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Stationary convective instability in a superfluid ^{3}He-^{4}He mixture

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The stationary convection instability of a superfluid mixture is considered. The criteria for the instability onset in a superfluid region of the ^{3}He-^{4}He phase diagram and in different limiting cases are obtained. It is shown that in different regions of the phase diagram, depending on the magnitude of the kinetic coefficients, the stationary-instability criterion is similar to that of either the regular binary mixture with abnormal thermal diffusion, to the pure liquid, or to the compressible pure liquid. The stability is also strongly dependent on concentration and temperature, and the critical temperature gradient rises sharply with decreasing temperature and concentration. At very low temperature (below 0.5 K) and concentration (dilute solution) the superfluid solution becomes stable with respect to stationary convection. Asymptotic behavior of the criterion in the vicinity of the λ line, the tricritical point and for infinitely dilute solutions is also estimated.

I. INTRODUCTION

The Rayleigh-Bénard problem has been the subject of considerable interest in recent years both theoretically[1,2] and experimentally.[3,4] From an experimental point of view, as shown by Ahlers,[5] cryogenic fluids have an advantage for investigation of hydrodynamic instabilities due to a high-temperature resolution and low extraneous heat transport. The He^3-He^4 mixture in a superfluid region between the λ line and coexistence curve of separation represents an unusual two-component system due to a very wide variation (several orders of magnitude) of the thermodynamic and kinetic properties.[6] The superfluid nature of this mixture also renders it a unique Bénard system.

It is well known that in an He^3-He^4 superfluid mixture a temperature gradient can exist in equilibrium in contrast to pure He II (Ref. 7). This temperature gradient leads to a corresponding concentration gradient

$$\nabla C_0 = K \nabla T_0\,, \quad K = \frac{c\dfrac{\partial}{\partial c}\left(\dfrac{S}{\rho c}\right)_{P,T}}{\dfrac{\partial}{\partial c}\left(\dfrac{Z}{\rho}\right)_{P,T}} < 0\,, \tag{1}$$

where ρ, T, and C are the density, the temperature, and the weight concentration respectively, S is the entropy per 1 cm^3 of solution and $Z = \rho(\mu_3 - \mu_4)$, μ is the chemical potential, and subscript "0" corresponds to mechanical-equilibrium conditions.

Thus, the superfluid He^4 component moves to the warm boundary and causes the light He^3 atoms to be concentrated near the cold boundary. A similar concentration distribution occurs in a regular binary mixture with a large abnormal thermodiffusion effect $k_T > 0$ (Ref. 8); in our case we also have

$$\frac{k_T}{T} = -\frac{\nabla C_0}{\nabla T_0} = -K > 0\,. \tag{2}$$

Such systems are unstable with respect to stationary convection when heated from above and with respect to oscillatory convection when heated from below,[9,10] and differ significantly from the Rayleigh-Bénard instability in a pure liquid. In the latter case the instability occurs due to a density gradient which becomes unstable in the gravitational field. But in a binary mixture it also may be caused by the separation of the time scales in relaxation of temperature and concentration fluctuations.[11]

The physical reason for this instability is clear; in the system heated from above the concentration perturbations are destabilizing whereas the temperature perturbations tend to stabilize the system. In the case where the relaxation time of the concentration fluctuations is much larger than the relaxation time of the temperature fluctuations the system becomes unstable with respect to stationary convection at the certain critical value of the temperature gradient. A similar explanation holds for the oscillatory instability (overstability) onset in the system heated from below.

As for a regular binary mixture with abnormal thermal diffusion, it is natural to expect the onset of a stationary instability in the superfluid mixture when heated from above and the onset of an oscillatory instability when heated from below. In this paper I will discuss just the stationary stability of a horizontal layer of a superfluid He^3-He^4 mixture. The oscillatory instability of this system will be the subject of the following paper.

The hydrodynamic equations in the Boussinesq approximation are discussed in Sec. II. In Sec. III the two limiting cases of stationary instability are considered: for one the dissipation of superfluid motion due to the second viscosity is large ($m > 1$), and for the other it is small ($m < 1$). In the first case, the superfluid motion is negligible, and the fluctuations of the chemical potential μ_4 relax diffusively like in a regular binary mixture. The difference from a regular mixture shows up only in relation (1) as a result of superfluidity. This causes a difference in the expression for the instability criterion compared to the regular mixture case.

In the second case ($m < 1$) the superfluid motion is essential for the convection onset. It means that fluctuations of the chemical potential μ_4 are very small and negligible. Thus, in this case only the entropy fluctuations are considered. This is analogous to the pure liquid, where only the temperature fluctuations are important. But unlike pure liquid the steady-state density gradient depends on both temperature and concentration gradients. On the other hand, due to the superfluid motion $\operatorname{div} V_n \neq 0$ even for an incompressible mixture. As a consequence of this the convection equations of a superfluid mixture are similar to the convection equations of a compressible viscous heat-conducting fluid. As shown for a compressible liquid,[12] consideration of the relation $\operatorname{div} V \neq 0$ in hydrodynamics leads to an additional dissipation, and as a result it leads to additional mechanical stability. In Sec. IV the results are given in convenient thermodynamic variables and numerical estimates and comparison with experiment are presented.

II. THE CONVECTION EQUATIONS FOR A SUPERFLUID MIXTURE

Let us consider for the sake of simplicity a typical Bénard geometry with free boundaries separated by a distance l along the vertical Z axis which are good heat conductors.

As is usual in the Boussinesq approximation of the Rayleigh-Bénard problem[13] it is assumed that the perturbations of the total mass density ρ are small and can be neglected in the hydrodynamic equations except in the buoyancy term describing the influence of the gravitational field in the Navier-Stokes equation.[2] Also any variation of properties with pressure can be neglected. The convection equations of a superfluid mixture in a general form were first presented in Ref. 14. But in Ref. 14 the fluctuations of chemical potential μ_4 are neglected, and, at the same time, it is assumed that $\operatorname{div} V_n = 0$. As will be shown below, this approximation is valid only in a certain concentration and temperature range of the He^3-He^4 phase diagram.

In the convection-free steady state the density gradient is

$$\frac{d\rho_0}{dz} = -\rho\alpha_T(1+\varphi)\frac{dT_0}{dz} . \qquad (3)$$

Here

$$\alpha_T = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_{P,c} , \quad \beta = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial c}\right)_{P,T}$$

and

$$\varphi = \frac{\beta}{\alpha_T}K$$

is the separation parameter like in a regular mixture.[9,11] In the concentration and temperature range of the phase diagram considered below the separation factor is $|\phi| \gg 1$. Therefore, the density gradient is mainly determined by the concentration gradient from (1) and is independent on sign of α_T:

$$\frac{d\rho_0}{dz} \simeq -\rho\beta K\frac{dT_0}{dz} .$$

Thus heating from above generates an unstable stratification ($d\rho_0/dz > 0$) as in a regular mixture with a large abnormal thermodiffusion effect.[9,11] An important point in solving the problem is choosing the appropriate variables in the convective equations. Two variables follow from the conditions of mechanical equilibrium[7]

$$\nabla P_0 = \rho_0\vec{g} , \quad \nabla\mu_{40} = \vec{g} \quad (\vec{g} = -g\vec{\gamma}) , \qquad (4)$$

where P is the pressure and $\vec{\gamma}$ is the unit vector along z axis. The third variable $\sigma = S/\rho c$ is the entropy per gram of He^3 atoms and is chosen in order to simplify the energy equation as shown in Ref. 14. Thus the thermodynamic perturbations can be written as

$$\begin{aligned} \rho' &= \left(\frac{\partial\rho}{\partial\sigma}\right)_{P,\mu_4}\sigma' + \left(\frac{\partial\rho}{\partial\mu_4}\right)_{P,\sigma}\mu_4' , \\ c' &= \left(\frac{\partial c}{\partial\sigma}\right)_{P,\mu_4}\sigma' + \left(\frac{\partial c}{\partial\mu_4}\right)_{P,\sigma}\mu_4' , \qquad (5) \\ T' &= \left(\frac{\partial T}{\partial\sigma}\right)_{P,\mu_4}\sigma' + \left(\frac{\partial T}{\partial\mu_4}\right)_{P,\sigma}\mu_4' . \end{aligned}$$

