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Integrable Hamiltonian systems and the Painlevé property

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A direct method is described for obtaining conditions under which certain N-degree-of-freedom Hamiltonian systems are *integrable,* i.e., possess N integrals in involution. This method consists of requiring that the general solutions have the Painlevé property, i.e., no movable singularities other than poles. We apply this method to several Hamiltonian systems of physical significance such as the generalized Hénon-Heiles problem and the Toda lattice with $N=2$ and 3, and recover all known integrable cases together with a few new ones. For some of these cases the second integral is written down explicitly while for others integrability is confirmed by numerical experiments.

I. INTRODUCTION

The question of integrability of a dynamical system was raised soon after Newton formulated the equations of motion of three bodies in a gravitational field. By integrability (often referred to as complete integrability) of a Hamiltonian system with N degrees of freedom we mean the existence of N-analytic, single-valued, global integrals of the motion which are functionally independent and in involution.[1–3] When this is the case, the equations of motion are (in principle at least) *separable* and the solutions can be obtained by the method of quadratures.[4]

To date, however, there exists no general method for determining whether a given dynamical system, Hamiltonian or not, is integrable. Since Newton, a number of "jewel" results have been obtained in this direction, notably Jacobi's solution of the geodesic motion on an ellipsoid[5] and Euler's and Kovalevskaya's integration of the rotation of a rigid body in some special cases.[6] More recently, several examples of integrable Hamiltonian systems have been discovered mainly by ingenious applications of the theory of Inverse Scattering Transforms (IST).[7–10]

In this paper a direct method is used to identify integrable Hamiltonian systems. In particular, we look for all values of the parameters of the system such that its solutions will have the Painlevé property, i.e., that the only *movable* singularities they can have are poles. (A singularity is movable if its location in the complex plane depends on the initial conditions; an equation whose only movable singularities are poles is said to be of P-type.[11,12])

It is known that a deep connection exists between IST and ordinary differential equations of P-type.[11,12] This connection suggests that dynamical systems with the Painlevé property might also be integrable. Here we demonstrate the effectiveness of our direct method—originally adopted by Kovalevskaya[6] — on several Hamiltonian systems of two or three degrees of freedom, which arise in a variety of physical problems. In some of the cases presented here we are able to provide the integrals explicitly, while in others we offer numerical evidence supporting integrability.

This paper is concerned only with Hamiltonian systems mainly because it is not clear how to numerically investigate integrability in dissipative systems. For example, while in integrable Hamiltonian systems all orbits are known to lie on invariant tori whose presence is evidenced by their intersection with Poincaré Maps,[13,14] etc., the corresponding situation in nonconservative systems is not as well understood. However, the method itself is not restricted to Hamiltonian systems. For instance, the Painlevé property has already identified integrable cases in dissipative systems such as the Lorentz equations[16] and Fisher's equation.[17]

Strictly speaking, of course, integrability is a mathematical property. From a physicist's point of view "globally" stable motion, i.e., absence of large scale "chaotic" regions is often equally important. Our numerical investigations, in agreement with many other studies, indicate that for considerably large ranges of parameter values near the integrable (Painlevé) cases the general behavior of the system is remarkably stable. Thus we expect that the Painlèvé method of identifying integrable systems can become a practical tool for locating "global" stability, a highly desirable property in many physical problems as, e.g., fusion research[18] or high-energy accelerators.[19]

II. SUMMARY OF RESULTS

We apply here the Painlevé analysis to a number of Hamiltonian systems of physical significance and find all parameter values for which their solutions have the Painlevé property. In some of these cases we demonstrate complete integrability by explicitly writing down the integral, while in others we offer numerical evidence which confirms the existence of the second integral. Specifically we have analyzed among other systems:

(1) The Hénon-Heiles Hamiltonian

$$H=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2+Ax^2+By^2)-x^2y-\frac{\epsilon}{3}y^3 , \tag{2.1}$$

well-known from a variety of problems in celestial mechanics,[20,13] statistical mechanics,[14] and quantum mechanics.[15]

