpha |\vec{\mathcal{E}}_\alpha(\vec{x})|^2,\end{aligned} \tag{3.11}$$

where $\vec{E}_0(\vec{x},t)$ is again the source-free electric field operator. This shows that the Casimir-Polder interaction depends only on the modal properties of the field. It may therefore be derived using a zero-point field having an energy $\frac{1}{2}\hbar\omega_\alpha$ per normal mode and satisfying the appropriate boundary conditions. We obtain the same result when we account for the fact that the radiation reaction field, like the zero-point field, must also depend on the modal characteristics of the space surrounding the atom.

B. Necessity of both radiation reaction and zero-point field

It is worth emphasizing that, although the interpretation of the Lamb shift and the Casimir-Polder force can be given exclusively in terms of either radiation reaction or the zero-point field, *both fields are in fact necessary for the formal consis-*

tency of the theory. Perhaps the simplest way to demonstrate this is to consider the Heisenberg equation of motion for a nonrelativistic electron in free space

$$\frac{d^2x(t)}{dt^2} - \gamma\frac{d^3x(t)}{dt^3} = \frac{e}{m}E_{0x}(t)\,, \tag{3.12}$$

where $\gamma = 2e^2/3mc^3$, m being the renormalized mass, and $E_{0x}(t)$ is the x component of the free electric field operator. Equation (3.12) can be used to determine the electron coordinate x and linear momentum p_x, and therefore the commutator $[x,p_x]$. It is easily shown that[30]

$$[x(t),p_x(t)] = \frac{8\pi^2 i}{3m}\int_0^\infty \frac{d\omega\,\rho_0(\omega)}{\omega^3(1+\gamma^2\omega^2)} = i\hbar\,, \tag{3.13}$$

where $\rho_0(\omega) = \hbar\omega^3/2\pi^2c^3$ is the spectral energy density of the vacuum field. This shows that both radiation reaction and the free field are necessary for the preservation of the canonical commutation relation for the electron. The relation between the dissipation force $m\gamma d^3x/dt^3$ and the spectral density of the fluctuation force eE_{0x} may in fact be inferred from the general theorem of Callen and Welton[31] for linearly dissipative systems.

C. Explicit form of radiation reaction field

The radiation reaction field $\vec{E}_{RR}$ for an atom at a distance z from the conducting plane is easily obtained from Eq. (2.34). For the z component, for example, we obtain

$$\hat{z}\cdot\vec{E}_{RR}(z,t) = \frac{2e}{3c^3}\frac{d^3r_z(t)}{dt^3} - \frac{4\Omega}{3\pi c^3}\frac{d^2r_z(t)}{dt^2} - 2e\frac{dr_z}{dt}\left(t-\frac{2z}{c}\right)\frac{1}{4z^2c} - 2er_z\left(t-\frac{2z}{c}\right)\frac{1}{8z^3}\,. \tag{3.14}$$

$\vec{E}_{RR}$ is in fact the free-space ($z\to\infty$) result (2.29) plus the retarded dipole field from the "image" dipole at $-z$.

The formal identification of the *operator* equation (3.14) with the classical result permits a simple, intuitive understanding of the Casimir-Polder force. The positive-frequency part of the field (3.14) may be obtained from Eq. (2.34)

$$\hat{z}\cdot\vec{E}^{(+)}_{RR}(z,t) = -\frac{2e}{\pi c^3}\int d\omega\,\omega^2\left(\frac{1}{3} - \frac{\cos\omega x}{\omega^2x^2} + \frac{\sin\omega x}{\omega^3x^3}\right) \times \int_0^t dt_1\frac{dr_z(t)}{dt}\,e^{i\omega(t_1-t)}\,, \tag{3.15}$$

where $x = 2z/c$. From Eq. (2.35), therefore, the z-dependent portion of ΔE_j is

$$\Delta E_j(z) - \Delta E_j = -\frac{2e^2}{\pi c^3}\int d\omega\,\omega^2\left(\frac{\cos\omega x}{\omega^2x^2} - \frac{\sin\omega x}{\omega^3x^3}\right)\int_0^t dt_1\left\langle j\left|\, r_z(t)\frac{dr_z(t_1)}{dt}\,\right|j\right\rangle e^{i\omega(t_1-t)} + \cdots\,, \tag{3.16}$$

where the ellipsis represents terms involving $\left\langle j\left|\, r_{x,y}(t)\left(\frac{dr_{x,y}(t_1)}{dt}\right)\right|j\right\rangle$.