According to the linear stability theory the convection equations for superfluid solutions after some transformation can be written as (all symbols are the same as in Ref. 7)

$$\operatorname{div}\vec{j}' = 0\,, \quad \vec{j}' = \rho_n \vec{V}_n' + \rho_S \vec{V}_S'\,,$$

$$\frac{\partial \vec{j}'}{\partial t} + \nabla P' = \eta \Delta \vec{V}' + \vec{g}\rho' + (\tfrac{1}{3}\eta - \zeta_1 \rho - \zeta_2)\nabla \operatorname{div}\vec{V}_n'\,,$$

$$\frac{\rho_n}{\rho_S}\frac{\partial}{\partial t}\operatorname{div}\vec{V}_n' = \Delta\mu_4' - (\zeta_4 - \rho\zeta_3)\Delta \operatorname{div}\vec{V}_n'\,, \tag{6}$$

$$\frac{1}{c}\left(\frac{\partial c}{\partial \sigma}\right)_{P,\mu_4}\frac{\partial \sigma'}{\partial t} + \frac{1}{c}\left(\frac{\partial c}{\partial \mu_4}\right)_{P,\sigma}\frac{\partial \mu_4'}{\partial t} + \frac{\nabla(\rho c)}{\rho c}\vec{V}_n' + \operatorname{div}\vec{V}_n' = \frac{D}{c}\left\{\left[\left(\frac{\partial c}{\partial \sigma}\right)_{P,\mu_4} + \frac{k_T}{T}\left(\frac{\partial T}{\partial \sigma}\right)_{P,\mu_4}\right]\Delta\sigma' + \left[\left(\frac{\partial c}{\partial \mu_4}\right)_{P,\sigma} + \frac{k_T}{T}\left(\frac{\partial T}{\partial \mu_4}\right)_{P,\sigma}\right]\Delta\mu_4'\right\},$$

$$\frac{\partial \sigma'}{\partial t} + \Delta\sigma_0 \vec{V}_n' = \frac{\chi_{\text{eff}}}{\rho c T}\left[\left(\frac{\partial T}{\partial \sigma}\right)_{P,\mu_4}\Delta\sigma' + \left(\frac{\partial T}{\partial \mu_4}\right)_{P,\sigma}\Delta\mu_4'\right].$$

Since in this communication we will discuss the onset of stationary convection, only the set of equations (6) may be written for stationary perturbations in the scaled variables as[15]

$$(\nabla \operatorname{div} - \Delta)\Delta\vec{V}_n' = (\nabla \operatorname{div} - \Delta)\left[R_a\sigma + \left(\frac{l}{l_0}\right)^3 L\mu_4\right]\vec{\gamma}\,,$$

$$\left(\frac{l}{l_0}\right)^3 \Delta\mu_4 = m\Delta \operatorname{div}\vec{V}_n\left[\text{or}\left(\frac{l}{l_0}\right)^3 \mu_4 = m \operatorname{div}\vec{V}_n\right], \tag{7}$$

$$aR_a\vec{V}_n\vec{\gamma} + \left(\frac{l}{l_0}\right)^3 \operatorname{div}\vec{V}_n = \frac{P_T}{P_c}a_1R_a\Delta\sigma + \left(\frac{l}{l_0}\right)^3\frac{P_T}{P_c}n_1L\Delta\mu_4\,,$$

$$\vec{V}_n\vec{\gamma} = \Delta\sigma + \left(\frac{l}{l_0}\right)^3\frac{Ld}{R_a}\Delta\mu_4\,.$$

Here the scaled variables are used, z by l, t by $\rho n l^2/\eta$, V_n' by κ/l, σ' by $(d\sigma_0/dz)l$, and μ_4' by gl; also the following symbols are used:

$$R_a = \frac{1}{\rho_n}\left(\frac{\partial \rho}{\partial \sigma}\right)_{P,\mu_4}\frac{\rho_n g l^4}{\eta\kappa}\cdot\frac{d\sigma_0}{dz}\,, \quad \kappa = \frac{\chi_{\text{eff}}}{\rho c T}\left(\frac{\partial T}{\partial c}\right)_{P,\mu_4}\,,$$

$$P_T = \frac{\eta}{\rho_n\kappa}\,, \quad P_c = \frac{\eta}{\rho_n D}\,, \quad l_0^3 = \frac{\eta\kappa}{g\rho_n}\,,$$

$$m = \rho_n\frac{\zeta_4 - \rho\zeta_3}{\eta}\,, \quad L = \frac{gl}{\rho_n}\left(\frac{\partial \rho}{\partial \mu_4}\right)_{P,\sigma}\,, \tag{8}$$

$$\nabla\sigma_0 = \frac{d\sigma_0}{dz}\vec{\gamma}, \quad \nabla T_0 = \frac{dT_0}{dz}\vec{\gamma}, \quad \nabla C_0 = \frac{dC_0}{dz}\vec{\gamma},$$

$$a = \frac{\rho_n}{c}\frac{(\partial c/\partial\sigma)_{P,\mu_4}}{(\partial\rho/\partial\sigma)_{P,\mu_4}}\,, \quad a_1 = a\left[1 + \frac{k_T}{T}\frac{(\partial T/\partial\sigma)_{P,\mu_4}}{(\partial c/\partial\sigma)_{P,\mu_4}}\right],$$

$$n = \frac{\rho_n}{c}\frac{(\partial c/\partial\mu_4)_{P,\sigma}}{(\partial\rho/\partial\mu_4)_{P,\sigma}}\,, \quad n_1 = n\left[1 + \frac{k_T}{T}\frac{(\partial T/\partial\mu_4)_{P,\sigma}}{(\partial c/\partial\mu_4)_{P,\sigma}}\right],$$

$$d = \frac{(\partial T/\partial\mu_4)_{P,\sigma}}{(\partial T/\partial\sigma)_{P,\mu_4}}\frac{(\partial\rho/\partial\sigma)_{P,\mu_4}}{(\partial\rho/\partial\mu_4)_{P,\sigma}}\,.$$

The difficulty in solving this problem resides in the fact that Eqs. (7) are non-self-adjoint. Since the eigenvalue problem is defined only for a self-adjoint operator, one would have to transform the non-self-adjoint operator to a self-adjoint form; however, there is no such procedure in general. For the set of Eqs. (7), I succeeded in performing this procedure in two limiting cases with respect to the parameter m.

If $m < 1$ it is possible to neglect the right part of the second equation in (7). Then one obtains

$$\mu_4 = 0\,, \quad \operatorname{div}\vec{V}_n \neq 0\,. \tag{9}$$

This case corresponds to small dissipation of superfluid motion. Therefore, the superfluid motion is essential, and any perturbations of the chemical potential relax with the second sound velocity. Then these perturbations can be neglected. The second case corresponds to large dissipation of superfluid motion ($m > 1$). So from the second equation of set (7) it follows that

$$\operatorname{div}\vec{V}_n = 0\,, \quad \mu_4 \neq 0\,. \tag{10}$$

Here the fluctuations of chemical potential relax diffusively, because superfluid motion is negligible.

III. CRITERIA FOR THE ONSET OF STATIONARY CONVECTION

Now let us obtain criteria for the stationary instability in both limiting cases.