(2) Two coupled quartic oscillators

$$H=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2+Ax^2+By^2)+\frac{x^4}{4}+\frac{\sigma y^4}{4}+\frac{\rho}{2}x^2y^2 , \tag{2.2}$$

arising, for example, in connection with problems in field theory.[21,22]

In Sec. III the analysis is described only for case 1, since that of case 2 proceeds along similar lines. The general solution (i.e., four arbitrary constants) is found to possess the Painlevé property only in the following cases:

Case 1 (a) $A=B$, $\epsilon=1$: Here the equations of motion decouple in $(x+y)$, $(x-y)$ variables and integrability has long been known.[23] (b) $\epsilon=6$, any A,B: This case is less trivial and the second integral is given explicitly in Sec. III. For the special choice $A=1$, $B=4$, it was known[18] that the equations of motion separate in parabolic coordinates. (c) $\epsilon=16$, $B=16A$ (see Refs. 32 and 33).

Case 2 (a) $A=B$, $\sigma=\rho=1$; (b) $A=B$, $\sigma=1$, $\rho=3$: For these examples, we prove that the Painlevé property implies integrability by explicitly deriving the second integral of the motion (see Sec. III).

We have also studied a one-dimensional lattice with nearest-neighbor exponential interactions for different boundary conditions:

Case 3 The periodic lattice[19] with three masses:

$$H=\frac{p_1^2}{2m_1}+\frac{p_2^2}{2m_2}+\frac{p_3^2}{2}+e^{\delta(q_1-q_2)}+e^{\epsilon(q_2-q_3)}+e^{q_3-q_1} , \tag{2.3}$$

(and three degrees of freedom) where q_1, q_2, q_3 are displacements from equilibrium, p_1, p_2, p_3 are their conjugate momenta, and m_1, m_2, δ, ϵ are positive parameters. There is only one case with the Painlevé property here: $m_1=m_2=\epsilon=\delta=1$. This is the well-known Toda case[24] which was first shown to be integrable numerically by Ford *et al.*[25] Integrability was then proved rigorously by Hénon,[26] Flaschka[8] and Manakov[27] for the case of N degrees of freedom.

Case 4 Fixed-end lattice with two masses:

$$H=\frac{p_1^2}{2m_1}+\frac{p_2^2}{2m_2}+e^{-\delta q_1}+e^{\epsilon(q_1-q_2)}+e^{q_2} . \tag{2.4}$$

Here we find three cases with the Painlevé property

(a) $m_1/m_2=1$, $\delta=\epsilon=1$
(b) $m_1/m_2=1$, $\delta=1$, $\epsilon=\frac{1}{2}$
(c) $m_1/m_2=\frac{1}{3}$, $\delta=1$, $\epsilon=\frac{1}{2}$.

Case 4 was studied by Casati and Ford[28] and by Bogoyavlenski.[29] In particular, Casati and Ford concentrated on $\delta=\epsilon=1$ and numerically explored the behavior of the solutions for different mass ratios (m_1/m_2) ranging between 0 and 1. They found evidence for complete integrability only for $m_1/m_2=1$ (and 0) which agrees with our result in Case 4 (a) above. Bogoyavlenski,[29] using group-theoretical methods, obtained several examples of Hamiltonians with exponential interactions admitting the Lax pair representation. For the fixed-end two-particle case considered here, he found exactly the same three cases (a), (b), and (c) listed above, and no more. The integrability of case 4 (b) and (c) is also suggested here numerically by the surface of section method (cf. Figs. 1–7).

