Comments

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Comment on "Electron diffusion under the influence of an electric field near absorbing boundaries." I

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A simple rigorous proof is given of the complementarity theorem introduced by Lowke, Parker, and Hall (LPH) to relate electron back diffusion to forward diffusion. It provides additional insight from which we conclude (i) that the theorem does not strictly apply to the situation discussed by LPH, and (ii) that for certain types of Monte Carlo simulations of electron-swarm behavior, the relaxation distance of the injected electrons can be minimized by injecting them into the forward hemisphere with a mean energy substantially higher than the anticipated equilibrium value. The Monte Carlo simulation data of Braglia and Lowke provide an interesting example of the effects discussed, and an improved version of this type of simulation is proposed. It employs an ideal source, having the same velocity distribution as the electrons arriving at an absorbing anode. In the steady-state distribution generated by such a source all the electrons carry useful information, even when the source is placed arbitrarily close to an absorbing anode.

The paper by Lowke, Parker, and Hall[1] (LPH) is devoted primarily to an analysis of the spatial dependence of the density and transport coefficients of a continuous stream of electrons impinging on an absorbing planar anode under the action of an applied uniform electric field. In Sec. V of the paper, the situation where electrons back diffuse upstream against the field, from a planar electron source situated in space, is shown to be complementary to the situation near an absorbing anode, where the electrons are diffusing forward, with the applied field. LPH assume that the complementarity applies when the source electrons have the equilibrium energy distribution appropriate to the applied field, and when there is an absorbing electrode a distance "h" upstream from the source. In this Comment we point out that, for the complementarity theorem to apply, the source must be specified very differently than assumed by Lowke *et al.*, and any upstream boundary must be sufficiently far from the source as to have no effect on the behavior of the electrons. These conclusions follow directly from a simple proof of the complementarity relationship, now given via a simple gedanken experiment, illustrated in Fig. 1.

Consider a uniform flow of electrons, represented by o in Fig. 1(a), far from boundaries. The number of electrons per unit volume having velocities in the range $\vec{v}$ to $(\vec{v}+d\vec{v})$ is represented by $f_{\infty}(\vec{v})d\vec{v}$, and their total density by

$$n_{\infty}=\int f_{\infty}(\vec{v})d\vec{v}\ .$$

Both n_{∞} and $f_{\infty}(\vec{v})$ are independent of position z and are appropriate to the applied field E. The electron flux is Γ_e, also independent of position. At some instant we insert into this situation a plane at $z=0$ having the following special properties: (a) all electrons pass freely through the plane, i.e., it has no effect on their velocity vector and (b) any o electron crossing the plane becomes an x electron, while any x electron crossing the plane remains an x electron. The presence of such a plane will result in the situation represented by Fig. 1(b).

At all times the sum of the o and x electrons gives the original equilibrium uniform distribution of electrons characterized by n_{∞} and $f_{\infty}(\vec{v})$. In particular, after the establishment of a new steady state we have

$$n_{o}(z)+n_{x}(z)=n_{\infty} \tag{1}$$

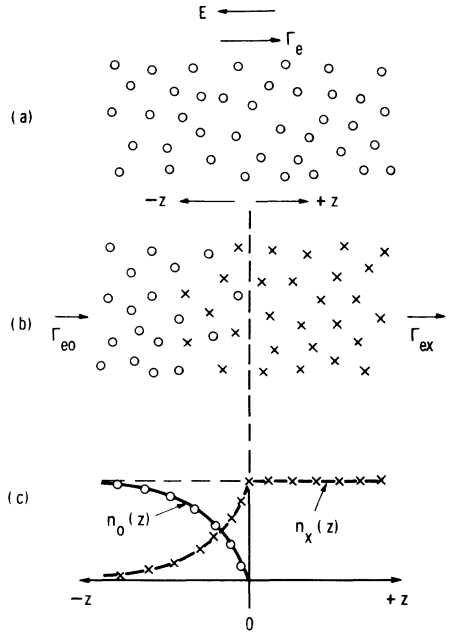

FIG. 1. Illustrating the gedanken experiment discussed in the text: (a) represents the situation in a uniform field far from boundaries and (b) represents the new steady state after the introduction of the plane at $z=0$ which transforms o electrons into x electrons. The resulting spatial variation of their densities is represented in (c). Note that $0<n_o(0)<<n_{oo}$.