A. $m < 1$

$m < 1$ means the small dissipation of superfluid motion. Assuming that the variable fluctuations are given by

$$[V_{nz}, \sigma] = [v(z), \sigma(z)]e^{i\vec{k}\vec{r}}\,, \tag{11}$$

where $\vec{k} = k_x\vec{i} + k_y\vec{j}$ is the horizontal wave vector and $\vec{r} = x\vec{i} + y\vec{j}$ is the horizontal radius vector, we obtain the following set of equations with eigenvalue R_a for variables $v(z)$ and $\sigma(z)$:

$$D^2 v = -R_a k^2\sigma + \delta R_a D^2\frac{d\sigma}{dz}\,,$$
$$D\sigma = v\,, \tag{12}$$

with the free boundary conditions

$$v=\frac{d^2v}{dz^2}=\sigma=0 \quad \text{at} \quad z=\pm\tfrac{1}{2}\,.$$

Here

$$\delta=\left(\frac{l_0}{l}\right)^3 a\left(\frac{a_1}{a}\frac{P_T}{P_c}-1\right)\,, \quad D=\frac{d^2}{dz^2}-k^2\,, \quad k^2=k_x^2+k_y^2\,.$$

The system (12) is non-self-adjoint. In order to transform it to a self-adjoint form let us rewrite the set (12) in the following form:

$$\begin{aligned}&D^2v=\psi\,,\\ &\hat{L}\psi=-R_ak^2v\,, \quad \hat{L}\equiv\frac{d^2}{dz^2}-\delta R_a\frac{d}{dz}-k^2\,.\end{aligned} \tag{13}$$

By substituting $\psi=\varphi\exp[(\delta R_a/2)z]$, the operator $\hat{L}$ can be written in the self-adjoint form

$$\begin{aligned}&D^2v=e^{\delta R_a z/2}\varphi\,,\\ &e^{\delta R_a z/2}\left[D-\left(\frac{\delta R_a}{2}\right)^2\right]\varphi=-R_ak^2v\end{aligned} \tag{14}$$

and

$$v=\frac{d^2v}{dz^2}=\varphi=0 \quad \text{at} \quad z=\pm\tfrac{1}{2}\,.$$

Now to find the eigenvalue $R_a(k)$ it is possible to use the direct variational method (Galerkin's method).[6] As the trial function one can use

$$\varphi=A\cos\Pi z\,,$$

which satisfies the free boundary conditions (14). As a result of the application of Galerkin's method to Eqs. (14) one obtains the following equation for $R_a(k)$ as a function of wave number k:

$$R_a^2-R_a\frac{4k^2}{\delta^2(\Pi^2+k^2)^2}+4\frac{\Pi^2+k^2}{\delta^2}=0\,. \tag{15}$$

The minimal value of $R_a(k)$ is the criterion for the onset of stationary convection. Let us analyze (15). As seen from (12), $\delta\sim(l_0/l)^3$ and in the case of a relatively small "dissipation" length l_0 we have $\delta\ll1$. From Eq. (15) it is clear that for $\delta=0$ the criterion is[17]

$$R_a^{(s)}=\frac{27\Pi^4}{4} \quad \text{at} \quad k_c^2=\frac{\Pi^2}{2}\,. \tag{16}$$

This approach implies small dissipation of both superfluid and normal motion. Equations (12) with $\delta=0$ and the corresponding criterion (16) are the same as for a pure liquid[18]; as seen from (8),

$$R_a=\frac{gl^4}{\nu\kappa}\frac{d\rho_0}{dz}\ \left(\nu\equiv\frac{\eta}{\rho_n}\right).$$

The same result can be obtained for a pure liquid.[13] However, for a pure liquid $d\rho_0/dz=-\rho\beta(dT_0/dz)$ and unstable stratification occurs only when heated from below, unlike the present case where it occurs when heated from above as shown in (3).

By increasing δ, the value of $R_a^{(s)}$ increases from $27(\Pi^4/4)$ proportional to δ^4, but the value of critical wave number k_c decreases from $\Pi/2$ proportional to δ. The maximal value of δ for which a solution $R_a^{(s)}$ exists is

$$\delta^*=0.006\,. \tag{17}$$

Corresponding numbers for the criterion and the critical wave number are[19]

$$R_a^{(s)}=13.45\Pi^4 \quad \text{at} \quad k_c^2=0.45\Pi^2\,. \tag{18}$$

Therefore the criterion for stationary instability changes with changing δ or l_0 as

$$\frac{27\Pi^4}{4}\leqslant R_a^{(s)}\leqslant 13.45\Pi^4\,, \tag{19}$$

and the critical wave number as

$$\frac{\Pi^2}{2}\geqslant k_c^2\geqslant 0.45\Pi^2\,. \tag{20}$$

Thus, the convective stability of the superfluid mixture arises due to additional dissipation connected with the superfluid motion ($m<1$, the small dissipation of superfluid motion) when the dissipation of the normal motion increases and becomes comparatively large ($\delta\neq0$). The fluctuations of thermodynamical variables cause the superfluid flow that, in its turn, dissipates through the normal motion connected with it. For a large dissipation of the normal motion $\delta>\delta^*$ the system becomes mechanically stable. This effect is similar to the influence of compressibility on the convective instability of a compressible, viscous, and heat-conducting pure liquid.[12]

B. $m>1$

$m>1$ means the large dissipation of a superfluid motion. Since in this case the superfluid motion is insignificant, the fluctuations of chemical potential have to be taken into account. The mechanical stability of the system should be reduced because the chemical potential perturbations are dangerous for stability when heated from above. Thus, there are two thermodynamic variables, σ and μ_4, in this case, as in a regular mixture. Assuming as noted above that their perturbations are given by

$$[V_{nz},\sigma,\mu_4]=[v(z),\sigma(z),\xi(z)]e^{i\vec{k}\cdot\vec{r}}\,, \tag{21}$$

one can obtain the following set of equations:

$$D^2v = -R_a k^2\sigma - \left(\frac{l}{l_0}\right)^3 Lk^2\xi \,,$$

$$aP_cR_av = a_1P_TR_aD\sigma + n_1P_TL\left(\frac{l}{l_0}\right)^3 D\xi \,, \tag{22}$$

$$v = D\sigma + \frac{Ld}{R_a}\left(\frac{l}{l_0}\right)^3 D\xi \,.$$

Hence the equation for the eigenvalue problem is

$$D^3v = -\tilde{R}_a k^2 v \,,$$

where (23)

$$\tilde{R}_a = R_a\psi_1, \quad \psi_1 = \frac{P_T(n_1 - a_1) - aP_c(d-1)}{P_T(n_1 - a_1 d)} \,.$$

For free boundary conditions, the criterion of stationary instability from (23) is given by

$$\tilde{R}_a^{(s)} = \frac{27\Pi^4}{4} \quad \text{at} \quad k_c^2 = \frac{\Pi^2}{2} \tag{24}$$

and looks similar to one for a regular binary mixture.[9]

IV. ESTIMATIONS, COMPARISON WITH EXPERIMENT

A. Obtained results

Let us write the obtained results in convenient thermodynamic variables and estimate the value of the critical temperature gradient. The expressions from (8), (12), and (23) can be written as (see Appendix A)

$$R_a = -\frac{\alpha_T g\rho^2\left[C_{P,c} + K^2T\,\frac{\partial(Z/\rho)}{\partial c}\Big/_{P,T}\right]}{\eta\chi_{\text{eff}}}(1+\varphi_0)l^4\frac{dT_0}{dz} \,,$$

$$\psi_1 = \frac{1+\varphi+\frac{P_c}{P_T}\varphi_0}{1+\varphi_0} \,, \quad \varphi_0 = -\frac{\beta}{\alpha_T}\,\frac{(\partial\mu_4/\partial T)_{P,c}}{(\partial\mu_4/\partial c)_{P,T}} \,,$$

$$\varphi = -\frac{\beta}{\alpha_T}\,\frac{k_T}{T} \,, \quad l_0^3 = \frac{\eta\chi_{\text{eff}}}{g\rho_n\rho C_{P,c}\left(1-\frac{ad}{n}\right)} \,, \tag{25}$$

$$\frac{ad}{n} = \frac{(\partial\mu_4/\partial T)_{P,c}}{(\partial\mu_4/\partial c)_{P,T}}\,\frac{(\partial\sigma/\partial c)_{P,T}}{(\partial\sigma/\partial T)_{P,c}} \,,$$

$$\delta = \left(\frac{l_0}{l}\right)^3\frac{\rho_n}{\rho c\beta}\,\frac{\varphi P_cP_T^{-1} + \varphi_0(1-P_cP_T^{-1})}{1+\varphi_0} \,.$$

Typical values of parameters affecting the onset of convection for several concentrations and temperatures below the λ line and to the left of the coexistence separation line of the He^3-He^4 phase diagram are tabulated in Table I (see also Appendix B).