Case 5 Free-end lattice with three masses (center-of-mass frame):

$$H=\frac{p_1^2}{2m_1}+\frac{p_2^2}{2m_2}+\frac{p_3^2}{2}+e^{\epsilon(q_1-q_2)}+e^{q_2-q_3} . \quad (2.5)$$

There are three families of lattices, which have the Painlevé property (within each family all members are equivalent under scaling):

$$\text{(a)}\ m_1=\frac{\epsilon(2\epsilon-1)}{2-\epsilon},\quad m_2=2\epsilon-1 ,\qquad \tfrac{1}{2}<\epsilon<2 ,$$

$$\text{(b)}\ m_1=\frac{\epsilon(\epsilon-1)}{2-\epsilon},\quad m_2=\epsilon-1,\qquad 1<\epsilon<2 ,$$

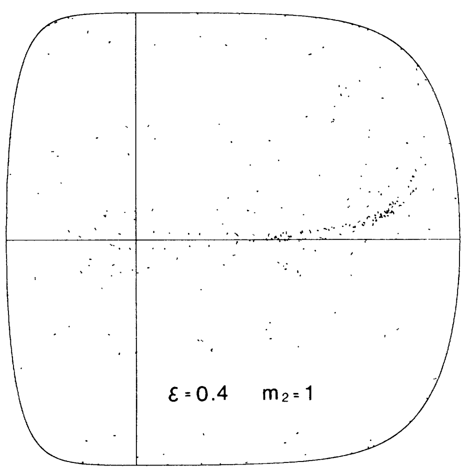

FIG. 1. Surfaces of section for the fixed-end lattice (2.4) with $\delta=1$, $m_1=1$, and energy $E=1000$. Orbit intersections are plotted in the p_1, q_1 plane at $q_2=0$, $p_2\geq 0$. In the Painlevé cases, Fig. 3 and 7, invariant curves everywhere suggest integrability, in agreement with Ref. 29. Nonintegrability in Figs. 4, 5, and 2 is indicated by some of the invariant curves "breaking" into chains of "islands". Note the sizable range of ϵ values over which the motion is "globally" stable, i.e., free from large-scale "stochastic" regions.

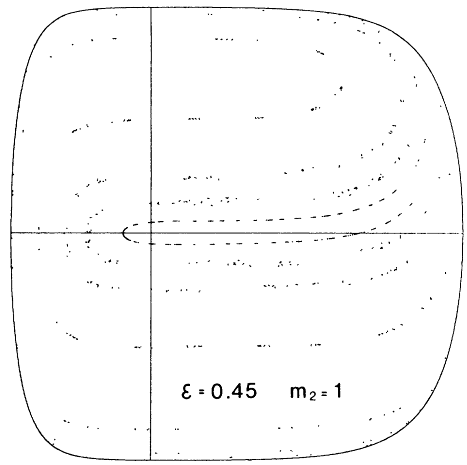

FIG. 2. See Fig. 1.

$$\text{(c)}\ m_1=\frac{3\epsilon(2\epsilon-1)}{2-3\epsilon},\quad m_2=2\epsilon-1,\qquad \tfrac{1}{2}<\epsilon<\tfrac{2}{3} .$$

The integrability of case 5 (a) was shown rigorously by Moser[7] and Bogoyavlenski.[29] Their results apply to the N degree-of-freedom case and yield the N integrals of the motion. Their methods, however, rely on one's ingenuity to find for a given system the appropriate symmetry group or Lax pair if it exists. The advantage of the Painlevé analysis is that it is directly applicable to any system provided that the equations of motion can be written in (or transformed into) polynomial form.

Cases 5 (b) and 5 (c) above are, to our knowledge, new. Their integrability, however, cannot be verified numerically by the surface of section method, since (2.5) describes a "scattering" problem, much like the familiar collisions of steel balls, for which the energy surface $H=E$ is not compact.[7]

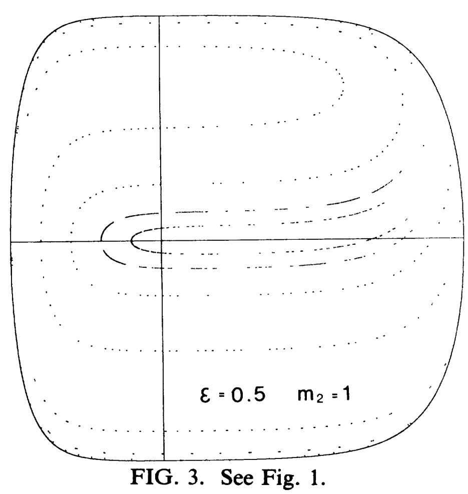

FIG. 3. See Fig. 1.

