Dirac equation of the electron in a magnetic field

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To every solution of the time-independent Dirac equation for the electron in an arbitrary magnetic field there belongs a second solution. A characteristic twofold degeneracy, suggested by this circumstance, is shown to be excluded for the ground state at the energy $E=mc^2$, with the further distinction of this state that the current density vanishes here at all points in space. The conditions under which the excited states exhibit such a degeneracy are discussed and in the case of its occurrence are shown to allow the construction of a particularly symmetrical pair of mutually orthogonal solutions.

Among the earliest applications of the Dirac equation, Rabi[1] obtained a complete set of solutions for the stationary states of the electron in a homogeneous magnetic field. Based upon a suitable gauge in the choice of the vector potential, his treatment allows each state at a given energy to be further characterized by the angular momentum around an axis in the field direction. One deals here with an infinitely high accidental degeneracy insofar as the same energy permits to assign to the angular momentum any one of its eigenvalues. It is of particular relevance to the following remarks that such an assignment uniquely defines a state at the lowest energy $E=mc^2$ but leaves a twofold degeneracy for all excited states.

A detailed discussion of the special aspects, encountered for a homogeneous field, is found in two papers by Johnson and Lippmann.[2] In the second paper they further point out that the existence of a certain constant of motion in an arbitrary magnetic field permits a more general twofold degeneracy and they specify the conditions necessary to actually deal with a degenerate state. In a more recent paper by Aharanov and Casher,[3] the case of a homogeneous field is extended to include a magnetic field which is still unidirectional but allowed to maintain an arbitrary magnitude along any line of force. The exceptional property of the ground state is utilized to show that the choice of a divergence-free vector potential directly yields here a set of solutions . In analogy to the separate states in a homogeneous field, characterized by the angular momentum, they may likewise exhibit a degeneracy with a multiplicity, however, which is limited by the total flux of the field. While no such solutions are available for the stationary states at higher energies, it is observed for those, selected to remain constant in the field direction, that they are degenerate with respect to a reversal of the spin orientation.

It will be shown that the procedure of Aharonov and Casher can be generalized to lead in a different way to conclusions about the stationary states in an arbitrary magnetic field, including those which have previously been reached by Johnson and Lippmann. In particular, it becomes evident from this procedure that the type of twofold degeneracy possible for higher energies cannot occur at the ground state of energy $E=mc^2$ and that the current density is here necessarily zero in all points of space.

A convenient albeit less concise formulation is obtained by separating the four components of the wave function ψ into functions ψ^a and ψ^b of two components each, writing $\psi^a_{1,2}$ for $\psi_{1,2}$ and $\psi^b_{1,2}$ for $\psi_{3,4}$. For a vector potential $\vec{A}$ and in the absence of a scalar potential, the time-independent Dirac equation then appears in the form of the simultaneous equations,

$$c\vec{\sigma}\cdot\vec{\pi}\psi^a=(E+mc^2)\psi^b\ , \quad (1a)$$

$$c\vec{\sigma}\cdot\vec{\pi}\psi^b=(E-mc^2)\psi^a\ , \quad (1b)$$

where $\vec{\sigma}$ is the vector of the Pauli matrices $\sigma_{x,y,z}$

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and where

$$\vec{\pi}=\vec{p}+\frac{e}{c}\vec{A} \tag{2}$$

(see Ref. 4). The discussion will refer to states of positive energy E but applies equally to those of negative energy since the same equations with the opposite sign of E are obtained by a mere change of notation, replacing ψ^a by ψ^b and ψ^b by $-\psi^a$.