As it has been shown both criteria R_a and $\tilde{R}_a$ ensure the positive critical temperature gradient irrespective of the sign of α_T (Ref. 20) because $|\phi_0| > 1$ always. Besides, as seen from (25), $\psi_1 > 1$ and therefore the chemical potential fluctuations decrease the stability like in a regular binary mixture. The difference in the expression for ψ_1 is that for a regular mixture $\phi = \phi_0$.[9]

It is important to estimate where on the phase diagram the parameter m is greater or less than one. This is not easy because experiments for second viscosity coefficients in He^3-He^4 mixtures are almost lacking. One can only use the first and second sound-absorption measurements for the estimation of the second viscosity coefficients, but also not separately.[7] As η is known from flow experiments, measurements of the first sound absorption determines ζ_2. Then, since thermal conductance and the value $(\frac{4}{3})\eta + \zeta_2$ are known, measurements of second sound absorption determine $(\rho\zeta_3 - \zeta_4)$. These data are sufficient to evalutate the paramenter m. The estimations for pure HeII show that ζ_2 is a complicated function of temperature, pressure, and frequency. Above 1.2 K and not close to T_λ experiments reveal that ζ_2 is independent of ω and approximately ten times larger than η but decreases when the temperature increases.[21] Below 0.6 K theoretical calculations give for all second viscosity coefficients negligibly small values.[7] Between 0.8 K and 1.2 K, ζ_2 has a weak maximum for low frequencies. Although there are no measurements, it is expected from theory that ζ_1, ζ_2, and ζ_3 all diverge near T_λ as $\epsilon_\lambda^{-3\nu/2}$; $\nu \simeq \frac{2}{3}$, and that η remains finite ($\epsilon_\lambda = T_\lambda - T/T_\lambda$).[22]

The second viscosity coefficients decrease rapidly with increasing concentration and pressure.[6] Also it is expected from theory that all second viscosity coefficients diverge near the λ line as ϵ_λ^{-1},[22] but near the tricritical points their behavior is not known at all. The temperature and concentration dependence of the parameter $(l_0/l)^3$ is much simpler. It increases with decreasing temperature and concentration, but becomes large only for a very dilute solution and $T < 1$ K. The result of calculations of the critical temperature gradient according to the estimates of the parameters are exhibited in Figs. 2 and 3. The only experiment work that represents results of convective instability investigations in the He^3-He^4 mixture below the λ line is that of Ref. 23.

The onset of stationary convection in a cylindrical cell with diamter 2.5 cm and spacing 0.2 cm between two horizontal copper end plates heated from above was observed. The experiments were performed with a 15 molar percent mixture of He^3 in He^4, the lower plate was maintained at $T_\lambda - 0.095$ K, and a heat current was directed downward through the upper plate. The authors assert that the convection occurs at a heat current less

TABLE I. Typical values of parameters affecting the onset of stationary convection at different temperatures and concentrations: (a) $x=0.01\%$ mol He^3, (b) $x=0.1\%$ mol He^3, (c) $x=1\%$ mol He^3, (d) $x=10\%$ mol He^3, (e) $x=30\%$ mol He^3, (f) $x=50\%$ mol He^3.

(a) $x=0.0001$

T	$C_{P,x}$	S	ρ_n/ρ	$(\partial\mu_4/\partial x)_{P,T}$	$(\partial\mu_4/\partial T)_{P,x}$	$\frac{\partial(Z/\rho)}{\partial x}\Big/_{P,T}$	$\frac{\partial(Z/\rho)}{\partial T}\Big/_{P,x}$	α_T	χ_{eff}	D
K	$J\,\mathrm{mol}^{-1}\,\mathrm{K}^{-1}$	$J\,\mathrm{mol}^{-1}\,\mathrm{K}^{-1}$		$J\,\mathrm{mol}^{-1}$	$J\,\mathrm{mol}^{-1}\,\mathrm{K}^{-1}$	$J\,\mathrm{mol}^{-1}\times10^{4}$	$J\,\mathrm{mol}^{-1}\,\mathrm{K}^{-1}$	$\mathrm{K}^{-1}\times10^{-3}$	$\mathrm{erg\,cm^{-1}\,sec^{-1}\,K^{-1}}$ $\times10^{5}$	$\mathrm{cm^2\,sec^{-1}}$
0.5	0.01	0.013	7.5×10^{-5}	−4.18	−0.0034	4.18	−112	0.13	8	8
1	0.4	0.077	0.01	−8.36	−0.067	8.36	−114.7	0.46	0.8	10^{-2}
1.5	4.52	0.8	0.12	−12.54	−0.8	12.54	−116.7	−2	2	10^{-3}
2	20.7	3.76	0.55	−16.72	−3.76	16.72	−118.7	−12	8	2×10^{-4}

η	$\frac{k_T}{T}$	$\frac{(\partial\sigma/\partial c)_{P,T}}{(\partial\sigma/\partial T)_{P,c}}$	$\frac{(\partial\mu_4/\partial c)_{P,T}}{(\partial\mu_4/\partial T)_{P,c}}$	$\frac{ad}{n}$	P_T	P_c	l_0^3	δl^3	φ	φ_0	ψ_1	$R_a/l^4(dT_0/dZ)$
μP	K^{-1}	K	K				cm^3	cm^3				$\mathrm{cm^{-3}\,K^{-1}}$
1000	6×10^{-4}	−4050	922	−4.4	2.3	11.5	3.6	18	−1.38	−2.5	8.8	0.2
40	6×10^{-3}	−3035	93.6	−32.4	1.7	2.9	5×10^{-7}	1.8×10^{-4}	−3.9	−7	2.4	1.8×10^{5}
12	4.8×10^{-2}	−3503	15.6	−224.5	1.2	0.7	4×10^{-10}	10^{-6}	7.2	9.6	1.3	2.5×10^{8}
14	0.17	−4854	4.4	−1103	1.8	0.9	2.2×10^{-11}	2.5×10^{-7}	4.2	5.7	1.2	4.45×10^{9}

(b) $x=0.001$

T	$C_{P,x}$	S	ρ_n/ρ	$(\partial\mu_4/\partial x)_{P,T}$	$(\partial\mu_4/\partial T)_{P,x}$	$\frac{\partial(Z/\rho)}{\partial x}\Big/_{P,T}$	$\frac{\partial(Z/\rho)}{\partial T}\Big/_{P,x}$	α_T	χ_{eff}	D	η	$\frac{k_T}{T}$
K	$J\,\mathrm{mol}^{-1}\,\mathrm{K}^{-1}$	$J\,\mathrm{mol}^{-1}\,\mathrm{K}^{-1}$		$J\,\mathrm{mol}^{-1}$	$J\,\mathrm{mol}^{-1}\,\mathrm{K}^{-1}$	$J\,\mathrm{mol}^{-1}\times10^{3}$	$J\,\mathrm{mol}^{-1}\,\mathrm{K}^{-1}$	$\mathrm{K}^{-1}\times10^{-3}$	$\mathrm{erg\,cm^{-1}}$ $\mathrm{sec^{-1}\,K^{-1}}$ $\times10^{5}$	$\mathrm{cm^2\,sec^{-1}}$	μP	K^{-1}
0.5	0.013	0.07	10^{-3}	−4.18	−0.012	4.18	−64.2	0.13	2	8	50	1.3×10^{-3}
1	0.4	0.14	10^{-2}	−8.36	−0.075	8.36	−66.6	0.46	0.7	0.01	25	2.7×10^{-3}
1.5	4.52	0.88	0.12	−12.54	−0.8	12.54	−67.9	−2	2	10^{-3}	13	1.6×10^{-2}
2	20.7	3.84	0.55	−16.72	−3.77	16.72	−66.8	−12	8	2×10^{-4}	13	0.056

