Kinetic equation for a weakly coupled test particle. II. Approach to equilibrium

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We study the Fokker-Planck equation for the distribution in velocity of a test particle uniformly distributed through a weakly coupled classical fluid. The diffusion tensor, described first by Landau and by Chandrasekhar, is a complicated function of velocity. We discuss the manner in which the distribution function approaches, in time, its final Maxwellian form. The angle-averaged ($l = 0$) and the ($l = 1$) components are particularly interesting, the latter governing the autocorrelation function for velocity. Both relax as $t^a \exp(-t^b)$. Our analysis is based upon R. E. Langer's method of comparison equations. The two-dimensional case is remarkably like the three-dimensional.

I. INTRODUCTION

Although the Markovian kinetic equation for the diffusion of a weakly coupled test particle (the Fokker-Planck equation) has been in the literature for some time,[1,2] little is known about its solutions. In this paper we discuss the approach to equilibrium of distributions $f(\vec{r},\vec{u},t)$ that are uniformly distributed in space. We are particularly interested in the isotropic component ($l=0$) and the ($l=1$) component of the velocity distribution. The latter is related to the autocorrelation function for the velocity of the test particle. Finally, we shall comment about the relation between two- and three-dimensional velocity spaces.

The kinetic equation is, in dimensionless variables,

$$\frac{\partial}{\partial t} f(\vec{u},t) = \epsilon \frac{\partial}{\partial \vec{u}} \cdot \overleftrightarrow{D}(\vec{u}) \cdot \left(\frac{\partial}{\partial \vec{u}} + 2\vec{u}\right) f(\vec{u},t), \qquad (1)$$

with $f(\vec{u},t=0)$ given. The distribution function is subject to

$$\int d^3\vec{u} f(\vec{u},t) = \text{const.}$$

This condition will supply a boundary condition for the behavior of the isotropic component of $f(\vec{u},t)$ at $u=0$ when we make a spherical-harmonics decomposition. Other angular modes will enter when we study the relaxation of certain correlation functions. Then, additional conditions like

$$\int d\vec{u}\, \varphi(\vec{u}) f(\vec{u},t) < \infty$$

will guarantee the existence of the correlation function and provide a boundary condition.

The diffusion tensor is

$$D_{\alpha\beta}(\vec{u}) = D_{\parallel}(u)\frac{u_\alpha u_\beta}{u^2} + D_{\perp}(u)\left(\delta_{\alpha\beta} - \frac{u_\alpha u_\beta}{u^2}\right), \qquad (2)$$

with

$$D_{\parallel}(u) = \frac{4}{\sqrt{\pi}} \frac{1}{u^3} \int_0^x dt\, t^2 e^{-t^2}$$

and

$$D_{\perp}(u) + \tfrac{1}{2} D_{\parallel}(u) = \tfrac{1}{2}\,\mathrm{tr} D_{\alpha\beta} = \frac{1}{\theta^3}\frac{\mathrm{erf} x}{x} .$$

Here, $x = u/\theta$, $\theta^2 = m_1/m_2$, where m_1 is the mass of the test particle and m_2 is that of the host particle. The dimensionless velocity $\vec{u}$ is given by $\vec{v} = v_B \vec{u}$, $\frac{1}{2} m_1 v_B^2 = k_B T$. The weak-coupling parameter ϵ is compounded of the usual quantities: the host density n_0, the range of force a, the strength of force λ, to which we have added θ^2, the mass ratio, and ϕ_3, a shape factor, to get $\epsilon = n_0 a^3 (\lambda/k_B T)^2 \phi_3 (\theta^2/16)$. The dimensionless time t is given by $v_B \bar{t} = at$, where $\bar{t}$ is the physical time. Our earlier paper[2] contains much information about $D_{\alpha\beta}$, the kinetic equation, and their non-Markovian cousins. We have reprinted the graphs of $D_{\parallel}$ and $D_{\perp}$, ($\theta=1$) as Fig. 1, for convenience. Our present definition of $D_{\alpha\beta}$ differs from the earlier by the factor $\frac{1}{4}\phi_3\theta^2$.

Equation (1) has the form of a Fokker-Planck equation with velocity-dependent diffusion tensor. Another interesting form, suggested by Kuščer and Illner,[3] is the self-adjoint

$$\frac{\partial}{\partial \vec{u}} \cdot \overleftrightarrow{D}(\vec{u}) \cdot \frac{\partial}{\partial \vec{u}} \psi(\vec{u},t) - V(u)\psi = \frac{1}{\epsilon}\frac{\partial}{\partial t}\psi(\vec{u},t), \qquad (3)$$

with

$$f(\vec{u},t) = e^{-u^2/2}\psi(\vec{u},t),$$

and

$$V(u) = \vec{u}\cdot\overleftrightarrow{D}\cdot\vec{u} - \mathrm{tr}\overleftrightarrow{D} - \vec{u}\cdot\vec{\nabla}\cdot\overleftrightarrow{D}$$
$$= u^2 D_{\parallel} - \frac{1}{u^2}\frac{d}{du} u^3 D_{\parallel} . \qquad (4)$$

Equation (3) reminds one of a Schrödinger equation with a variable mass. Kuščer and Illner[3] point out that a crucial condition for the spectrum of eigenvalues is that $V(u) \to \infty$ as $u \to \infty$. This is not so in weak-coupling kinetic theory, where $D_{\parallel} \sim u^{-3}$ and

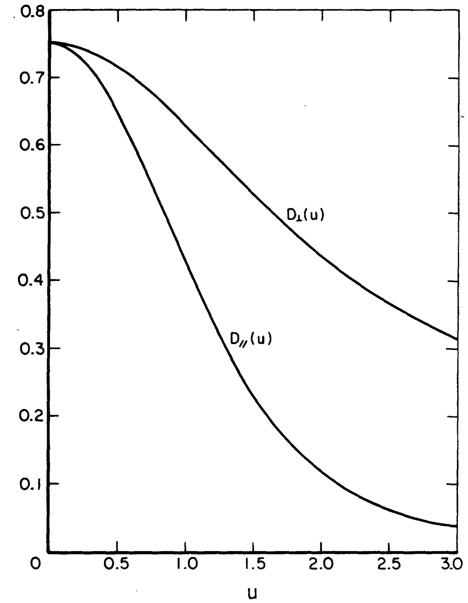

FIG. 1. $D_\parallel$, $D_\perp$, the components of the Markovian tensor.