and

$$f_o(z,\vec{v})+f_x(z,\vec{v})=f_{oo}(\vec{v})\ . \tag{2}$$

Equation (1) is illustrated in Fig. 1(c). The o electrons are distributed in space as if there were an absorbing boundary at $z=0$, while the x electrons are distributed as if there were a planar source of strength Γ_e at $z=0$. Hence, Eqs. (1) and (2) represent the complementarity relationship invoked by LPH. Note, however, that our gedanken experiment does not exactly simulate their situation. For it to do so we must add a second plane at $z=-h$, which absorbs x electrons but is transparent to o electrons. However, if this plane is situated such that it absorbs any electrons, Eqs. (1) and (2) can no longer hold. To restore complementarity the plane at $z=-h$ must be removed or made conservative. For example, it could be redefined to transmit x electrons as o electrons, rather than absorb them, but again this does not simulate the situation discussed by LPH. We must conclude that the complementarity theorem of LPH does not strictly apply to the situation they discuss.

Returning to our gedanken experiment in its original form, the characteristics of the planar source may be prescribed, in principle, by noting that in the new steady state, represented by Figs. 1(b) and 1(c), this source is due to the conversion to x electrons of the o electrons arriving at $z=0$. Hence, the source required for the complementarity theorem to hold is characterized by the velocity distribution of the arriving o electrons $f_o(o,\vec{v})$. One of the important results from LPH is that $f_o(o,\vec{v})$ differs substantially from $f_{oo}(\vec{v})$, as may be seen in their Figs. 1, 2, and 3. Hence, we would not expect complementarity to hold if we were to adopt LPH's assumption that the planar source is characterized by $f_{oo}(\vec{v})$.

Since in the new steady state, represented by Figs. 1(b) and 1(c), there can be no o electrons downstream from $z=0$, it follows from Eqs. (1) and (2) that the x electrons have the equilibrium velocity distribution $f_{oo}(v)$ and density n_{oo} at all downstream positions. Thus, with the source characterized by $f_o(o,v)$, there is no relaxation distance, making it ideal for use in Monte Carlo modeling studies of this situation, provided it can be adequately defined.

We know *a priori* that $f_o(o,\vec{v})$ contains only forward directed velocities. Furthermore, $f_x(o,\vec{v})$ consists of relatively few source electrons having mean energy $\bar{\epsilon}_o(o)$, and predominantly of electrons which, since their introduction as source electrons, have been scattered through adjacent regions of z space and returned to $z=0$. Their net energy gain from the field is zero, and they will have lost energy to the gas via collisions. It follows that every x electron at $z=0$ either has its original source energy, or a lower energy. Hence, we expect that $\epsilon_{oo}<\bar{\epsilon}_o(o)$.

This prediction is entirely consistent with the numerical results of LPH, who found that the electrons immediately adjacent to an absorbing anode had a mean energy substantially higher than the equilibrium value, the degree of enhancement depending somewhat on the energy dependence of the momentum-transfer scattering cross section σ_m. Braglia and Lowke[2] treated the case of $\sigma_m \propto \epsilon^{1/2}$ using the numerical techniques of LPH to solve the Boltzmann equation and also via a Monte Carlo simulation. The two methods agreed, giving $\bar{\epsilon}_{oo}<\epsilon_o(o)$, as in the cases treated by LPH. However, additional effects can arise from the energy dependences of the electron drift and diffusion processes.[1] In the cases treated by LPH[1] and Braglia and Lowke,[2] these effects appear to be relatively small. In more recent work, Robson[3] discusses the constant collision-frequency case ($\sigma_m \propto \epsilon^{-1/2}$), and finds that in this case

$\bar{\epsilon}_{oo}=(\frac{3}{5})\bar{\epsilon}_o(o)$. Thus the consensus of these[1-3] and earlier works[4,5] is that, in general, $\bar{\epsilon}_{oo}<\bar{\epsilon}_o(o)$.