$\frac{(\partial\sigma/\partial c)_{P,T}}{(\partial\sigma/\partial T)_{P,c}}$	$\frac{(\partial\mu_4/\partial c)_{P,T}}{(\partial\mu_4/\partial T)_{P,c}}$	$\frac{ad}{n}$	P_T	P_c	l_0^3	δl^3	φ	φ_0	ψ_1	$R_a/l^4(dT_0/dz)$
K	K				cm^3	cm^3				$\mathrm{cm^{-3}\,K^{-1}}$
−1100	463	−2.4	0.03	0.04	4.7×10^{-3}	0.06	−3	−4.98	2.4	40
−300	148	−2	0.1	1.8	3.0×10^{-7}	4×10^{-4}	−1.8	−4.4	23.5	3.3×10^{5}
−370	20.6	−18	0.12	0.75	5.3×10^{-9}	6×10^{-6}	2.4	7.3	6	3.3×10^{6}
−490	5.9	−83	0.13	0.8	2.3×10^{-10}	10^{-6}	1.4	4.2	5.7	4.6×10^{7}

TABLE I. (*Continued*).

(c) $x = 0.01$

T	$C_{P,x}$	S	ρ_n/ρ	$(\partial\mu_4/\partial x)_{P,T}$	$(\partial\mu_4/\partial T)_{P,x}$	$\frac{\partial(Z/\rho)}{\partial x}\Big/_{P,T}$	$\frac{\partial(Z/\rho)}{\partial T}\Big/_{P,x}$	α_T	χ_{eff}	D	η	$\frac{k_T}{T}$
K	J $\text{mol}^{-1}\,\text{K}^{-1}$	J $\text{mol}^{-1}\,\text{K}^{-1}$		J mol^{-1}	J $\text{mol}^{-1}\,\text{K}^{-1}$	J mol^{-1}	J $\text{mol}^{-1}\,\text{K}^{-1}$	$\text{K}^{-1}\times10^{-3}$	erg cm^{-1} $\text{sec}^{-1}\,\text{K}^{-1}\times10^{4}$	$\text{cm}^2\,\text{sec}^{-1}$	μP	$\text{K}^{-1}\times10^{-2}$
0.5	0.043	0.54	0.01	−4.18	−0.087	418	−45	0.13	2	8	25	4.8
1	0.44	0.63	0.02	−8.36	−0.15	836	−47.3	0.58	1	0.01	17	3.3
1.5	4.6	1.4	0.13	−12.54	−0.88	1254	−48.6	−1.7	0.6	10^{-3}	12	8
2	20.8	4.4	0.56	−16.72	−3.84	1672	−47.6	−11.4	1	2×10^{-4}	13	24

$\frac{(\partial\sigma/\partial c)_{P,T}}{(\partial\sigma/\sigma T)_{P,c}}$	$\frac{(\partial\mu_4/\partial c)_{P,T}}{(\partial\mu_4/\partial T)_{P,c}}$	$\frac{ad}{n}$	P_T	P_c	l_0^3	δl^3	φ	φ_0	ψ_1	$R_a/l^4(dT_0/dZ)$
K	K				cm^3	cm^3				$\text{cm}^{-3}\,\text{K}^{-1}$
−308	64	−4.8	0.09	0.002	4×10^{-6}	1.3×10^{-3}	−110.8	−36	3.2	2.2×10^{4}
−81.6	74	−1.1	0.2	0.6	1.9×10^{-7}	7×10^{-6}	−17	−7	6.2	1.7×10^{6}
−45	19	−2.4	0.6	0.6	7.4×10^{-10}	4.2×10^{-8}	14	9.3	2.4	2.4×10^{8}
−52	5.8	−9	1.2	0.8	2.3×10^{-11}	6.3×10^{-9}	6.3	4.5	1.9	6.4×10^{9}

(d) $x = 0.1$

T	$C_{P,x}$	S	ρ_n/ρ	$(\partial\mu_4/\partial x)_{P,T}$	$(\partial\mu_4/\partial T)_{P,x}$	$\frac{(\partial Z/\rho)}{\partial x}\Big/_{P,T}$	$\frac{(\partial Z/\rho)}{\partial T}\Big/_{P,x}$	α_T	χ_{eff}	D	η	$\frac{k_T}{T}$
K	J $\text{mol}^{-1}\,\text{K}^{-1}$	J $\text{mol}^{-1}\,\text{K}^{-1}$		J mol^{-1}	J $\text{mol}^{-1}\,\text{K}^{-1}$	J mol^{-1}	J $\text{mol}^{-1}\,\text{K}^{-1}$	$\text{K}^{-1}\times10^{-3}$	erg cm^{-1} $\text{sec}^{-1}\,\text{K}^{-1}$	$\text{cm}^2\,\text{sec}^{-1}$	μP	K^{-1}
0.5	1	3.4	0.17	−2.14	−0.92	21.4	−24.9	0.12	10^{4}	10^{-2}	16	0.7
1	1.5	3.7	0.2	−6.8	−0.99	68	−27.3	1.7	10^{4}	10^{-3}	14	0.2
1.5	7	4.5	0.35	−11.4	−1.72	114	−28.6	1	10^{4}	2.5×10^{-4}	12	0.21
2	35	7.4	0.93	−16.1	−4.68	161	−27.6	−5.6	10^{4}	10^{-4}	18	0.33

$\frac{(\partial\sigma/\partial c)_{P,T}}{(\partial\sigma/\partial T)_{P,c}}$	$\frac{(\partial\mu_4/\partial c)_{P,T}}{(\partial\mu_4/\partial T)_{P,c}}$	$\frac{ad}{n}$	P_T	P_c	l_0^3	δl^3	φ	φ_0	ψ_1	$R_a/l^4(dT_0/dZ)$
K	K				cm^3	cm^3				$\text{cm}^{-3}\,\text{K}^{-1}$
−10	2.9	−3.4	0.025	0.07	4.5×10^{-9}	1.6×10^{-7}	-1.75×10^{3}	−862	4.8	2.5×10^{8}
−14.8	8.6	−1.7	0.07	0.5	3.6×10^{-9}	10^{-7}	−35.3	−20.5	9.3	6×10^{7}
−6.8	8.3	−0.8	0.1	1	5.7×10^{-10}	2.3×10^{-8}	−63	−36	12	2.4×10^{8}
−4	4.3	−0.93	0.33	1.4	6×10^{-11}	7×10^{-8}	17.7	12.5	5.3	1.65×10^{9}

TABLE I. (*Continued*).