$D_\perp \sim u^{-1}$. The vanishing of the tensor components as $u \to \infty$ complicates the structure of the solution, as we shall see. It is already known (Ref. 2 and references therein) that the spectrum of eigenvalues is continuous, extending from zero to $-\infty$, and includes a point, at zero, corresponding to the stable Maxwellian distribution. Thus, the question of *how* the equilibrium is approached, as $t \to \infty$, is particularly interesting.

II. CALCULATION

A. The equation, $l=0$

We begin with distributions that are isotropic in velocity ($l=0$). Then only $D_\parallel(u)$ enters the kinetic equation, and

$$\frac{\partial}{\partial t} f(u,t) = \frac{\epsilon}{u^2}\frac{\partial}{\partial u} u^2 D_\parallel(u)\left(\frac{\partial}{\partial u} + 2u\right) f(u,t) \quad (0 \leq u < \infty) \tag{5}$$

concerns us. Its solutions are controlled by $D_\parallel$, which (along with $D_\perp$) is positive and decreasing on $0 \leq u < \infty$. The integrability condition on $f(\vec{u},t)$ produces a boundary condition at $u=0$ through the following argument: Integrate Eq. (1) through a region in $\vec{u}$ space that is a large sphere surrounding $\vec{u}=0$, from which a small concentric sphere of radius u_0 has been removed. The integral, which becomes an integral over the two spherical surfaces, must vanish as one goes to the obvious limits. The outer integral vanishes when we choose initial distributions localized in velocity space, and the inner integral leads to the condition

$$\lim_{u_0 \to 0} u_0^2 D_\parallel(u_0)\left(\frac{\partial}{\partial u_0} + 2u_0\right) f(u_0,t) = 0$$

or

$$\lim_{u_0 \to 0} u_0^2 \frac{\partial}{\partial u_0} e^{u_0^2} f(u_0,t) = 0\,,$$

with $f(u_0,t)$ being the average of $f(\vec{u},t)$ over the sphere, the isotropic component.

The change of variable

$$h(u,t) = \sqrt{\Delta(u)}\, e^{u^2} f(u,t) = \sqrt{\Delta(u)}\, g(u,t)\,,$$

and the introduction of the Laplace-transformed distribution function

$$\tilde{f}(u,s) = \int_0^\infty dt\, e^{-\epsilon s t} f(u,t)$$

brings the kinetic equation to the convenient forms

$$\frac{1}{u^2}\frac{\partial}{\partial u} u^2 D_\parallel(u)\left(\frac{\partial}{\partial u} + 2u\right)\tilde{f}(u,s) - s\tilde{f}(u,s) = -\frac{1}{\epsilon} f(u,t=0)\,,$$

$$\frac{\partial}{\partial u}\Delta(u)\frac{\partial}{\partial u}\tilde{g}(u,s) - su^2\tilde{g}(u,s) = -\frac{u^2}{\epsilon} f(u,t=0)\,, \tag{6a}$$

and

$$\frac{d^2}{du^2} h(u,s) - [A(u) + sB(u)]h(u,s) = -H_0(u)\,, \tag{6b}$$

with

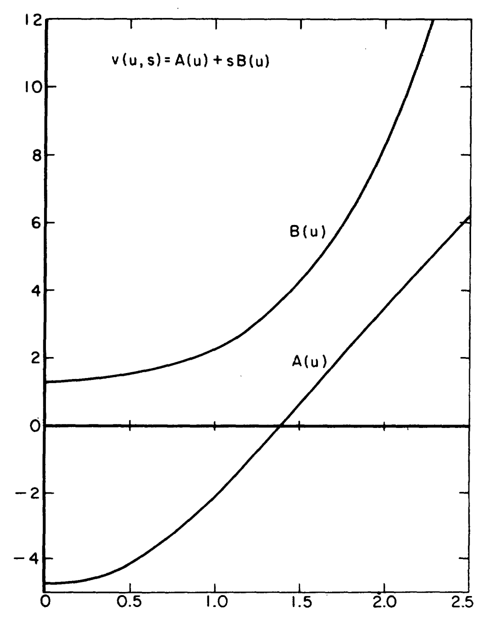

FIG. 2. $A(u) = 1/\sqrt{\Delta} = (\sqrt{\Delta})_{uu}$; $B(u) = 1/D_\parallel(u)$, $\Delta = u^2 e^{-u^2} D_\parallel(u)$.

$$A(u)=\frac{1}{\sqrt{\Delta}}(\sqrt{\Delta})_{uu},\quad B(u)=\frac{1}{D_{\parallel}(u)}\ ,$$

$$H_0(u)=\frac{u^2 f(u,t=0)}{\sqrt{\Delta}}\ ,\quad \Delta=u^2e^{-u^2}D_{\parallel}(u)\,.$$

The boundary condition at the origin is now $h(0,s)=0$.