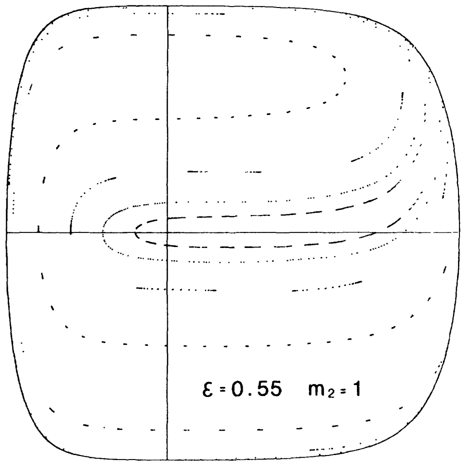

FIG. 4. See Fig. 1.

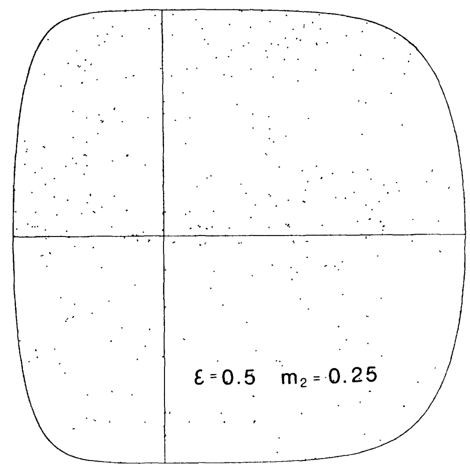

FIG. 6. See Fig. 1.

III. PAINLEVÉ ANALYSIS OF THE HÉNON-HEILES PROBLEM

The equations of motion for Hamiltonian (2.1), written as a system of first order o.d.e.'s are

$$\begin{aligned} &(a)\ \dot{x}=u, \\ &(b)\ \dot{y}=v, \\ &(c)\ \dot{u}=-Ax+2xy\ , \\ &(d)\ \dot{v}=-By+\epsilon y^2+x^2\ . \end{aligned} \tag{3.1}$$

The Painlevé property requires that the solutions of (3.1) may be written as Laurent series expansions in the complex variable (see also Refs. 11, 12, 16, and 32)

$$\tau\equiv t-t_0\ , \tag{3.2}$$

with leading-order behavior

$$x\sim a_p\tau^p,\ \ y\sim b_q\tau^q,\ \ \tau\to 0 \tag{3.3}$$

(likewise for u,v) where p,q are as yet undetermined integers.

Inserting the *dominant*[12] terms (3.3) in (3.1) one finds two possibilities:

(i) $p=q=-2$,
(ii) $p>-2,\ q=-2$.

For (i) Eqs. (3.1c) and (3.1d) yield $a_{-2}=3(2-\epsilon)^{1/2}$, $b_{-2}=3$. Now, to find higher-order terms we write

$$\begin{aligned} &x\sim 3(2-\epsilon)^{1/2}\tau^{-2}+\alpha\tau^{-2+r}\ , \\ &y\sim 3\tau^{-2}+\beta\tau^{-2+r}\ , \\ &u\sim -6(2-\epsilon)^{1/2}\tau^{-3}+\gamma\tau^{-3+r}\ , \\ &v\sim -6\tau^{-3}+\delta\tau^{-3+r}\ , \end{aligned} \tag{3.4}$$