(e) $x=0.3$

T	$C_{P,x}$	S	ρ_n/ρ	$(\partial\mu_4/\partial x)_{P,T}$	$(\partial\mu_4/\partial T)_{P,x}$	$\left.\frac{\partial(Z/\rho)}{\partial x}\right/_{P,T}$	$\left.\frac{\partial(Z/\rho)}{\partial T}\right/_{P,x}$	α_T	χ_{eff}	D	η	$\frac{k_T}{T}$
K	$J\,\text{mol}^{-1}\,\text{K}^{-1}$	$J\,\text{mol}^{-1}\,\text{K}^{-1}$		$J\,\text{mol}^{-1}$	$J\,\text{mol}^{-1}\,\text{K}^{-1}$	$J\,\text{mol}^{-1}$	$J\,\text{mol}^{-1}\,\text{K}^{-1}$	$\text{K}^{-1}\times10^{-3}$	$\text{erg}\,\text{cm}^{-1}\,\text{sec}^{-1}\,\text{K}^{-1}$	$\text{cm}^2\,\text{sec}^{-1}$	μP	K^{-1}
1	4	7.8	0.5	−4.4	−3.1	14.7	−16	4.5	10^4	2×10^{-4}	15	0.95
1.5	12	9	0.75	−10.4	−3.8	34.7	−17.3	8	10^4	10^{-4}	17	0.5

$\frac{(\partial\sigma/\partial c)_{P,T}}{(\partial\sigma/\partial T)_{P,c}}$	$\frac{(\partial\mu_4/\partial c)_{P,T}}{(\partial\mu_4/\partial T)_{P,c}}$	$\frac{ad}{n}$	P_T	P_c	l_0^3	δl^3	φ	φ_0	ψ_1	$R_a/l^4(dT_0/dZ)$
K	K				cm^3	cm^3				$\text{cm}^{-3}\,\text{K}^{-1}$
−4.6	1.8	−2.5	0.12	1.2	4.8×10^{-10}	3.5×10^{-9}	−66	−38.5	12	1.1×10^{9}
−2.8	3.4	−0.8	0.14	1.8	2.3×10^{-10}	2.4×10^{-9}	−19.5	−11.5	15.8	0.65×10^{9}

(f) $x=0.5$

T	$C_{P,x}$	S	ρ_n/ρ	$(\partial\mu_4/\partial x)_{P,T}$	$(\partial\mu_4/\partial T)_{P,x}$	$\left.\frac{\partial(Z/\rho)}{\partial x}\right/_{P,T}$	$\left.\frac{\partial(Z/\rho)}{\partial T}\right/_{P,x}$	α_T	χ_{eff}	η	$\frac{k_T}{T}$	D
K	$J\,\text{mol}^{-1}\,\text{K}^{-1}$	$J\,\text{mol}^{-1}\,\text{K}^{-1}$		$J\,\text{mol}^{-1}$	$J\,\text{mol}^{-1}\,\text{K}^{-1}$	$J\,\text{mol}^{-1}$	$J\,\text{mol}^{-1}\,\text{K}^{-1}$	$\text{K}^{-1}\times10^{-3}$	$\text{erg}\,\text{cm}^{-1}\,\text{sec}^{-1}\,\text{K}^{-1}$	μP	K^{-1}	$\text{cm}^2\,\text{sec}^{-1}$
1	7	10.3	0.8	−4.5	−5.9	9	−10	7.7	10^4	15	1.45	10^{-4}

$\frac{(\partial\sigma/\partial c)_{P,T}}{(\partial\sigma/\partial\theta)_{P,c}}$	$\frac{(\partial\mu_4/\partial c)_{P,T}}{(\partial\mu_4/\partial T)_{P,c}}$	$\frac{ad}{n}$	P_T	P_c	l_0^3	δl^3	φ	φ_0	ψ_1	$R_a/l^4(dT_0/dZ)$
K	K				cm^3	cm^3				$\text{cm}^{-3}\,\text{K}^{-1}$
−2.5	0.8	−3	0.17	1.7	1.7×10^{-10}	10^{-9}	−56.5	−48.7	11.4	3.1×10^{9}

than 0.2 μW. This number corresponds to a critical temperature difference $\Delta T_c < 10$ μK. The temperature and concentration at which the experiments were run correspond to values of the parameters which yield $m > 1$ and $(l_0/l)^3 < 1$, as seen from Fig. 1. Therefore, we have to use the criterion (23). As mentioned above, for the rigid boundary conditions the eigenvalue has to be in this case, like for a regular mixture, 1708.[9] For the critical temperature difference it gives $\Delta T_c \simeq 8$ μK ($\tilde{R}_a/l^3 \Delta T \simeq 1\times 10^{10}$) in good agreement with the experimental data.

B. Mechanical stability near λ-line and tricritical point

As is apparent from Fig. 2 the mechanical stability of the system rises with reducing temperature. When the temperature approaches the λ line or the tricritical point ($T_t = 0.867$ K, $X_t = 0.675$ K),[22] the stability drops. In these two limiting cases it is easy to derive the temperature dependence of the criterion (its anomalous part).

Using the singularities of the thermodynamic and kinetic properties in the vicinity of the λ line of He^3-He^4 mixtures[22,24] one can easily find the anomaly for the criterion. According to Refs. 22 and 24 one has

$$k_T \sim \epsilon_\lambda^{-\alpha}, \quad \left(\frac{\partial(Z/\rho)}{\partial c}\right)_{P,T}^{-1} \sim \epsilon_\lambda^{-\alpha},$$

$$\left(\frac{\partial \mu_4}{\partial c}\right)_{P,T}^{-1} \sim \epsilon_\lambda^{-\alpha}, \quad D \sim \epsilon_\lambda^{-0.34},$$

$$\zeta_1, \zeta_2, \zeta_3 \sim \epsilon_\lambda^{-1},$$

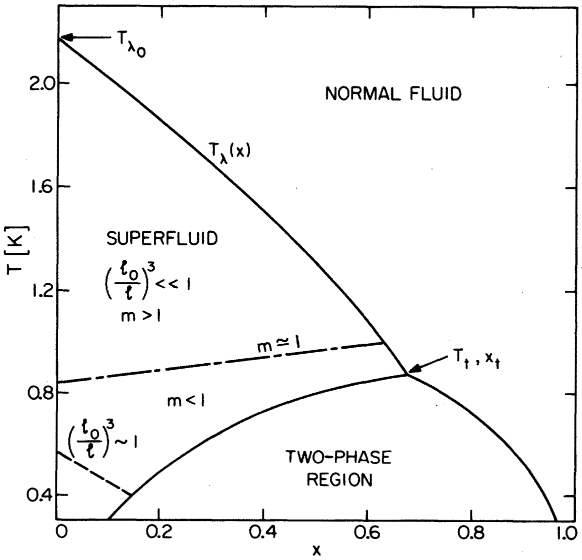

FIG. 1. The phase diagram of He^3-He^4 mixtures in the T-x plane and different regions of parameter values [m and $(l_0/l)^3$] affecting the onset of stationary convection.

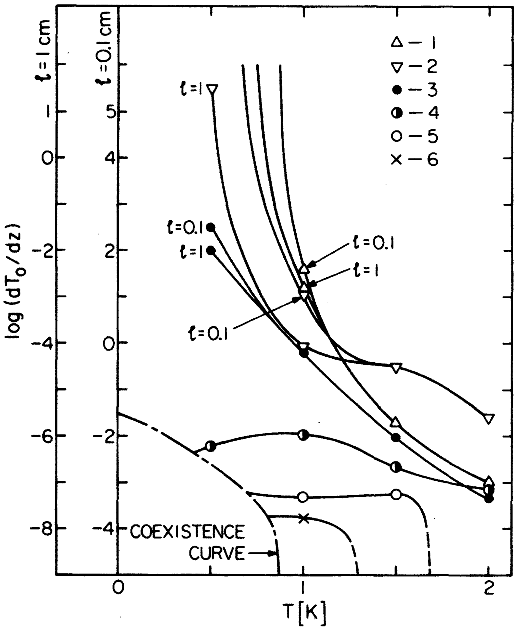

FIG. 2. Temperature dependence of the critical temperature gradient defining the onset of stationary convection for different He^3 concentrations: (1) 0.01% mol He^3, (2) 0.1% mol He^3, (3) 1% mol He^3, (4) 10% mol He^3, (5) 30% mol He^3, (6) 50% mol He^3.

where

$$\epsilon_\lambda = \frac{T_\lambda(x) - T}{T_\lambda(x)}, \quad \alpha = -0.02.$$

Then we have

$$m \sim \epsilon_\lambda^{-1}, \quad l_0^3 \sim \epsilon_\lambda^{\alpha}, \quad \delta \sim \epsilon_\lambda^{0.34+\alpha}, \quad \tilde{R}_a \sim \epsilon_\lambda^{-2\alpha}. \qquad (26)$$