Both $A(u)$ and $B(u)$ are monotone increasing functions of u^2. They are displayed in Fig. 2. $B(u)$ is positive; for large u, asymptotically,

$$B(u)=u^3+\cdots\ ,$$

where the ellipsis represent exponentially small terms. $A(u)$ rises from its initial value, $A(0)=-3[1+\frac{2}{5}(1/\theta^2)]$ to

$$A(u)=u^2+\frac{3}{4u^2}+\cdots\ ,$$

where, again, the ellipsis represent exponentially small terms when $u\gg\theta$. The fact that the important coefficient in Eq. (6b) is proportional to su^3 as $u\to\infty$ means that we face a singular perturbation when we consider solutions at long time (small $|s|$). The approach to equilibrium is controlled by particles of high energy, a fact that might have been deduced from the dynamics of scattering in the impulse, or linear trajectory approximation.[4]

Since large u is controlling, and since the functions $A(u)$ and $B(u)$ are not simple, one might study the long-time behavior of $f(u,t)$ by replacing $D_{\parallel}(u)$ by its large-u form, $1/u^3$. In mathematical terms, one neglects exponential terms $\sim\exp(-u^2/\theta^2)$ relative to algebraic terms; a corresponding physical model might be one in which the test particles were relatively light ($\theta\to 0$). In any case, the homogeneous equation becomes

$$\frac{d^2}{du^2}h(u,s)-\left(su^3+u^2+\frac{3}{4u^2}\right)h(u,s)=0\ , \qquad (7a)$$

or, after

$$u^2=2\xi,\quad \sigma=\sqrt{2}s,\quad h=\frac{1}{\sqrt{u}}y\ ,$$
$$\frac{d^2}{d\xi^2}y(\xi)-(1+\sigma\sqrt{\xi})y(\xi)=0\,. \qquad (7b)$$

Both equations show the singular perturbation quite clearly. The second was discussed earlier by Mazo and Resibois.[5] The true approach is even more complicated than Ref. 5 would suggest. While its essence is described by Eqs. (7a) and (7b), the analysis of the full equation (6) can be carried out without undue additional affort.

B. Green's function ($l=0$)

We shall discuss the solution to Eq. (6b) for $H_0(u)=\delta(u-u_0)$, $h(0,s)=0$, and $h(u,s)\to 0$ as $u\to\infty$. This function is proportional to the Green's function for Eq. (6a), a function which, for fixed u, u_0 will be analytic in the s plane cut by the negative real axis. We can write it concisely as

$$g(u,u_0,s)=\frac{h_0(u_<,s)h_2(u_>,s)}{W(0,2)}\ , \qquad (8a)$$

where

$$h_0(0,s)=0,\quad \frac{d}{du}h_0(0,s)=1\ ,$$

$h_2(u,s)$ is the "falling" solution, and $W(0,2)$ is the Wronskian. In particular, h_2 may be seen to behave as $\exp(-C_1\sqrt{s}\,u^{5/2})$ for $|\arg s|<\pi$.

The coefficient of $h(u,s)$ in Eq. (6b) is analytic in s for all fixed u, and C^∞ in u for $0\leq u<\infty$. Thus, fundamental solutions like $h_0(u,s)$ and $h_1(u,s)$ [defined through $h_1(0,s)=1$, $(d/du)h_1(0,s)=0$], will be analytic throughout the s plane.[6] It is convenient, then, to represent the falling solution as $h_2=h_0+m(s)h_1$. The Green's function becomes

$$g(u,u_0,s)=-\frac{h_0(u_<,s)}{m(s)}[h_0(u_>,s)+m(s)h_1(u_<,s)]\ , \qquad (8b)$$

and $m(s)$, the only nonanalytic quantity in the expression, generates the spectrum through its zeros and its singularities. For example, $m(s)$ vanishes at $s=0$; at that point h_2 is simply proportional to h_0, and Eq. (6b) has solutions

$$h_0^{(0)}(u)=ue^{-u^2/2}\sqrt{\delta(u)}$$

and

$$h_1^{(0)}(u)=\left[ue^{-u^2/2}\int_0^u du_1 e^{u_1^2}\left(2+\frac{1-\delta(u_1)}{u_1^2\delta(u_1)}\right)-e^{u^2/2}\right]\sqrt{\delta(u)},$$

where we use the convenient notation $\Delta(u)=u^2e^{-u^2}D_{\parallel}(u)=u^2e^{-u^2}D_{\parallel}(0)\delta(u)$. We are describing the Maxwellian equilibrium solution and the point ($s=0$) in the spectrum that corresponds to it. In this self-adjoint and "physical" problem, the spectrum will lie only on the negative real axis. Thus, $m(s)$ is analytic in the cut plane.[6]

C. Inversion of the Green's function

The Laplace inversion of $g(u,u_0,s)$ is carried out along the conventional contour of Fig. 3. We shall discuss the behavior of g on the three sections of the contour; I, $|s|$ large; II, $|s|$ small; III, s real and negative. The integration over I can be neglected because the exponential in the inversion integral easily dominates the large-$|s|$ behavior of h_0 and h_1. The integration for $|s|$

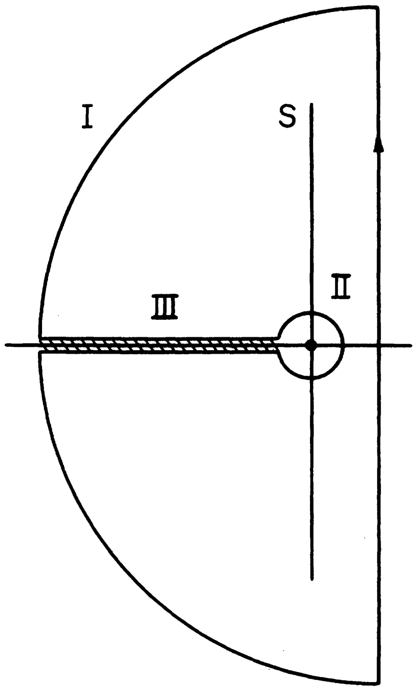

FIG. 3. Contour for Laplace inversion.