Upon application of the operator $c\vec{\sigma}\cdot\vec{\pi}$ on both sides of Eqs. (1), it is seen that they are likewise satisfied by the functions

$$\phi^a=\kappa c\vec{\sigma}\cdot\vec{\pi}\psi^a , \tag{3a}$$

$$\phi^b=\kappa c\vec{\sigma}\cdot\vec{\pi}\psi^b , \tag{3b}$$

with κ as a constant of proportionality.[5] In view of Eqs. (1), the equivalent connection between the solution $\psi^{a,b}$ and $\phi^{a,b}$ is given by

$$\phi^a=\kappa(E+mc^2)\psi^b , \tag{4a}$$

$$\phi^b=\kappa(E-mc^2)\psi^a . \tag{4b}$$

Although the existence of both solutions at the same energy is suggestive of a twofold degeneracy, it would be erroneous to thereby conclude upon its inevitable occurrence. The conclusions actually to be drawn from this fact will be separately discussed for $E=mc^2$ and for all higher energies.

Starting with $E=mc^2$, this value of the energy is distinguished from any other positive value by the fact that both ϕ^a and ϕ^b are found to be zero so that the corresponding state is ruled out. Indeed, it follows here directly from Eq. (4b) that $\phi^b=0$. To see that ϕ^a vanishes, consider the conjugate complex of Eq. (1a), multiplied on both sides with ψ^b. Since $\vec{\sigma}\cdot\vec{\pi}$ is a Hermitian operator, one then obtains

$$(\psi^{a*}c\vec{\sigma}\cdot\vec{\pi}\psi^b)=2mc^2(\psi^{b*}\psi^b) , \tag{5}$$

where the round parentheses here and in the subsequent formulas indicate integration over the space variables and summation over both indices used to label functions of two components. Since for $\phi^b=0$, according to Eq. (3b),

$$c\vec{\sigma}\cdot\vec{\pi}\psi^b=0 ,$$

it follows from Eq. (5) that

$$(\psi^{b*}\psi^b)=0 ,$$

and therefore necessarily that

$$\psi^b=0 . \tag{6}$$

In view of Eq. (4a) the result $\phi^a=0$ is thus confirmed.

Before considering ψ^a, the preceding result can be used to arrive at a separate conclusion. At the end of Rabi's paper[1] it is noted that the current density vanishes in the ground state. This fact is not limited, however, to the special case of a homogeneous field which he considered, but can be seen to describe a characteristic property of the ground state in an arbitrary magnetic field. Indeed, the notation used here leads to the expression

$$\vec{i}=ec(\psi^{a*}\vec{\sigma}\psi^b+\psi^{b*}\vec{\sigma}\psi^a)$$

for the current density and hence to the result $\vec{i}=0$ as a consequence of Eq. (6). This result can be regarded as the counterpart to the classical fact that the lowest energy corresponds to an electron at rest and remaining at rest in any magnetic field due to the absence of a Lorentz force.

As a further consequence of Eq. (6), one obtains from Eq. (1a) the equation

$$c\vec{\sigma}\cdot\vec{\pi}\psi^a=0 \tag{7}$$

for the determination of ψ^a (see Ref. 6). Combined with the result of Eq. (6) for ψ^b, a single-valued normalizable solution of Eq. (7) describes a stationary ground state at the energy $E=mc^2$. In case of an accidental degeneracy, each of the different solutions is to be characterized by the eigenvalues Ω' of an operator Ω such that

$$\Omega\psi^a=\Omega'\psi^a . \tag{8}$$

The choice of the operator depends on the particular symmetry properties of the vector potential so as to fulfill the condition

$$[\Omega,\vec{\sigma}\cdot\vec{\pi}]\psi^a=0 , \tag{9}$$

which is necessary for Eq. (8) to be compatible with Eq. (7).[7] In the absence of any such symmetry, Eq. (7) is sufficient, on the other hand, to uniquely determine the function ψ^a so that one deals then with a nondegenerate ground state.

In contrast to the ground state, it follows for $E>mc^2$, in view of Eqs. (4), that to every single-valued and normalizable solution $\psi^{a,b}$ of Eqs. (1) there belongs a solution $\phi^{a,b}$ which is likewise single-valued and normalizable with a finite value of the factor κ. This fact alone is not sufficient, however, to prove a twofold degeneracy since it includes the possibility of a mere repetition of $\phi^a=\psi^a$ and $\phi^b=\psi^b$, seen to be allowed by Eqs. (4) with $\kappa=\pm(E^2-m^2c^4)^{-1/2}$. One therefore has to conclude upon the absence of such a degeneracy if there is no other choice than the identity of $\phi^{a,b}$ and $\psi^{a,b}$. It will be seen, on the other hand, that this identity may appear merely as a particular solution among a set of linearly independent functions which then allow to select a pair of normalized orthogonal solutions representing two different states at the same energy.