In approximation (2.39),

$$\left\langle j\left|\, r_z(t)\frac{dr_z(t_1)}{dt}\,\right|j\right\rangle \cong -i\sum_i \omega_{ji}|r_{zji}|^2e^{-i\omega_{ji}(t_1-t)} \tag{3.17}$$

so that

$$\Delta E_j(z) - \Delta E_j \cong \sum_i\int_0^\infty dk\,\frac{1}{2\pi i}\,\frac{k^2\omega_{ji}}{kc-\omega_{ji}}\left[|r_{zji}|^2\frac{e^{2ikz}}{2kz}\left(\frac{2i}{2kz} - \frac{2}{4k^2z^2}\right) + \text{c.c.} + \cdots\right], \tag{3.18}$$

where the ellipsis represent terms involving $|r_{xji}|^2$ and $|r_{yji}|^2$. This is precisely the expression obtained by Casimir and Polder,[26] who evaluated (3.18) by including a factor $\exp(-\gamma k)$ in the integrand and then taking the limit $\gamma\to 0+$ at the end of the calculation.

After presenting the derivation of their result, Casimir and Polder noted the appearance of quantities that have the same form as the electric field of the image dipole (at $-z$) evaluated at z. They stated, however, that "... we have not been able to find a general consideration, based on the correspondence idea, by means of which at least the form of [Eq. (3.18)] could be foretold."[31] The difficulty they encountered in attempting to give a more classical interpretation of the (Casimir-Polder) force on the atom stems from the profound difference between the roles of positive- and negative-frequency parts of the field in the quantum theory of radiation. There would appear to be no simple correspondence principle applicable to cases where this difference is crucial. In classi-

cal theory, on the other hand, the positive- and negative-frequency components of the field are on the same footing, and are introduced only for mathematical convenience.[32] Thus the Casimir-Polder interaction can be understood, as we have seen, on the basis of the classically familiar concept of radiation reaction. *However, the radiation reaction must be treated according to the quantum theory, where the positive- and negative-frequency parts of the radiation reaction field must be carefully distinguished.*

D. Radiative level shifts in murium

In a recent interesting paper, Shakeshaft and Spruch[16] have considered the radiative level shifts of the system they call "murium," an electron bound to a wall by its image charge. The radiative level shift in this system is due predominantly to the A^2 term in the Hamiltonian.[16] They interpret the level shifts in terms of the electron acquiring, "through vacuum fluctuations, a zero-point kinetic energy whose magnitude depends on the distance of the electron from the wall."[16] However, their level shifts may also be obtained from our Eq. (2.40) in the free-electron limit of neglecting the transition frequencies ω_{ji} in the denominator. We have already shown in Sec. II [Eq. (2.43)] how the A^2 contribution to the level shift results from this limit in the case of free space. The extension to the half-space bounded by a conducting wall is easily made, and we obtain the level shifts of Shakeshaft and Spruch. In our approach, however, the shifts are due entirely to radiation reaction rather than vacuum-field fluctuations.

IV. VAN DER WAALS INTERACTION BETWEEN TWO ATOMS

The van der Waals interaction between two ground-state atoms may be derived using standard fourth-order perturbation theory. London's derivation[33] of the R^{-6} interaction was considered an important accomplishment of the quantum theory. Casimir and Polder[26] included retardation and showed that, for separations large compared with wavelengths of transitions connected to the ground state, the interaction potential actually goes as R^{-7}. The van der Waals interaction is frequently interpreted physically using the notion of a zero-point radiation.[15-28] Indeed, the van der Waals force is often cited as evidence of the physical reality of the vacuum field. We now show that this interaction may also be understood physically from radiation reaction, with no explicit reference to any vacuum field.

Whereas in the preceding section we were concerned with the modification of free-space plane-wave modes by a conducting wall, we are now interested in the effect of a single atom B. The effect of atom B on the modes of the field will in turn effect a change in the radiation reaction field of atom A from its free-space form, and therefore a change in radiative level shifts of atom A from their free-space values. The R-dependent portion of the modified radiative level shift of the ground state will be shown to be the van der Waals potential.

A. Effect of a single atom on the field modes

We remarked in Sec. II that in the standard field quantization procedure the mode functions may be determined using classical electromagnetic theory. Field quantization for free space provides an obvious example; another is the quantization in the half-space of Sec. III. We may also approach the quantization of the field in dielectric media in this way. For example, field quantization in the presence of a dielectric interface has been considered by Carniglia and Mandel.[34] What is implied by such treatments, of course, is that the medium is passive and characterized, for instance, by a simple dielectric constant. The quantum-mechanical state of the medium is assumed to be essentially unchanging. Characterization of a medium by a dielectric constant $\epsilon(\omega)$ in order to determine field mode functions assumes that the molecules of the medium remain, to a good approximation, in a single state, usually the ground state. With each molecule is associated the polarizability of that state. Obviously this polarizability must be determined quantum mechanically, e.g., by the Kramers-Heisenberg dispersion formula.