small involves us in a study of $m(s)$ near the origin. Noting the analyticity of h_0 and h_1, we have

$$f^{\mathrm{II}}(u,t)=\frac{e^{-u^2}}{4\pi[\Delta(u)\Delta(u_0)]^{1/2}}\int_{\mathrm{II}}\frac{ds}{2\pi i}\,e^{\epsilon st}\frac{h_0(u,s)h_0(u_0,s)}{m(s)}$$

becoming

$$f^{\mathrm{II}}(u,t)=\frac{e^{-u^2}}{4\pi D_{\parallel}(0)}\int_{\mathrm{II}}\frac{ds}{2\pi i}\frac{1}{m(s)}$$

as the contour shrinks. Although the singularity of $m(s)$ at the origin is not isolated, and is not easy to describe precisely, a formal treatment gives the dominant behavior of $m(s)$—enough to evaluate the integral. One merely considers the expansion

$$h_0(u,s)=h_0^{(0)}(u)+sh_0^{(1)}(u)+\cdots,$$

$$h_1(u,s)=h_1^{(0)}(u)+sh_1^{(1)}(u)+\cdots,$$

of the analytic basis functions. $h_0^{(0)}$ and $h_1^{(0)}$ have been noted earlier. The first-order solutions are arrived at by iteration. Then, assuming that near $s=0$, $M(s)=m_1s+\cdots$, one finds that the growing portion $(u\to\infty)$ of

$$h_2(u,s)=h_0^{(0)}(u)+[m_1h_1^{(0)}(u)+h_0^{(1)}(u)]\,s+\cdots$$

can be suppressed—at least through terms proportional to s—by proper choice of m_1. We get, finally,

$$f^{\mathrm{II}}(u,t)=\frac{e^{-u^2}}{4\pi\int_0^\infty du\,u^2e^{-u^2}}\,, \tag{9}$$

as expected.

The contribution from III is the crux of this paper. It is

$$f^{\mathrm{III}}(u,t)=\frac{e^{-u^2}}{4\pi[\Delta(u)\Delta(u_0)]^{1/2}}\times\int_0^\infty\frac{d\eta}{\pi}\,e^{-\epsilon\eta t}h_0(u,-\eta)h_0(u_0,-\eta)\,\mathrm{Im}\frac{1}{m(\eta e^{i\pi})}\,, \tag{10}$$

where $s=\eta e^{i\theta}$. The functions $h_0(u,-\eta)$ and $h_1(u,-\eta)$ are now solutions to

$$\frac{d^2}{du^2}h(u,-\eta)-[A(u)-\eta B(u)]h(u,-\eta)=0\,, \tag{11}$$

an equation related trivially to Eq. (6b), but of quite different nature because of the change in sign. The quantity $v(u,\eta)=A(u)-\eta B(u)$ is displayed in Fig. 4. The boundary value at infinity, or, the analytic continuation of $h_2(u,s)$, is expressed through

$$h_2(u,\eta e^{\pm i\pi})=h_0(u,-\eta)+m(\eta e^{\pm i\pi})h_1(u,-\eta)\equiv h^{(\pm)}(u,\eta)\,. \tag{12}$$

The solutions $h^{(\pm)}$ behave, at infinity, as $\exp(\mp i\frac{2}{5}\sqrt{\eta}u^{5/2})$.

Since

$$\int_0^\infty du\,u^2f^{\mathrm{III}}(u,t)$$

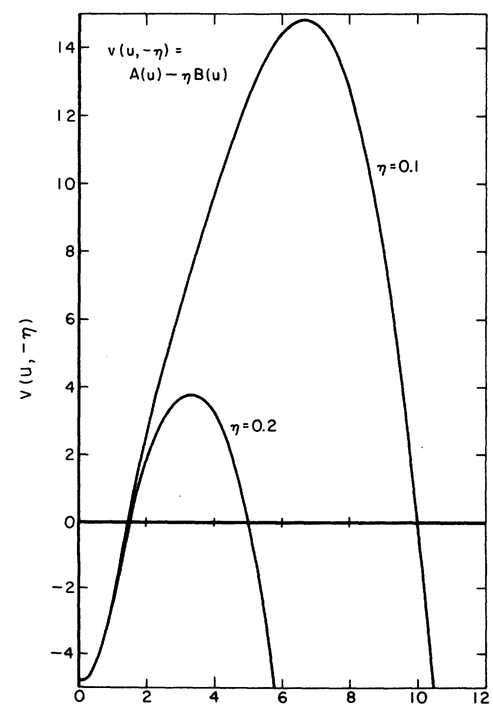

FIG. 4. The potential $v(u)=A(u)-\eta B(u)$.

must vanish, we expect that the integral of the oscillating function $h_0(u, -\eta)$, appropriately weighted, will vanish, too. This follows at once from the direct integration of the homogeneous version of Eq. (6a) or (6b), taking into account the behavior at zero and infinity.

D. Solution via "comparison equation"

Equation (11), $h_{uu} - v(u,\eta)h = 0$, does not have simple solutions. It is an equation characterized by two turning points. There is no "small parameter" for generating an expansion. We have a small parameter if we consider the approach to equilibrium as $t \to \infty$, for that behavior is controlled by the small-η portion of $m(\eta e^{i\pi})$.