For this purpose, let

$$\psi^a=\lambda f^a , \quad (10a)$$

$$\psi^b=\frac{1}{\lambda}f^b , \quad (10b)$$

with

$$\lambda=\left[\frac{E+mc^2}{E-mc^2}\right]^{1/4} , \quad (11)$$

so as to obtain the reformulation

$$c\vec{\sigma}\cdot\vec{\pi}f^a=(E^2-m^2c^4)^{1/2}f^b , \quad (12a)$$

$$c\vec{\sigma}\cdot\vec{\pi}f^b=(E^2-m^2c^4)^{1/2}f^a , \quad (12b)$$

of Eqs. (1). The previous recognition of $\phi^{a,b}$ as an alternate solution to $\psi^{a,b}$ then reappears in the fact that Eqs. (12) remain satisfied if f^a and f^b are interchanged. Indeed, with

$$\phi^a=\lambda f^b , \quad (13a)$$

$$\phi^b=\frac{1}{\lambda}f^a , \quad (13b)$$

obtained from this interchange in Eqs. (10) and with λ from Eq. (11), one verifies the relation between these two solutions, expressed in Eqs. (4), by choosing $\kappa=(E^2-m^2c^4)^{-1/2}$. Moreover, the general solution of Eqs. (12) is seen to be given by

$$f^a=\alpha u+\beta v , \quad (14a)$$

$$f^b=\alpha u-\beta v , \quad (14b)$$

where α and β are arbitrary constants and where u and v are solutions of the equations

$$c\vec{\sigma}\cdot\vec{\pi}u=+(E^2-m^2c^4)^{1/2}u , \quad (15a)$$

$$c\vec{\sigma}\cdot\vec{\pi}v=-(E^2-m^2c^4)^{1/2}v , \quad (15b)$$

which arise from those for the sum or difference of corresponding sides in Eqs. (12a) and (12b).[8]

The property of a stationary state at the energy E is thus determined by the solutions of Eqs. (15). Similarly to the ground state, the occurrence of an accidental degeneracy leads to the characterization of different solutions by the eigenvalues Ω' of an operator Ω, to be chosen according to the particular symmetry properties of the vector potential. Denoting by w either of the functions u and v, one has then,

$$\Omega w=\Omega' w , \quad (16)$$

with the requirement

$$[\Omega,\vec{\sigma}\cdot\vec{\pi}]w=0 , \quad (17)$$

in order to be compatible with Eqs. (15). No further degeneracy occurs if either Eq. (15a) or Eq. (15b), but not both, have a single-valued normalizable solution at a given value of E, so that either $v=0$ or $u=0$, respectively. In the first case one has from Eqs. (14), $f^a=f^b$ with the consequence, in view of Eqs. (10) and (13), that the alternate solution $\phi^{a,b}$ amounts merely to a repetition of $\psi^{a,b}$. Whereas in the second case $f^a=-f^b$, no different solution results here either since it yields no more than an irrelevant change of sign.

On the other hand, there remains a twofold degeneracy if Eqs. (15) permit a single-valued normalizable solution for u as well as v at the same value of E, both uniquely defined either in the absence of an accidental degeneracy or pertaining to a given eigenvalue Ω', according to Eq. (16).[9] Even then it does not follow that $\phi^{a,b}$ necessarily represents an alternate solution to $\psi^{a,b}$ since the particular choice $\beta=0$ or $\alpha=0$ of the constants in Eqs. (14) still results in $f^a=f^b$ or $f^a=-f^b$, respectively. The fact that, nevertheless, one deals here with a degeneracy shall be explicitly demonstrated by means of another suitable choice of these constants which leads to a pair of mutually orthogonal functions, $\psi^{a,b}$ and $\phi^{a,b}$, related to each other by Eqs. (3) or (4).