The Monte Carlo results of Braglia and Lowke[2] are particularly pertinent to the present discussion since they serve to illustrate the behavior adjacent to an absorbing anode, and the behavior in the vicinity of a planar source. Figure 2 is reproduced from the paper by Braglia and Lowke. We note first that in this example the electrons impinging on the anode have a mean energy which is almost twice the equilibrium value. Braglia and Lowke used a planar source 1 cm from the anode and injected electrons isotropically with the equilibrium energy distribution. The resulting steady-state electron mean energy at the source position was substantially lower than the injected mean energy, which the electrons regained only after drifting downstream approximately 0.4 cm. In the light of the preceding discussion we would expect that in this example this relaxation distance could be essentially eliminated by employing an anisotropic source (forward directed) with a mean energy of approximately 0.6 eV, based on Braglia and Lowke's own data in Fig. 2.

In general, for Monte Carlo simulation of real gases, the information required for prescribing the ideal source discussed above, namely, the velocity distribution of the electrons impinging on an absorbing anode, is not available. One may perform a Monte Carlo simulation to generate this information,[6] but in view of the computer costs involved, it is desirable to design the simulation to generate the maximum amount of information, without simulating more collisions than is necessary. We arrive at such a design via a second gedanken experiment, represented in Fig. 3.

The experiment is an extension of that represented in Fig. 1, and involves two conservative planes. The first of these, at $z=-d$ converts o electrons to □ electrons, while the second plane at $z=$o converts □ electrons to x electrons.

In this situation the complementarity relationships become

$$f_o(z,\vec{v})+f_\square(z,\vec{v})+f_x(z,\vec{v})=f_{oo}(\vec{v}) \tag{3}$$

and

$$n_o(z)+n_\square(z)+n_x(z)=n_{oo}\ . \tag{4}$$

The source of the □ electrons is the flux of o electrons impinging on the plane at $z=-d$, while the source of the x electrons is the flux of □ electrons arriving at $z=0$. These two sources are identical and are characterized by the distribution

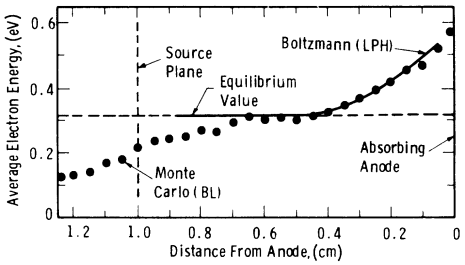

FIG. 2. A reproduction of Fig. 2 from Ref. 2 (Braglia and Lowke). The Monte Carlo results were obtained by injecting electrons isotropically at the source plane with the equilibrium energy distribution, whose mean energy is indicated (0.32 eV). Using equilibrium values $(D_L/w_e)=0.09$ cm and $(D_T/w_e)=0.31$ cm for the conditions used.

impinging on an absorbing anode. Thus the spatial and velocity distribution of the □ electrons produced by this gedanken experiment could be produced by an appropriate Monte Carlo simulation in which this ideal source is placed an arbitrary distance d from an absorbing anode. Let us now consider the information available from such a simulation, i.e., contained in the distribution $f_\square(z,v)$.

In terms of our second gedanken experiment it is desirable to evaluate

$$f(z,\vec{v})=f_\square(z,\vec{v})+f_o(z,\vec{v})$$

for negative values of z, since this sum represents the behavior of a steady electron flux approaching an absorbing anode. To do so requires that $f_o(z,\vec{v})$ be expressed in terms of $f_\square(z,\vec{v})$, since only the

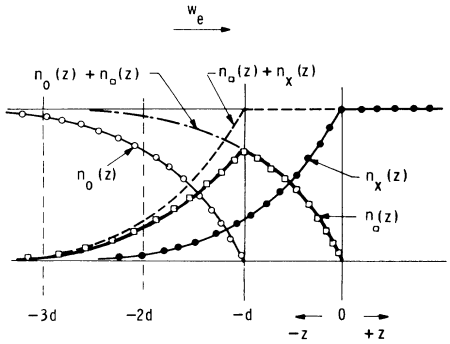

FIG. 3. Illustrating the second gedanken experiment discussed in the text. The conservative plane at $z=-d$ transforms o elecrons to □ electrons, with no other effects, while that at $z=0$ transforms □ electrons to x electrons. As in Fig. 1(c), these curves represent the local integrated densities $n(z)$. Similar sets could be drawn for a particular $f(z,\vec{v})$, but the shapes would depend on $\vec{v}$.

latter would be known from the proposed simulation.