When η is small, the two zeros (turning points) of $v(u,\eta)$ are separated widely. The smaller, u_1, is essentially the zero of $A(u)$, and is of minor importance in the approach to equilibrium. The larger, u_2, occurs at $u_2 \gg 1$. In fact, $u_2 \to 1/\eta$ as $\eta \to 0$. Thus, we may write explicitly $[R(u) > 0]$

$$v(u,\eta) = R(u)(u-u_1)(u_2-u)$$
$$\cong \frac{1}{\eta} R(u)(u-u_1)(1-\eta u), \tag{13}$$

and consider the difficulty introduced by the last factor. [See also Eq. (7b).] It is clear that any simple expansion in η is doomed to nonuniformity of convergence when we consider the velocity variable as well. Since we seek a formula connecting large-u and small-u behavior, through Eq. (12), the flaw is deadly. In addition, we expect complicated behavior in $m(s)$ near $s=0$, behavior that will not appear in a power-series expansion. These features appear clearly enough in the model $v_1 = (1-\eta u)$, which can be solved precisely, in terms of Airy functions of argument $\eta^{-2/3}(1-\eta u)$.[2] One finds

$$\mathrm{Im}\frac{1}{m(\eta e^{i\pi})} \equiv \mathrm{Im}\frac{1}{m_+(\eta)}$$
$$= \frac{1}{\pi\sqrt{p}}\frac{1}{[\mathrm{B_i}(p)]^2 + [\mathrm{A_i}(p)]^2},$$
$$p = \frac{1}{\eta^{2/3}} \to \exp\left(-\frac{4}{3\eta}\right) \text{ as } \eta \to 0.$$

One needs a powerful, nonperturbative technique. The method of comparison equations, sometimes called Langer's method, provides the nonperturbative technique.[7] It is used frequently to extract uniform approximations to functions defined by differential equations of second order.[8] The method is based upon the mapping of the differential equation in question onto a simpler equation, whose solutions are known. The situation is such that one seeks an expansion in a large (or small) parameter and recognizes that turning points play an exceptional role. They dominate the asymptotics. Thus, the mapping is onto a simpler equation having the same number of turning points as the original equations. The accuracy and success of the technique is impressive. One sees a simple version of it in elementary treatments of the WKB method, when the Schrödinger equation is mapped onto Airy's equation.[9]

The mapping is carried out via the Schwarz transformation; thus, with the change of variables $\xi = \xi(u)$, $h(u) = 1/\sqrt{\xi_u}\, y(\xi)$, and $\xi_u = d\xi/du > 0$,

$$\frac{d^2}{du^2} h(u) - v(u)h(u) = 0$$

becomes

$$\frac{d^2}{d\xi^2} y(\xi) - V(\xi) y(\xi) = 0,$$

with

$$\xi_u^2 V(\xi) = v(u) - \frac{Q_{uu}}{Q} \tag{14}$$

and $Q = 1/\sqrt{\xi_u}$. With $V(\xi)$ chosen to be "simple", Eq. (14) becomes a difficult differential equation to determine $\xi(u)$. The troublesome term is the second, which may be written

$$\frac{Q_{uu}}{Q} = [(\ln Q)_u]^2 + (\ln Q)_{uu}.$$

One expects the derivatives of the logarithm of the transformation to be small if the transformation is gentle enough. Indeed, the term vanishes for linear fractional transformations. Our strategy, then, is to treat Q_{uu}/Q as a small correction. We shall do the calculation first, neglecting Q_{uu}/Q. Then, by means of estimates and examples, we shall convince the reader of its minor role in the behavior of solutions as $\eta \to 0$. That limit is controlled by the exceptional points.

1. Calculation

The simple equation for the problem of two turning points is the equation for the parabolic cylinder functions,[10]

$$\frac{d^2y}{d\xi^2} + (\tfrac{1}{4}\xi^2 - a)y = 0. \tag{15}$$

The differential equation for the transformation is, with neglect of Q_{uu}/Q,

$$(\tfrac{1}{4}\xi^2 - \beta^2)\xi_u^2 = R(u)(u-u_1)(u-u_2)$$

or

$$V(\xi)\xi_u^2 = v(u),$$

with $\beta^2 = a$, and $R(u)$, u_1, and u_2 depending upon η.

The solution giving a smooth mapping is

$$(0\leq u\leq u_1)\quad \int_{\xi}^{-2\beta} d\xi_1\,|V(\xi_1)|^{1/2}=\int_u^{u_1} d\bar{u}\,|v(\bar{u})|^{1/2},$$

$$(u_1\leq u\leq u_2)\quad \int_{-2\beta}^{\xi} d\xi_1\,|V(\xi_1)|^{1/2}=\int_{u_1}^{u} d\bar{u}\,|v(\bar{u})|^{1/2},\qquad(16)$$

$$(u_2\leq u)\quad \int_{2\beta}^{\xi} d\xi_1\,|V(\xi_1)|^{1/2}=\int_{u_2}^{u} d\bar{u}\,|v(\bar{u})|^{1/2},$$

with $u=0$ mapped onto $\xi_0<0$, u_1 onto $\xi=-2\beta$, and u_2 onto $\xi=2\beta$. ξ_0 and β are determined as functions of η by the equations

$$2\beta^2\int_{-1}^{1} ds(1-s^2)^{1/2}=\pi a=\int_{u_1}^{u_2} d\bar{u}\,|v(\bar{u})|^{1/2},$$
$$2\beta^2\int_{1}^{\zeta_0} ds(s^2-1)^{1/2}=\int_{0}^{u_1} d\bar{u}\,|v(\bar{u})|^{1/2},\qquad(17a)$$

with $2\beta\zeta_0=|\xi_0|$. These stem from evaluating Eqs. (17) at the end points of their intervals. $\zeta(\eta)$ plays an important role in the asymptotics, as $\eta\to0$. We shall also use the notation

$$\tfrac{2}{3}t^{3/2}=\int_u^{u_1} d\bar{u}\,|v(\bar{u})|^{1/2}=2\beta^2\int_1^{\zeta} ds(s^2-1)^{1/2}.\qquad(17b)$$

When $u=0$, $\frac{2}{3}t_0^{3/2}(\eta)$ denotes the integral in the second equation of Eqs. (17).