Assuming u and v to be normalized so that

$$(u^*u)=(v^*v)=1 , \quad (18)$$

and considering, in view of Eqs. (15), that they pertain to different eigenvalues of the Hermitian operator $c\vec{\sigma}\cdot\vec{\pi}$, u and v further satisfy the relation of orthogonality

$$(u^*v)=(v^*u)=0 . \quad (19)$$

Using these relations, it follows from Eqs. (14) that

$$(f^{a*}f^a)=(f^{b*}f^b)=|\alpha|^2+|\beta|^2 ,$$

and hence, from Eqs. (10) and (13) that

$$(\psi^{a*}\psi^a)+(\psi^{b*}\psi^b)=(\phi^{a*}\phi^a)+(\phi^{b*}\phi^b)$$
$$=(\lambda^2+1/\lambda^2)(|\alpha|^2+|\beta|^2) .$$

The normalization of both $\psi^{a,b}$ and $\phi^{a,b}$ is thus achieved if

$$(\lambda^2+1/\lambda^2)(|\alpha|^2+|\beta|^2)=1\ ,$$

or, in view of Eq. (11), by letting

$$|\alpha|^2+|\beta|^2=(E^2-m^2c^4)^{1/2}/2E\ . \tag{20}$$

Similarly, the mutual orthogonality of these two solutions requires

$$(\psi^{a*}\phi^a)+(\psi^{b*}\phi^b)=(\lambda^2+1/\lambda^2)(|\alpha|^2-|\beta|^2)=0\ ,$$

and hence

$$|\alpha|^2-|\beta|^2=0\ . \tag{21}$$

By means of the special assignment

$$\alpha=\beta=(E^2-m^2c^4)^{1/4}/2E^{1/2}\ , \tag{22}$$

satisfying Eqs. (20) and (21), and upon insertion of (22) in Eqs. (14), one thus arrives through Eqs. (10), (11), and (13) at the functions

$$\psi^a=\frac{1}{2}\left[\frac{E+mc^2}{E}\right]^{1/2}(u+v)\ , \tag{23a}$$

$$\psi^b=\frac{1}{2}\left[\frac{E-mc^2}{E}\right]^{1/2}(u-v)\ , \tag{23b}$$

$$\phi^a=\frac{1}{2}\left[\frac{E+mc^2}{E}\right]^{1/2}(u-v)\ , \tag{24a}$$

$$\phi^b=\frac{1}{2}\left[\frac{E-mc^2}{E}\right]^{1/2}(u+v)\ , \tag{24b}$$

as a particularly symmetrical pair of normalized orthogonal solutions.[10]

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[1] I. I. Rabi, Z. Phys. 49, 507 (1928).

[2] M. H. Johnson and B. A. Lippmann, Phys. Rev. 75, 828 (1949); 77, 702 (1950).

[3] Y. Aharonov and A. Casher, Phys. Rev. A 19, 2461 (1979).

[4] The form corresponding to Eqs. (1) appears in Eq. (18) of Ref. 3 for the functions $U_{\rm up}$ and $U_{\rm down}$ with the subscripts chosen to indicate opposite orientations of the spin as a consequence of the special circumstances considered. It is not possible, however, to generally assign a definite spin orientation to the functions denoted here by ψ^a and ψ^b.

[5] Equations (3) represent the generalization of Eq. (16) in Ref. 3, both being valid due to the absence of a scalar potential Φ. While its presence would merely add a term $e\Phi$ to the energy E in our Eqs. (1), it would prevent the operator $\vec{\sigma}\cdot\vec{\pi}$ from commuting with the expression contained in the parentheses on the right-hand side of these equations and hence invalidate the recognition of ϕ^a and ϕ^b as alternate solutions.

[6] Equation (7) is the generalized form of Eq. (7) in Ref. 3.