Comparison of the two gedanken experiments shows that $f_x(z,\vec{v})$ is the same in both, while $f_o(z,\vec{v})$ in the first experiment [see Fig. 1(c)] is the same as $f_o((z-d),\vec{v})$ in the second experiment. It follows that in our second experiment

$$f_x(z,\vec{v})+f_o((z-d),\vec{v})=f_\infty(\vec{v}) . \quad (5)$$

Equations (3) and (5) give for our desired function

$$f(z,\vec{v})=f_o(z,\vec{v})+f_\square(z,\vec{v})$$
$$=f_o((z-d),\vec{v}) . \quad (6)$$

This equation may be used as a general recurrence formula

$$f_o((z+ld),\vec{v})+f_\square((z+ld),\vec{v})$$
$$=f_o((z+ld-d),\vec{v}) , \quad (7)$$

where l is an integer. Successive application of Eq. (7) to Eq. (6) gives

$$f(z,\vec{v})=f_o(z,\vec{v})+f_\square(z,\vec{v})$$
$$=\sum_{l=0}^{\infty} f_\square((z+ld),\vec{v}) . \quad (8)$$

In general, the number of terms in the series will be finite, depending on the value of z. For $-d<z<0$ all terms other than $l=0$ are zero, meaning that in this region $f_\square(z,\vec{v})$ gives the desired function directly, as indicated by Fig. 3. For $z\to-\infty$ the series has an infinite number of terms whose sum gives $f_\infty(\vec{v})$, the equilibrium distribution. This infinite summation is best represented by reversing the order of the terms in which case we may write

$$f_\infty(\vec{v})=\sum_{l=1}^{\infty} f_\square((z-ld),\vec{v}) \quad (9)$$

with the stipulation that $0<z<d$. Integrating over this range of z gives

$$df_\infty(\vec{v})=\int_o^d \sum_{l=1}^{\infty} f_\square((z-ld),\vec{v})dz ,$$

where we recognize the right-hand side as simply the totality of $\square$ electrons. Hence, we may write

$$f_\infty(\vec{v})=d^{-1}\int_{-\infty}^{0} f_\square(z,\vec{v})dz . \quad (10)$$

Thus, although $f_\square(z,\vec{v})$ is nowhere identical locally in character to $f_\infty(\vec{v})$, it can provide $f_\infty(v)$ via Eq. (9) or (10). Use of the latter is clearly to be preferred since it makes use of all the electrons in the simulation space, thereby optimizing the statistics of the Monte Carlo result.

In summary, the distribution of $\square$ electrons contains the following information.

(i) At $z=0$ it gives the velocity distribution impinging on an absorbing anode, which prescribes the ideal source required at $z=-d$ to maintain the desired steady state.

(ii) In the region $-d<z<0$ it represents directly the spatial and velocity distribution adjacent to an absorbing anode.

(iii) In the region $z<-d$ this distribution may be derived from $f_\square(z,\vec{v})$ via Eq. (8).

(iv) The totality of $\square$ electrons gives $f_\infty(\vec{v})$, the position independent velocity distribution which would apply far from boundaries.

This distribution of $\square$ electrons can be generated in a Monte Carlo simulation by using a planar source at distance d upstream from an absorbing anode. The proposed simulation is started by using an initial guess at the appropriate source. The strength (electrons/cm^2/sec) of this source is maintained constant throughout the simulation, but its character is continually modified by recycling the "trained" electrons arriving at the absorbing anode, reintroducing them at the source plane, at $z=-d$, with the velocity they had on arrival at $z=0$. After a time comparable to d/w_e the simulation will become increasingly regenerative, with the source consisting predominantly of trained recycled electrons. Sampling of the electron distribution at suitably chosen time intervals will establish when it is acceptably close to steady state. Subsequent sampling can then provide all or part of the information listed above as desired.

In conclusion, a simple proof has been given of the complementarity theorem introduced by LPH,[1] and the theorem has been used to show that the ideal source for Monte Carlo simulations of the type performed by Braglia and Lowke[2] has a distribution of velocities the same as the electrons impinging on an absorbing anode. Previous work[1-5] indicates that this ideal source will have a mean energy substantially higher than the anticipated equilibrium energy that would exist far from boundaries. A simulation procedure is proposed which generates the required ideal source, and which should converge optimally to the desired steady state, which is such that all the electrons carry useful information. A particularly important conclusion is that, with the proposed simulation procedure, the source may be placed arbitrarily close to the absorbing anode.

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