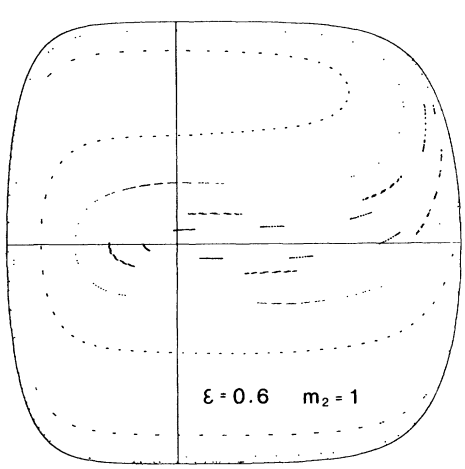

FIG. 5. See Fig. 1.

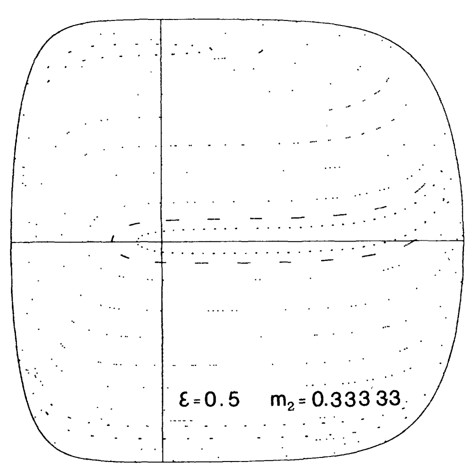

FIG. 7. See Fig. 1.

and substitute in (3.1) to obtain "resonances",[11,12] i.e., conditions such that *arbitrary* constants may enter in the expansions (3.4). This will happen if

$$\det\begin{vmatrix} r-2 & 0 & -1 & 0 \\ 0 & r-2 & 0 & -1 \\ -6 & -6(2-\epsilon)^{1/2} & r-3 & 0 \\ -6(2-\epsilon)^{1/2} & -6\epsilon & 0 & r-3 \end{vmatrix} =0\,, \tag{3.5}$$

or

$$[(r-2)(r-3)-6\epsilon+6] \times[(r-2)(r-3)-12]=0\,. \tag{3.6}$$

For Painlevé all roots of (3.6) must be integers. The second square-bracket term gives $r=-1$ (corresponds to t_0) and $r=6$ which implies that a *second* arbitrary constant exists, besides t_0. Requiring that the first square bracket in (3.6) also have integer roots, we arrive at

$$\epsilon=[(2n-5)^2+23]/24\,, \tag{3.7}$$

where $n=0, \pm1, \pm2, \ldots$. Equation (3.7) imposes a restriction on ϵ which will be taken into account below.

Turning to possibility (ii) we find that the dominant terms in (3.1) balance provided

$$\epsilon p(p-1)=12\,. \tag{3.8}$$

It is easy to see that there are three values of ϵ which satisfy both (3.7) and (3.8):

(a) $\epsilon=1$, with resonances $r=n=2,3$ [cf. (3.7); in addition to $r=-1$ and $r=6$ mentioned earlier)]; hence 4 arbitrary constants appear in the series solution of (3.1), which is a genuine Laurent series.

(b) $\epsilon=6$, with resonances $r=n=-3$ (no information) and 8 [cf. (3.7)]. Together with $r=-1$ and $r=6$ we have only three arbitrary constants here. Checking possibility (ii) we find the resonance condition

$$r(r-3)(r-6)(r+1)=0\,, \tag{3.9}$$

whence four arbitrary constants: t_0, and three at $r=0$, 3, 6 and integrability conditions are established in this case also. The second integral, due to John Greene[30] is

$$x^4+4x^2y^2+4\dot{x}(\dot{x}y-\dot{y}x) -4Ax^2y+(4A-B)(\dot{x}^2+Ax^2)=\text{const}\,,$$

as can be verified by direct differentiation. Because this integral happens to be quadratic in the momenta, it also can be obtained by a special method of Whittaker (see Ref. 4, Sec. 152). Whether there is any connection between his method and the Painlevé property is open.