The modes of the field in the presence of a single ground-state atom B may be determined from the "refractive index" of the medium consisting of the vacuum plus the atom, i.e., from the scattering of each plane wave by the atom. The basic idea here is no different from the case of a dielectric medium, where the scattering from individual molecules is responsible for the refractive index. The modes of the field in the case of a single ground-state atom B are the plane-wave modes of free space plus the dipole field produced by the single scatterer. In other words, the free-space mode functions (2.16) are altered according to the formula

$$\hat{e}_{\vec{k}\lambda}e^{i\vec{k}\cdot\vec{x}} \to \hat{e}_{\vec{k}\lambda}e^{i\vec{k}\cdot\vec{x}} + \alpha_B(\omega_k)k^3 e^{i\vec{k}\cdot\vec{x}_B}e^{ikr} \times\left[\hat{e}_{\vec{k}\lambda}\left(\frac{1}{kr}+\frac{i}{k^2r^2}-\frac{1}{k^3r^3}\right) -(\hat{e}_{\vec{k}\lambda}\cdot\hat{r})\hat{r}\left(\frac{1}{kr}+\frac{3i}{k^2r^2}-\frac{3}{k^3r^3}\right)\right], \qquad (4.1)$$

where $\vec{x}_B$ specifies the location of the atom, $\vec{r} = \vec{x} - \vec{x}_B = r\hat{r}$, and

$$\alpha_B(\omega) = \frac{2}{3\hbar}\sum_m \frac{e^2|\vec{r}_{mg}|^2\omega_{mg}}{\omega_{mg}^2 - \omega^2} \tag{4.2}$$

is the polarizability of the ground state $|g\rangle$.

Using the mode functions (4.1), we can calculate the radiation reaction field of atom A in the presence of atom B, and therefore the radiation level shifts of atom A. This leads us, of course, to Eq. (2.40), in which we require the mode functions evaluated at $\vec{x}_A$:

$$\vec{r}_{ji}\cdot\vec{F}_{\vec{k}\lambda}(\vec{x}_A) \cong L^{-3/2}\left\{\vec{r}_{ji}\cdot\hat{e}_{\vec{k}\lambda}e^{i\vec{k}\cdot\vec{x}_A} + \alpha_B(\omega_k)k^3e^{i\vec{k}\cdot\vec{x}_B}e^{ikR}\left[(\vec{r}_{ji}\cdot\hat{e}_{\vec{k}\lambda})\left(\frac{1}{kR}+\frac{i}{(kR)^2}-\frac{1}{(kR)^3}\right)\right.\right.$$
$$\left.\left. - (\hat{e}_{\vec{k}\lambda}\cdot\hat{R})(\vec{r}_{ji}\cdot\hat{R})\left(\frac{1}{kR}+\frac{3i}{(kR)^2}-\frac{3}{(kR)^3}\right)\right]\right\}, \tag{4.3}$$

where $\vec{R} = \vec{x}_A - \vec{x}_B = R\hat{R}$. The correction to the free-space modes due to the presence of atom B is a small one, so that we shall retain only corrections linear in α_B in our calculations. Thus we take

$$|\vec{r}_{ji}\cdot\vec{F}_{\vec{k}\lambda}(\vec{x}_A)|^2 \cong L^{-3}|\vec{r}_{ji}\cdot\hat{e}_{\vec{k}\lambda}|^2 + 2/L^{-3}\mathrm{Re}\{\alpha_B(\omega_k)k^3e^{ikR}e^{i\vec{k}\cdot\vec{R}}[(\vec{r}_{ji}\cdot\hat{e}_{\vec{k}\lambda})^2A^*(kR)$$
$$- (\hat{e}_{\vec{k}\lambda}\cdot\vec{r}_{ji})(\hat{e}_{\vec{k}\lambda}\cdot\hat{R})(\vec{r}_{ji}\cdot\hat{R})B^*(kR)]\}, \tag{4.4}$$

where

$$A(x) = \frac{1}{x}+\frac{i}{x^2}-\frac{1}{x^3},$$

and

$$B(x) = \frac{1}{x}+\frac{3i}{x^2}-\frac{3}{x^3}.$$

It is convenient to again write the level shift in the form (3.7). The second term, when added to the electrostatic dipole-dipole interaction, gives a contribution which, except for Lamb-shift calculations, may be omitted for our purposes.[35] As in the preceding section, therefore, we need only consider the first term of (3.7). We will make use of this simplification at the outset and write the level shift in the ground state of atom A as

$$\Delta E_g^{(A)}(\vec{x}_A) = -2\pi e^2\sum_i\sum_{\vec{k},\lambda}\frac{\omega_k}{\omega_k+\omega_{ig}}|\vec{r}_{ig}\cdot\vec{F}_{\vec{k}\lambda}(\vec{x}_A)|^2, \tag{4.5}$$

with the understanding that for the "Lamb shift"—the position-independent part $\Delta E_g^{(A)}$ of (4.5)—the contribution of the term $2\pi\int d^3x\vec{p}^{\perp}(\vec{x})^2$ must be included.[35,22]