We are particularly interested in $E(a,\xi)$ and $E^*(a,\xi)$, the complex solutions to Eq. (15). For $\xi\gg|a|$, $E(a,\xi)\sim(\frac{2}{\xi})^{1/2}\exp(i\frac{1}{4}\xi^2)$.[10] Since the transformation, Eq. (16), gives $\frac{1}{4}\xi^2\sim\frac{2}{5}\sqrt{\eta}\,u^{5/2}$, we note that $h^{(+)}(u,\eta)$ of Eq. (12) corresponds to $E^*(a,\xi)$. Then, if we write

$$h_0(u)=Ae(a,\xi)+A^*e^*(a,\xi),$$
$$h_1(u)=Be(a,\xi)+B^*e^*(a,\xi),$$

with $\sqrt{\xi_u}\,e(a,\xi)=E(a,\xi)$, consideration of the boundary conditions at $u=0$ gives

$$h_0(u,-\eta)=\mathrm{Im}[e^*(a,\xi_0)e(a,\xi)],$$
$$m(\eta e^{i\pi})\equiv m_+(\eta)=\frac{e^*(a,\xi_0)}{e_u^*(a,\xi_0)},\qquad(18)$$
$$\mathrm{Im}\frac{1}{m_+(\eta)}=-\frac{1}{|e(a,\xi_0)|^2},$$

where we have used the Wronskian of E and E^*.[10] The expressions (18) are accurate only in the limit $\eta\to0$, where a and $|\xi_0|\to\infty$. In this limit, in the notation of Ref. 10 we have

$$-\mathrm{Im}\frac{1}{m_+(\eta)}=\frac{1}{2}\frac{e^{-\pi a}}{W^2(a,-|\xi_0|)}(1+\cdots),$$

where the ellipsis represent exponentially small terms. $W(a,-|\xi_0|)$ itself requires a uniform representation when both its variable and its parameter are large and are functions of a single parameter. The expansion is available, in terms of Airy functions [10]

$$W(a,-|\xi_0|)\sim2\sqrt{\pi}\,(4a)^{-1/4}e^{\pi a/2}\left(\frac{t_0}{\zeta_0^2-1}\right)^{1/4}\mathrm{Ai}(-t_0),\qquad(19)$$

where ζ_0 is noted in Eq. (17) and

$$t_0^{3/2}=\tfrac{3}{2}\int_0^{u_1} d\bar{u}\,|v(\bar{u})|^{1/2}.$$

As a final step, we turn to the evaluation of $a(\eta)$, $t_0(\eta)$, and $\zeta_0(\eta)$. These are described in Appendix A. We find

$$a(\eta)=\frac{4}{15}\frac{1}{\pi\eta^2}+a_0+o(1),$$
$$t_0(\eta)=t_{00}+o(1),\qquad(20)$$
$$\psi(\eta)=(\zeta_0^2-1)\frac{a}{t_0}=\left(\frac{4}{15\pi}\right)^{1/3}\frac{1}{\eta^{2/3}}[1+o(1)].$$

Then, we have

$$-\mathrm{Im}\frac{1}{m_+(\eta)}\sim\frac{1}{4\pi}\left(\frac{4}{15\pi}\right)^{1/6}\frac{e^{-2\pi a_0}}{\mathrm{Ai}^2(-t_0)}\frac{\exp[-\frac{8}{15}(1/\eta^2)]}{\eta^{1/3}}.\qquad(21)$$

If we return to Eq. (10) for $f^{\mathrm{III}}(u,t)$ we note that the crucial factor for the asymptotic evaluation is $\exp\{-[\epsilon\eta t+\frac{8}{15}(1/\eta^2)]\}$. With $t\to\infty$, the simplest application of the saddle-point method to f^{III} gives

$$f^{\mathrm{III}}(u,t)\sim-\left(\frac{2}{3\pi}\right)^{1/2}\gamma^{2/3}\frac{B}{4\pi}\frac{e^{-u^2}}{[\Delta(u)\Delta(u_0)]^{1/2}}h_0\left(u,-\frac{\gamma}{\tau^{1/3}}\right)h_0\left(u_0,-\frac{\gamma}{\tau^{1/3}}\right)\frac{e^{-3\tau^{(2/3)}/2}}{\tau^{5/9}},\qquad(22)$$

with $\gamma=4/\sqrt{15}$, $\tau=\gamma\epsilon t$. B is the constant factor in Eq. (21). Equation (22) with its unusual time dependence is one of the principal results of this paper. We have generated the expression

$$-\mathrm{Im}[1/m_+(\eta)]=\exp[-\tfrac{8}{15}(1/\eta^2)-\tfrac{1}{3}\ln\eta+\cdots]$$

to be used, for η small, in the evaluation of Eq. (10). We shall see that though the term $O(1)$ may change upon iteration of Eq. (14), the leading terms will not.

If one wished to evaluate Eq. (22) for some (u,u_0,τ), one would be advised to construct the analytic h_0 in a direct manner, making use, perhaps, of the smallness of $\gamma/\tau^{1/3}$. One would *not* want to use the approximate form given by Eq. (18), for the transformation $u \rightleftarrows \xi$ is singular as $\eta \to \infty$ ($\tau \to \infty$), and would introduce spurious effects into the calculation.

III. DISCUSSION OF ERROR

A careful discussion of the error in Eq. (22) would involve a difficult analysis of an integral equation for $h_0(u, -\eta)$, an equation whose kernel was constructed of parabolic cylinder functions.[8] We eschew this approach and shall rely on heuristic arguments to convince the reader that the leading terms of Eq. (22) give a true picture of the relaxation.