[7] The symmetry in the treatment of Ref. 1 for the homogeneous field calls for the operator

$$\Omega=\hbar\left[\frac{1}{i}\frac{\partial}{\partial\theta}+\frac{\sigma_z}{2}\right],$$

with eigenvalues $\Omega'=\hbar(m+\frac{1}{2})$ representing the angular momentum around a given axis in the z direction of the field and where θ is the angle measured around the axis. For the more general case, considered in Ref. 3, one is led to the replacement by the non-Hermitian operator

$$\Omega=\hbar\left[\frac{1}{i}\frac{\partial}{\partial\theta}+\frac{\sigma_z}{2}\right]+i\frac{e}{c}\frac{\partial\Phi}{\partial\theta}\sigma_z\ ,$$

where, in the notation of that reference, the vector potential is given by $A_x=-\partial\Phi/\partial y$, $A_y=\partial\Phi/\partial x$ with Φ as a function of $x(r,\theta)$ and $y(r,\theta)$. For the commutator of Eq. (9) one obtains here

$$[\Omega,\vec{\sigma}\cdot\vec{\pi}]=2i\frac{e}{c}\frac{\partial\Phi}{\partial\theta}\sigma_z(\sigma_x\pi_x+\sigma_y\pi_y)\ .$$

The vector potential in Ref. 1 can be derived by choosing $\Phi=Hr^2/4$ so that $\partial\Phi/\partial\theta=0$. As a result, the commutator vanishes in that case identically so that Eq. (9) is here applicable to any function ψ^a. For $\partial\Phi/\partial\theta\neq0$, however, Eq. (9) is valid only insofar as ψ^a satisfies the condition $(\sigma_x\pi_x+\sigma_y\pi_y)\psi^a=0$ which is characteristic for the ground state and its specific degeneracy noted by Aharonov and Casher.

[8] Equations (15) correspond to Eq. (9) in the discussion of an arbitrary magnetic field by Johnson and Lippmann in the second paper of Ref. 2. Indeed, the quantity F on the right-hand side of that equation represents the eigenvalue of $(\vec{\sigma}\cdot\vec{\pi})$, given in view of their Eq. (7) by $F=\pm(1/c)(E^2-m^2c^4)^{1/2}$. In the treatment presented here this is evident insofar as the

functions u and v can be interpreted to pertain to the positive and negative value of F, respectively.

[9]Given such a solution u of Eq. (15a), one can in this case introduce an operator R such that

$$v = Ru$$

yields the corresponding solution v of Eq. (15b). Equivalent to the content of Eq. (15) in the second paper of Ref. 2, a sufficient condition for the existence of this operator is then seen to be formulated by the requirement

$$\{R, \vec{\sigma}\cdot\vec{\pi}\}u = 0 ,$$

where the curly bracket indicates the anticommutator. In order for this requirement to be compatible with that for Ω, expressed in our Eq. (17) through application of the commutator to both u and v, it can further be seen that it is necessary to have

$$[R, \Omega]u = 0 .$$

[10]The distinction of this pair by an opposite orientation of the spin is a special feature of the circumstances considered in Ref. 3. Indeed, with $\pi_z = p_z$ for $A_z = 0$ and for the solutions assumed to depend only on x and y, the term $\sigma_z \pi_z$ in $\vec{\sigma}\cdot\vec{\pi}$ is to be omitted. Due to the anticommutation of the Pauli matrices, one has then

$$\{\sigma_z, \vec{\sigma}\cdot\vec{\pi}\} = 0 ,$$

thus allowing the operator R in Ref. 9 to be identified with σ_z and to yield $v = \sigma_z u$. Upon insertion of $(u \pm v) = (1 \pm \sigma_z)u$ in Eqs. (23) and (24), it then follows with $\sigma_z^2 = 1$ that $\sigma_z \psi^a = \psi^a$, $\sigma_z \psi^b = -\psi^b$, but $\sigma_z \phi^a = -\phi^a$, $\sigma_z \phi^b = \phi^b$ as the formulation for the solutions $\psi^{a,b}$ and $\phi^{a,b}$ to differ in this case through a reversal of the spin orientation.