Thus, the critical temperature gradient drops as $\epsilon_\lambda^{2\alpha}$ when the λ line is approached, but remains finite. Near the tricritical point the anomalies of thermodynamic properties are stronger,[22] but transport properties in this region have hardly been investigated at all. There is nothing known about singularities of the second viscosity coefficients. From the recent prediction confirmed by the analysis of acoustic attenuation experiments[24] the tricritical singularity of diffusion coefficients is expected to be $D \sim \epsilon_t^{1/3}$. The effective thermal conductivity shows only a weak variation with temperature for the tricritical mixture.[24] On the other hand, both temperature and concentration gradients relax with the same time constants, which indicates strong coupling via k_T.[24]

Therefore, in the tricritical region the chemical potential fluctuations are insignificant, $m < 1$, and the first limiting case pertains. According to Refs. 22 and 24, there are the following singularities of thermodynamic and kinetic properties near the tricritical point:

$$\left(\frac{\partial (Z/\rho)}{\partial c}\right)^{-1}_{P,T} \sim \epsilon_t^{-1}, \quad \left(\frac{\partial \mu_4}{\partial c}\right)^{-1}_{P,T} \sim \epsilon_t^{-1},$$

$$k_T \sim \epsilon_t^{-1}, \quad D \sim \epsilon_t^{1/3},$$

where

$$\epsilon_t = \frac{T_t - T}{T_t}.$$

Then we have

$$l_0^3 \sim \epsilon_t, \quad \delta \sim \epsilon_t^{5/3}, \quad R_a \sim \epsilon_t^{-2}, \tag{27}$$

and, respectively, the critical temperature gradient tends to zero as ϵ_t^2.

C. Concentration behavior of criteria

Another parameter whereby it is possible to consider a limiting behavior of the instability criterion is the concentration. Indeed as the concentration approaches zero the critical temperature gradient also must tend to zero because, as mentioned above, in pure He II mechanical equilibrium is possible only when $dT_0/dz = 0$. But as seen from Fig. 3 the critical temperature gradient does not tend to zero when the concentration is reduced. Taking into account that[7]

$$\chi_{\text{eff}} = \frac{\rho S_{40}^2 M_4 D}{Rc} \tag{28}$$

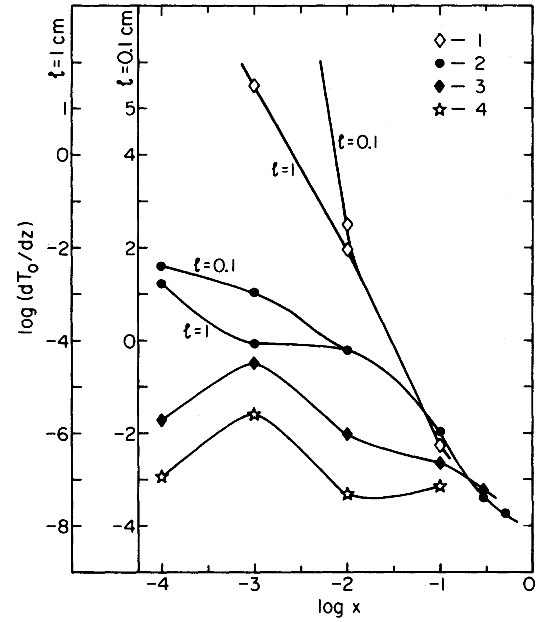

FIG. 3. Concentration dependence of the critical temperature gradient for different temperatures: (1) 0.5*K*, (2) 1*K*, (3) 1.5*K*, (4) 2*K*.

(R is the gas constant, M_4 is the He^4 molecular weight, and S_{40} is the He^4 entropy per gram), it is easy to derive for dilute superfluid solutions the following expressions (see Appendix C).

$$R_a = \frac{g\rho_n}{\eta D}\frac{S_{40}M_3^2}{RTM_4} l^4 \frac{dT_0}{dz} \quad \left[\text{or } R_a = \left(\frac{l}{l_0}\right)^3 \frac{k_T}{T} l \frac{dT_0}{dz}\right], \psi_1 = 1 + \frac{M_4}{M_3}\left(1 - \frac{S_{40}M_3}{3R\alpha_T T}\right)^{-1}, \; l_0^3 = \frac{\eta D}{g\rho_n}\frac{M_4}{M_3}, \quad \delta = \frac{9\eta D}{g\rho l^3}\cdot\frac{1}{c}. \tag{29}$$

In as much as value of formula (28) for χ_{eff} is available only in the approximate temperature range $1\,\text{K} \leq T \leq T_\lambda$,[6] the expressions (29) are also available only in this temperature range. For pure He II the condition $m > 1$ occurs at temperatures $T > 0.8 \div 1\,\text{K}$, while the condition $(l_0/l)^3 > 1$ for reasonable values l should be fulfilled at $T < 0.8\,\text{K}$. Thus, for dilute solutions we have to use the criterion $\tilde{R}_a$ at $T > 0.8\,\text{K}$ and the criterion (15) with $\delta \neq 0$ at $T < 0.8\,\text{K}$. For very low temperatures ($T < 0.5\,\text{K}$) a dilute solution becomes mechanically stable.

Both criteria $\tilde{R}_a$ and R_a tend to constant value when the concentration tends to zero. The contradiction with the assertion made above, that the critical temperature gradient should tend to zero, may be explained in the following way. On the one hand, for a dilute superfluid solution a thermodiffusion ratio is

$$\frac{k_T}{T} = \frac{S_{40}M_3}{RT} \tag{30}$$

and constant for a given T.[6] On the other hand, we have for the thermodiffusion ratio the expression (2). Since for an infinitely dilute solution in the convection-free steady state $dc/dz \sim c$, it follows from (2) and (30) that $dT/dz \sim dc/dz \sim c$. As a result the critical temperature gradient tends to zero as the concentration. It is in the thermodiffusion effect that a difference between a regular binary mixture and a superfluid one shows up clearly. Through the superfluid component the thermodiffusion ratio remains finite for a dilute solution as opposed to a regular binary solution where the thermodiffusion ratio $k_T \sim dc/dz \sim c$ and tends to zero, but $dT_0/dz \neq 0$.[8]

V. SUMMARY

I would like to emphasize here that the stationary convection in a superfluid He^3-He^4 mixture commences when heated from above only. Two parameters, m and $(l_0/l)^3$, define the areas of application of the criteria (15) and (23). Small dissipation of a superfluid motion $m < 1$ occurs at

TABLE II. Typical values of parameters affecting the onset of stationary convection in a dilute solution at different temperature.

T	C_{40}	S_{40}	ρ_n/ρ	α_T	D	η	$\frac{k_T}{T}$	P_c	l_0^3	ψ_1	$R_a/l^4(dT_0/dZ)$
K	$J\,\mathrm{mol}^{-1}\,\mathrm{K}^{-1}$	$J\,\mathrm{mol}^{-1}\,\mathrm{K}^{-1}$		$\mathrm{K}^{-1}\times10^{-3}$	$\mathrm{cm}^2\,\mathrm{sec}^{-1}$	μP	K^{-1}		cm^3		$\mathrm{cm}^{-3}\,\mathrm{K}^{-1}$
1	0.4	0.067	0.01	0.46	10^{-2}	40	6×10^{-3}	2.8	4×10^{-7}	2.7	1.5×10^{4}
1.5	4.52	0.8	0.12	−2	10^{-3}	13	4.8×10^{-2}	0.75	10^{-9}	2.2	4.8×10^{7}
2	20.7	3.76	0.55	−12	2×10^{-4}	14	0.17	0.9	5×10^{-11}	2.1	3.4×10^{9}
$T_{\lambda0}$	25	6.24	1	$\sim\epsilon_t^{-\alpha}$	10^{-4}	18.6	0.26	1.3	1.8×10^{-11}	1.74	1.4×10^{10}

lower temperatures and larger concentrations. In this region the superfluid motion is significant, hence the chemical potential fluctuations are negligible, and the temperature and concentration gradients are coupled via k_T. Only the thermodynamic parameter σ need be considered in the convection equations, just as only the temperature need be considered in a pure liquid.