(c) $\epsilon=2$, with resonances $r=n=0,5$, [cf. (3.7)] and possibility (i) reduces to (ii) since $a_2=0$ [cf. (3.4)]. The resonance condition here is

$$(r-5)(r+1)r(r-6)=0\,. \tag{3.10}$$

However, when the expansion in (3.4) is carried to higher orders only three arbitrary constants appear corresponding to $r=-1,5,6$. The fourth constant corresponding to $r=0$, may be obtained by replacing (3.3) with[31]

$$y\sim3\tau^{-2},\quad x\sim(-\tfrac{15}{2})^{1/2}\tau^{-2}(\ln\tau)^{-1/2}$$

as $\tau\to0$. Certainly this solution is not of *P*-type and numerical experiments confirm that (3.1) with $\epsilon=2$ is not integrable; see Fig. 8.

So far our analysis has not dealt at all with the harmonic square frequencies A, B which enter at higher order. Their values are determined by explicitly carrying out the expansions in each case and making sure that no contradictions arise in the evaluations of the coefficients a_k, b_k in the Laurent series of x and y. It turns out that in Cases 1(a) and 2 we must choose $A=B$, while in 1(b), and (c) any A, B are possible.

By a similar analysis we find that the general solution of the Hamiltonian (2.2) has the Painlevé

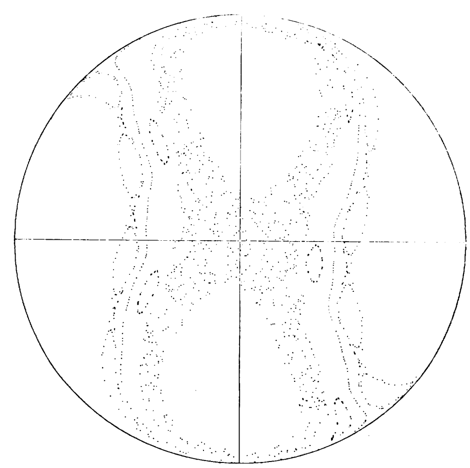

FIG. 8. Surface of section x, $\dot{x}$ ($y=0, \dot{y}\geq0$) of the generalized Hénon-Heiles (2.1) with $\epsilon=2$, $A=1$, $B=3$, and energy $E=0.205$. The presence of large "stochastic" regions indicates nonintegrability.

property only if

$$\begin{aligned}&(\mathrm{a})\ A=B \text{ and } \sigma=\rho=1,\\&(\mathrm{b})\ A=B,\ \sigma=1,\ \rho=3.\end{aligned} \tag{3.11}$$

In these cases, it is also possible to derive the second integral explicitly: written in terms of polar coordinates $x=r\cos\theta$, $y=r\sin\theta$, the Hamiltonian (2.2) [with (3.11a)] is independent of θ. This means that the "angular momentum"

$$r^2\dot{\theta}=x\dot{y}-y\dot{x}=\text{const} \tag{3.12}$$

is conserved, as can be directly verified by differentiating (3.12) and substituting in the equations of motion. In the case (3.11b) the equations uncouple in $(x+y)$, $(x-y)$ variables and the second integral is

$$\dot{x}\dot{y}+Axy+xy(x^2+y^2)=\text{const},$$

as was also observed by Yoshida.[34]

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APPENDIX: THE FREE-END TODA LATTICE ($N=3$)

In this appendix we find all cases for which the solutions of the three-particle, free-end Toda Hamiltonian,

$$H=\frac{1}{2m_1}p_1^2+\frac{1}{2m_2}p_2^2+\tfrac{1}{2}p_3^2+e^{\epsilon(q_1-q_2)}+e^{q_2-q_3}, \tag{A1}$$

admit no movable singularities, other than poles. We first change variables by defining[8]

$$\begin{aligned}&a_1\equiv\tfrac{1}{2}e^{\epsilon(q_1-q_2)/2},\quad a_2\equiv\tfrac{1}{2}e^{(q_2-q_3)/2},\\&b_k\equiv-\frac{p_k}{2m_k},\end{aligned} \tag{A2}$$