We begin by arguing the key role played by the turning points (exceptional points)—indeed, by the point $u_2(\eta)$, which is close to $1/\eta$ as $\eta \to 0$. Suppose $v(u,\eta) = R_1(u^2,\eta)[u_2(\eta) - u]$, with u_2 as before, and $R_1 > 0$. That is, we alter $v(u,\eta)$ when u is small, to remove the first turning point. Now the comparison equation is Airy's equation, $V(\xi) = -\xi$. The mapping takes u_2 to $\xi = 0$ and $u = 0$ to $\xi = \xi_0 < 0$. Thus

$$(0 \le u \le u_2 \,|\, \xi \le 0) \quad \int_{\xi_0}^{\xi} d\xi_1 |\xi_1|^{1/2} = \int_0^u d\bar{u}[v_1(\bar{u})]^{1/2},$$
$$(u \ge u_2 \,|\, \xi \ge 0) \quad \int_0^{\xi} d\xi_1 \sqrt{\xi_1} = \int_{u_2}^{u} d\bar{u}\, |v_1(\bar{u})|^{1/2} \tag{23}$$

and the function $\sqrt{\xi_u} e(\xi)$ now denotes the standard solution $\mathrm{Bi}(\xi) + i\,\mathrm{Ai}(\xi)$. The calculation of $\mathrm{Im}[1/m_+(\eta)]$ goes as before. We find, with the neglect of exponentially small terms, that as $\eta \to 0$,

$$-\mathrm{Im}\frac{1}{m_+(\eta)} \sim v_1(0,\eta)\exp(-\tfrac{4}{3}|\xi_0|^{3/2}),$$
$$\tfrac{2}{3}|\xi_0|^{3/2} = \int_0^{u_2} du[R_1(u^2,\eta)(u_2 - u)]^{1/2}, \tag{24}$$

which we estimate as $\eta \to 0$, $u_2 \to 1/\eta \to \infty$. Since $R_1(u^2,\eta)$ is a smooth, bounded function, approaching ηu^2 as $u \to \infty$, we are led to estimate Eq. (24) as

$$\tfrac{2}{3}|\xi_0|^{3/2} \sim \int_0^{u_2} du[\eta u^2(u_2 - u)]^{1/2} \sim \frac{4}{15}\frac{1}{\eta^2},$$

producing the $\exp[-\frac{8}{15}(1/\eta^2)]$ noted earlier. This is a rather general result. The turning point $u_2 = 1/\eta$ controls the asymptotic behavior. The other features of $\mathrm{Im}(1/m_+)$, the finite limiting value $[v_1(0,0)]^{1/2}$, and additional terms in the exponent depend upon details of the model. For example, let $v_1(u,\eta) = (a^2 + u^2)(1 - \eta u)$ so that, like the correct $v(u,\eta)$, the behavior when $\eta \ll 1$ is insensitive to η. Then calculation gives

$$\tfrac{2}{3}|\xi_0|^{3/2} = \frac{4}{15}\frac{1}{\eta^2} - \tfrac{1}{2}a^2\ln\eta + O(1)$$

as $\eta \to 0$. This leads to

$$-\mathrm{Im}\frac{1}{m_+(\eta)} = A\eta^{a^2}\exp\left(-\frac{8}{15}\frac{1}{\eta^2}\right)[1 + O(\eta)],$$

and the degree of sensitivity to model is apparent.

Let us turn to a more systematic discussion of the accuracy of Eq. (21). We seek the smooth transformation $\xi(u)$ that satisfies the differential Eq. (14) with $V(\xi) = (\beta^2 - \frac{1}{4}\xi^2)$ and $v(u) = A(u) - \eta B(u)$. Imagine that we obtain $\xi(u)$ via convergent iteration, beginning with $\xi_0(u)$, the approximate solution discussed above. Thus

$$\xi_{m,u}^2(\beta^2 - \tfrac{1}{4}\xi^2) = v_m(u), \quad m = 0, 1, \ldots$$
$$v_m(u) = v_{m-1}(u) - \frac{Q_{m-1,uu}}{Q_{m-1}}, \quad m = 1, 2, \ldots \tag{25}$$
$$v_0(u) = v(u),$$

and at each stage we proceed as above, Eqs. (16) onward. Of course, we require that all of the v_m are alike—in particular, that all have two turning points, so that the parabolic cylinder functions are always the correct comparison functions. Then, to prove that the asymptotic behavior of $\mathrm{Im}[1/m_+(\eta)]$ is indeed $\eta^{-1/3}\exp[-\frac{8}{15}(1/\eta^2)]$, we need to show that for every m,

$$a_m(\eta) = \frac{4}{15}\frac{1}{\pi\eta^2} + a_m + O(1),$$

and that

$$\lim_{\eta \to 0}\int_0^{u_{m1}(\eta)} du\, |v_m|^{1/2}$$

is finite and not zero. As the calculations of Appendix A indicate, these two facts guarantee the conjectured asymptotic form. We do not prove the result for arbitrary m. Instead, we consider the case $m = 1$, which suggests, strongly, the truth of the conjecture.

We shall evaluate the first correction

$$v_0(u) - v_1(u) = Q_{0,uu}/Q_0, \quad Q_0 = (\xi_{0,u})^{-1/2},$$

in the important region $u \gg 1$, where we may replace $v_0(u)$ by $u^2(1 - \eta u)$, and $V(\xi)$ by $\beta(\frac{1}{2}\xi - \beta)$. Then, calculation gives

$$\xi_{0,u} = 5^{1/2}\left(\frac{3\pi}{4}\right)^{1/6}\frac{\eta u}{(1 + \frac{3}{2}\eta u)^{1/3}}$$

and

$$\frac{Q_{0,uu}}{Q_0} = \frac{3 + 8\eta u + 4(\eta u)^2}{u^2(2 + 3\eta u)^2} = \eta^2 F(\eta u).$$

The correction, falling as u^{-2}, affects the small-η behavior of $a(\eta)$ (see Appendix A) only in the $O(1)$ term. It vanishes at the turning point, $\eta u = 1$, as

η^2. It is safe to conjecture that further iteration will not disturb this result.