As shown by calculations (see Table I and Figs. 2 and 3) in a major part of the considered temperature and concentration range of the He^3-He^4 phase diagram the dissipation length l_0 is rather small so that the condition $(l_0/l)^3 < 1$ should be fulfilled for a realistic layer height l. For $m < 1$ and $(l_0/l)^3 < 1$ (small dissipation of superfluid as well as normal motion) the criterion (16) is available ($\delta = 0$). For very low temperatures and small concentration the dissipation length increases drastically so that the condition $l_0/l)^3 > 1$ should be fulfilled for a realistic layer height. This is explained by a large kinetic coefficients increase (as D and κ) and the normal density ρ_n decrease (several orders of magnitude, see Tables I and II). For small dissipation of superfluid motion ($m < 1$) and large dissipation of a normal motion $[(l_0/l)^3 > 1)]$ the inclusion of the superfluid motion in the hydrodynamics ($\mathrm{div}\,\vec{V}_n \neq 0$) becomes necessary. This mechanism leads to an additional increase of dissipation of normal motion and, thus, an additional increase of mechanical stability. For very low temperatures ($T < 0.5$ K) the system becomes mechanically stable due to this effect. This amazing result is obtained for a dilute superfluid solution.

For $T > 0.8$ K in a major part of the superfluid He^3-He^4 phase diagram the condition $m > 1$ is fulfilled (Fig. 1). It means large dissipation of superfluid motion, and the chemical potential fluctuations become essential. The system and the stability criteria (23) in this case are very similar to a regular binary mixture with large abnormal thermal diffusion. In conclusion, it should be noted that in order to show the main features of a stationary instability in a He^3-He^4 superfluid mixture, I have considered here only the case of free boundaries. The analysis can be extended to deal with more realistic boundary conditions.

ACKNOWLEDGMENTS

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APPENDIX A

Let us write all thermodynamic derivatives used in (7) in convenient thermodynamic variables as follows:

$$a=\frac{\rho_n}{\rho_c}\frac{(\partial\mu_4/\partial T)_{P,c}}{(\partial\mu_4/\partial c)_{P,T}}\left(\alpha_T-\beta\frac{(\partial\mu_4/\partial T)_{P,c}}{(\partial\mu_4/\partial c)_{P,T}}\right)^{-1}, \tag{A1}$$

$$a_1=a\left(1-\frac{k_T}{T}\frac{(\partial\mu_4/\partial c)_{P,T}}{(\partial\mu_4/\partial T)_{P,c}}\right), \tag{A2}$$

$$n=\frac{\rho_n}{\rho_c}\frac{(\partial\sigma/\partial T)_{P,c}}{(\partial\sigma/\partial c)_{P,T}}\left(\alpha_T-\beta\frac{(\partial\sigma/\partial T)_{P,c}}{(\partial\sigma/\partial c)_{P,T}}\right)^{-1}, \tag{A3}$$

$$n_1=n\left(1-\frac{k_T}{T}\frac{(\partial\sigma/\partial c)_{P,T}}{(\partial\sigma/\partial T)_{P,c}}\right), \tag{A4}$$

$$\frac{ad}{n}=\frac{(\partial\mu_4/\partial T)_{P,c}}{(\partial\mu_4/\partial c)_{P,T}}\frac{(\partial\sigma/\partial c)_{P,T}}{(\partial\sigma/\partial T)_{P,c}}, \tag{A5}$$

$$\frac{d\sigma_0}{dz}=\frac{1}{c}\left[\frac{C_{P,c}}{T}+K^2\left(\frac{\partial(Z/\rho)}{\partial c}\right)_{P,T}\right]\frac{dT_0}{dz}. \tag{A6}$$

APPENDIX B

Since most of the experimental results are per mole of solution, let us write thermodynamic relations between the thermodynamic functions per gram of solution that have been used in the convection equations and per mole of solution. Thus, one has

$$c=\frac{xM_3}{xM_3+(1-x)M_4}\,,\quad \left(\frac{\partial c}{\partial x}\right)_{P,T}=\frac{M_4}{M_3}\frac{c^2}{x^2}\,,$$

$$\frac{c}{\rho}\left(\frac{\partial\rho}{\partial c}\right)_{P,T}=\frac{M_3}{M_4}\frac{x}{\rho}\left(\frac{\partial\rho}{\partial x}\right)_{P,T} \tag{B1}$$

($x=N_3/N_3+N_4$) is the molar concentration). The entropy per mole of solution is

$$\tilde{S}=[xM_3+(1-x)M_4]s,$$

where $\partial = S/\rho$ is the entropy per gram of solution and, respectively,

$$c\left(\frac{\partial s}{\partial c}\right)_{P,T}=\frac{x}{M_4}\left(\frac{\partial\tilde{S}}{\partial x}\right)_{P,T}+\frac{\tilde{S}c}{M_4}\left(1-\frac{M_3}{M_4}\right),$$

$$s-c\left(\frac{\partial s}{\partial c}\right)_{P,T}=\frac{\tilde{S}}{M_4}-\frac{x}{M_4}\left(\frac{\partial\tilde{S}}{\partial x}\right)_{P,T}\,. \tag{B2}$$

By analogy, there are for the heat capacity:

$$\left(\frac{\partial s}{\partial T}\right)_{P,c}=\frac{C_{P,c}}{T},\quad \left(\frac{\partial\tilde{S}}{\partial T}\right)_{P,x}=\frac{\tilde{C}_{P,x}}{T},$$

$$\tilde{C}_{P,x}=[xM_3+(1-x)M_4]C_{P,c}\,. \tag{B3}$$

APPENDIX C

Let us write some relations for infinite dilute solutions (here we use all functions per gram of solution):

$$\mu_4=\mu_{40}-\frac{RT}{M_4}x,\quad \mu_3=\mu_{30}-\frac{RT}{M_3}x, \tag{C1}$$

$$x\simeq\frac{M_4}{M_3}C, \tag{C2}$$

$$K=-\frac{S_{40}M_3}{RT}\,, \tag{C3}$$

$$\frac{\partial(s/c)}{\partial T}=\frac{C_{P,c}}{cT}, \tag{C4}$$

$$\frac{ad}{n}=-\frac{S_{40}^2M_3}{RC_{40}c},\quad \frac{ad}{n_1}=-1 \tag{C5}$$

$$\frac{a}{n}=\frac{3S_{40}\alpha_T T}{C_{40}c}\left(1-3\frac{\alpha_T TR}{S_{40}M_3}\right)^{-1},\quad a_1=0 \tag{C6}$$

$$n=-\frac{C_{40}}{\alpha_T TS_{40}}\cdot\frac{\rho_{40}}{\rho},\quad \frac{n_1}{n}=\frac{S_{40}^2M_3}{RC_{40}c}, \tag{C7}$$

$$L=\frac{gl\alpha_T}{S_{40}}\cdot\frac{\rho}{\rho_{40}}, \tag{C8}$$

$$P_c=\frac{\eta}{\rho_{40}D},\ P_T=P_c\frac{M_3}{M_4} \tag{C9}$$

$$\varphi=\varphi_0=-\frac{S_{40}M_3}{3\alpha_T TR}\,. \tag{C10}$$

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[15] In the third equation of Eqs. (6) as one passes from (6) to (7) we take into account that

$$\frac{\nabla(\rho c)_0}{\rho c}\simeq\frac{\nabla c_0}{c}=aR_a\left(\frac{l_0}{l}\right)^3,$$

because $\phi>1$ and $\beta<1/c$ as mentioned above.

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[17] This case was first considered in Ref. 14.

[18] As it is clear in this case the eigenvalue is the same as for a pure liquid for all kinds of the boundary conditions.

[19] These numbers are obtained as a solution of Eq. (15) and the equation $\partial R_a(k)/\partial k=0$.

[20] In He^3-He^4 mixtures α_T changes sign in the temperature and concentration ranges of the phase diagram considered (see Table I).

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