The evaluation of (4.5) using (4.4) is straightforward but tedious. As usual we replace the mode summation with an integration in k space. We incur some messy but easily evaluated integrals over solid angles, with the result that

$$\Delta E_g^{(A)}(R) - \Delta E_g^{(A)}(\infty) = -\frac{2e^2}{3\pi}\sum_i|\vec{r}_{ig}|^2\int_0^\infty\frac{dk\,k^6}{kc+\omega_{ig}}\alpha_B(kc)G(kR), \tag{4.6}$$

where

$$G(x) = \frac{\sin 2x}{x^2}+\frac{2\cos 2x}{x^3}-\frac{5\sin 2x}{x^4}$$
$$-\frac{6\cos 2x}{x^5}+\frac{3\sin 2x}{x^6}. \tag{4.7}$$

Using (4.2), therefore, we have

$$\Delta E_g^{(A)}(R) - \Delta E_g^{(A)}(\infty) = -\frac{1}{c}\left(\frac{2e^2}{3\pi}\right)^2\sum_i\sum_j|\vec{r}_{ig}^{(A)}|^2|\vec{r}_{jg}^{(B)}|^2k_j^{(B)}\int_0^\infty\frac{dk\,k^6G(kR)}{(k+k_i^{(A)})[(k_j^{(B)})^2-k^2]}, \tag{4.8}$$

where $k_i^{(A)}$ and $k_j^{(B)}$ are the wave numbers for transitions to the ground states of atoms A and B, respectively, and $e\vec{r}_{ig}^{(A)}$, $e\vec{r}_{jg}^{(B)}$ are the associated dipole matrix elements. By a simple deformation of the path of integration we may write

$$\int_0^\infty\frac{dk\,k^ne^{2ikR}}{(k+k_N)(k_M^2-k^2)} = i^n\int_0^\infty\frac{du\,u^ne^{-2uR}}{(u-ik_N)(u^2+k_M^2)}, \tag{4.9}$$

and this allows us to write (4.8) in the form

$$U(R)=\Delta E_g^{(A)}(R)-\Delta E_g^{(A)}(\infty)$$
$$=-\frac{4}{\pi\hbar c}\sum_{i,m}(q_i^{(A)}q_j^{(B)})^2 k_i^{(A)}k_j^{(B)}\int_0^\infty \frac{du\,e^{-2uR}}{[u^2+(k_i^{(A)})^2][u^2+(k_j^{(B)})^2]}\left(\frac{u^4}{R^2}+\frac{2u^3}{R^3}+\frac{5u^2}{R^4}+\frac{6u}{R^5}+\frac{3}{R^6}\right), \tag{4.10}$$

where $q_i^{(A,B)}=(e^2/3)|\vec{r}_{ig}^{(A,B)}|^2$. Equation (4.10) is precisely the result obtained by Casimir and Polder[25] using fourth-order perturbation theory. For $R\to 0$ the integration gives the London result $U(R)\propto R^{-6}$, whereas for large R,

$$U(R)=-\frac{23\hbar c}{4\pi R^7}\alpha_A\alpha_B\,, \tag{4.11}$$

with α_A,α_B the static polarizabilities of the ground-state atoms.

V. REMARKS

The Casimir-Polder and van der Waals interactions have been derived by considering the radiation reaction field of a bound electron. This radiation reaction field in general varies with the position of the electron. In free space the interaction of the bound electron with its radiation reaction field gives rise to the "Lamb shift," whereas near a conductor or another atom, for instance, the position dependence of the reaction field leads, respectively, to the Casimir-Polder and van der Waals potentials.

Unlike other derivations of these effects, ours does not invoke the vacuum-field fluctuations. Exactly as in the case of the "Lamb shift," these effects may be explained physically using either the zero-point electromagnetic field or the field of radiation reaction. These fields are connected by a fluctuation-dissipation relation, and *both* are necessary for the internal consistency of the quantum theory of radiation. We cannot point to the zero-point field as *the* agent responsible for these van der Waals effects, any more than we can say that they are due exclusively to radiation reaction. The two explanations "are merely two sides of the same quantum-mechanical coin,"[8] exactly as in the theory of spontaneous emission.[36]

Other well-known Casimir-type effects include forces on macroscopic objects. These may be considered to the manifestations of the van der Waals forces between individual particles, and therefore of the position-dependent reaction fields of the particles. For these macroscopic forces it is more convenient to determine the spatial dependence of the reaction field by solving macroscopic Maxwell equations subject to appropriate boundary conditions. This approach has been carried out by Schwinger, DeRaad, and Milton within the context of Schwinger's source theory, "where the vacuum is regarded as truly a state with all physical properties equal to zero."[37]

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