$k=1, 2, 3$, with m_3 normalized to one. In terms of the a_k, b_k's, the equations of motion become (in the center-of-mass frame)

$$\dot{a}_1=\epsilon a_1(b_2-b_1), \tag{A3a}$$

$$\dot{a}_2=-a_2[(1+m_2)b_2+m_1b_1], \tag{A3b}$$

$$\dot{b}_1=\frac{2\epsilon}{m_1}a_1^2, \tag{A3c}$$

$$\dot{b}_2=\frac{2}{m_2}(a_2^2-\epsilon a_1^2),\quad m_1,m_2,\epsilon>0 \tag{A3d}$$

where we have made use of the total momentum integral $m_1b_1+m_2b_2+b_3=0$ to eliminate b_3.

The dominant behavior[11] of $a_1(t)$, $a_2(t)$ near a pole in the complex t plane $t=t_0$ can be found by letting $a_1\sim c_1(t-t_0)^p$, $a_2\sim c_2(t-t_0)^q$ in (A3), where p, q are as yet undetermined. One easily finds that three choices are available

$$\begin{aligned}&(\mathrm{i})\ a_1\sim c_1\tau^p+\cdots,\quad a_2\sim c_2\tau^{-1}+\cdots,\quad p>-1,\\&(\mathrm{ii})\ a_1\sim c_1\tau^{-1}+\cdots,\quad a_2\sim c_2\tau^q\cdots,\quad q>-1\\&(\mathrm{iii})\ a_1\sim c_1\tau^{-1}+\cdots,\quad a_2\sim c_2\tau^{-1}+\cdots,\end{aligned} \tag{A4}$$

as $\tau\to 0$, with $\tau\equiv t-t_0$. We now examine these cases in detail and find all possible (p,q,ϵ,m_1,m_2) values for which no branch points or essential singularities can arise in the expansions (i), (ii), (iii) of a_1, a_2 (or in the expansions of simple functions of a_1 and a_2 as a_1^2 or a_2^2, etc.; see below).

Starting with (i) we find that equations (A3) yield the dominant behavior

$$\begin{aligned}&a_1\sim c_1\tau^p+\cdots,\\&a_2\sim i\left[\frac{m_2}{2(1+m_2)}\right]^{1/2}\frac{1}{\tau}+\cdots,\\&b_1\sim\frac{2c_1^2\epsilon}{(2p+1)m_1}\tau^{2p+1}+\cdots,\\&b_2\sim\frac{1}{(1+m_2)\tau}+\cdots,\end{aligned} \tag{A5}$$

where c_1 is an arbitrary constant and

$$p=\frac{\epsilon}{1+m_2},\quad 2p=\text{integer}(>0) \tag{A6}$$

so that the dominant behavior of b_1 does not introduce a branch point. Following the method described in Sec. III we find that the resonance equation associated with (A5), (A6) is

$$r(2p+1+r)(r+1)(r-2)=0\ . \tag{A7}$$

Thus the solutions of (A1) have the Painlevé property in this case with 3 arbitrary constants available: t_0, c_1, and one entering at $r=2$.

We now turn to case (ii), (A4). Examining the dominant behavior of the a's and b's we find

$$q=\frac{m_1}{\epsilon(m_1+m_2)},\quad 2q=\text{integer}(>0) \tag{A8}$$

analogous to (A6). The resonance condition (see Sec. II) in this case is

$$(r+1)r(r-1)(r-2)=0\ , \tag{A9}$$

and full dimensionality is obtained since there are four arbitrary constants here: t_0, c_2, f, and g entering at $r=-1$, 0, 1, and 2, respectively. Carrying out the expansions explicitly we find

$$\begin{aligned}
a_1 &\sim \frac{i}{\epsilon}\left[\frac{m_1m_2}{2(m_1+m_2)}\right]^{1/2}\left[\frac{1}{\tau}+g\tau+\cdots\right],\\
a_2 &\sim c_2\tau^q\left[1-\frac{\epsilon f}{2-\epsilon}\tau+\cdots\right],\\
b_1 &\sim \frac{m_2}{\epsilon(m_1+m_2)\tau}+f-\frac{2m_2g}{\epsilon(m_1+m_2)}\tau+\cdots\ ,\\
b_2 &\sim \frac{-m_1}{\epsilon(m_1+m_2)\tau}+f+\frac{2m_1g}{\epsilon(m_1+m_2)}\tau+\cdots\ .
\end{aligned} \tag{A10}$$

We remark here that, in the case of p in (A6) [or q in (A8)] being a half integer, the expansion of a_1 in (A5) [and/or that of a_2 in (A10)] would suffer from the presence of movable branch points. This could be remedied, however, with a simple change of variables $a\equiv a_1^2$ (or $a\equiv a_2^2$), which has only movable poles, resulting in the Painlevé property being recovered.

A similar analysis in case (iii), (A4) leads to the resonance condition

$$r^2-r-2M=0\ , \tag{A11}$$

where

$$M\equiv\frac{[m_1+\epsilon(m_1+m_2)](1+m_2+\epsilon)}{\epsilon m_2(m_1+m_2+1)} =\frac{(1+p)(1+q)}{1-pq}\ , \tag{A12}$$

cf. (A6), (A8). For the Painlevé property, (A11) must have integer roots only and this implies that

$$M=\frac{(1+p)(1+q)}{1-pq}=\frac{n(n+1)}{2}\ ,\quad n=0,1,2\ldots \tag{A13}$$

with $2p$ and $2q$ positive integers. Equation (A13) has five possible solutions:

(a) $p=q=\frac{1}{2}$, with $n=2$ and resonances at $r=-2$, -1, 2, 3. In this case (A6) and (A8) yield

$$m_1=\frac{\epsilon(2-1)}{2-\epsilon},\quad m_2=2\epsilon-1,\quad \tfrac{1}{2}<\epsilon<2\ . \tag{A14}$$

All these cases are equivalent within scaling to the equal mass Toda lattice $m_1=m_2=\epsilon=1$, which is known to be integrable.[24,25]

(b) $p=\frac{1}{2}, q=1$, or $p=1$, $q=\frac{1}{2}$ both with $n=3$ and resonances at $r=-3$, -1, 2, 4. For $p=\frac{1}{2}$, $q=1$ Eqs. (A6) and (A8) give

$$m_1=\frac{\epsilon(\epsilon-1)}{2-\epsilon},\quad m_2=\epsilon-1,\quad 1<\epsilon<2\ . \tag{A15}$$

All these are also equivalent within scaling. The case $p=1$, $q=\frac{1}{2}$ is identical to (A15) with $\epsilon\rightarrow 2\epsilon$.

(c) $p=\frac{1}{2}$, $q=\frac{3}{2}$ or $p=\frac{3}{2}$, $q=\frac{1}{2}$, both with $n=5$ and resonances at $r=-5$, -1, 2, 6. For $p=\frac{1}{2}$, $q=\frac{3}{2}$ (A6) and (A8) yield

$$m_1=\frac{3\epsilon(2\epsilon-1)}{2-3\epsilon},\quad m_2=2\epsilon-1,\quad \tfrac{1}{2}<\epsilon<\tfrac{2}{3}\ . \tag{A16}$$

Again ϵ can be scaled out of the equations of motion and the second case $p=\frac{3}{2}$, $q=\frac{1}{2}$ lead to (A16) with $\epsilon\rightarrow\epsilon/3$.

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