The first correction is a smooth function of u for all u. If we examine it in the neighborhood of the first turning point, we find that its value at u_1 is proportional to $\eta^{8/3}$. Thus, in the limit $\eta\to 0$, it disturbs neither u_1 nor u_2. Its value at $u=0$ may also be computed. It is nonzero in the limit, a few percent of the uncorrected value. Thus, the correction does not alter the picture of a (somewhat) parabolic potential with two turning points. It is extremely unlikely that higher corrections will do so. We conclude that $\eta^{-1/3}\exp[-\frac{8}{15}(1/\eta^2)]$ is undoubtedly correct, our estimate for the multiplying constant, the $O(1)$ term, being subject to some error.

IV. THE CASE $l>0$

A. General[11]

The case $l>0$ is complicated by the appearance of a new term in the kinetic equation. For example, Eq. (6b) is now

$$\frac{d^2}{du^2}h_l(u,s)-v_l(u,s)h_l(u,s)=-H_l(u)\,, \tag{26}$$

with $v_l(u,s)=A(u)+[l(l+1)/u^2]D_\perp(u)/D_\parallel(u)+sB(u)=A_l+sB$ and $H_l(u)=(u^2/\sqrt{\Delta})f_l(u,t=0)$. The new term, a centrifugal potential, causes the point $u=0$ to be singular and forces us to refer to another point in the construction of the analytic basis functions which are so helpful in the analysis of the Green's function $g_l(u,u_0,s)$. As before,

$$g_l(u,u_0 s)=\frac{h_0(u_<,s)h_2(u_>,s)}{w(0,2)}\,, \tag{27}$$

h_0 being regular at $u=0$ and h_2 falling as $u\to\infty$. [We shall suppress l, as we have on the right-hand side (rhs), whenever we can.] Now ($l>0$) neither h_0 *nor* h_2 is analytic in s. We write them as

$$h_0(u,s)=\phi_0(u,s)+m_0(s)\phi_1(u,s)\,,$$

$$h_2(u,s)=\phi_0(u,s)+m_2(s)\phi_1(u,s)\,,$$

the analytic basis functions $\phi_k(u,s)$ being defined through $\phi_0(a,s)=0$, $(d/du)\phi_0(a,s)=1$; $\phi_1(a,s)=1$, $(d/du)\phi_1(a,s)=0$. The point a is arbitrary; it should not affect the principal results of our calculation.

The analytic continuation of the Green's function goes as before, as does the discussion of the Laplace inversion contour. $m_0(s)$ and $m_2(s)$ are analytic in the s plane cut by the negative real axis. When $l\neq 0$ the point $s=0$ has no special significance. We are led to

$$g_l(u,u_0;t)=\int_0^\infty\frac{d\eta}{\pi}e^{-\epsilon\eta t}\{w_1(\eta)\phi_0(u,-\eta)\phi_0(u_0,-\eta)+w_2(\eta)\phi_1(u,-\eta)\phi_1(u_0,-\eta)+w_3(\eta)[\phi_0(u,-\eta)\phi_1(u_0,-\eta)+\phi_1(u,-\eta)\phi_0(u_0,-\eta)]\} \tag{28}$$

with

$$w_1(\eta)=\left(\operatorname{Im}\frac{1}{m_0-m_2}\right)_{s=\eta e^{i\pi}}\,,\quad w_2(\eta)=\left(\operatorname{Im}\frac{m_0m_2}{m_0-m_2}\right)_{s=\eta e^{i\pi}}\,,$$

$$w_3(\eta)=\left(\operatorname{Im}\frac{m_0}{m_0-m_2}\right)_{s=\eta e^{i\pi}}\,. \tag{29}$$

The weight functions $w_k(\eta)$ will control the relaxation of $g_l(\cdots t)$. A special case is worth noting here. If $m_0(\eta e^{i\pi})$ is real, we find $w_3(\eta)=m_0w_1$ and $w_2(\eta)=m_0^2w_1$, so that the compact equation

$$g_l(u,u_0;t)=\int_0^\infty\frac{d\eta}{\eta}e^{-\epsilon\eta t}w_1(\eta)h_0(u,\eta e^{i\pi})h_0(u_0,\eta e^{i\pi}) \tag{30}$$

obtains.

The most important member of the h_l is h_1 because of its connection with the autocorrelation function for velocity. In terms of $\vec{u}$, the dimensionless velocity,

$$\langle\vec{u}\cdot\vec{u}(t)\rangle=4\pi\int_0^\infty du\frac{u^3M(u)}{\sqrt{\Delta(u)}}h(u,t)\,, \tag{31}$$

with $h(u,t)$ the solution corresponding to $l=1$ and $h(u,t=0)=u\sqrt{\Delta}$. When the Green's function for the differential operator of Eq. (26) is introduced,

$$\langle\vec{u}\cdot\vec{u}(t)\rangle=4\pi\int_0^\infty du\,u^2\left(\frac{M}{D_\parallel}\right)^{1/2}\times\int_0^\infty du_0u_0^2\left(\frac{M}{D_\parallel}\right)^{1/2}g_1(u,u_0;s)\,,$$

leading finally to

$$\langle\vec{u}\cdot\vec{u}(t)\rangle=\int_0^\infty\frac{d\eta}{\pi}e^{-\epsilon\eta t}w_1(\eta)[H(\eta)]^2\,,$$
$$H(\eta)=\int_0^\infty du\,u^3\frac{M}{D_\parallel}h_0(u,\eta e^{i\pi})\,, \tag{32}$$

if $m_0(\eta e^{i\pi})$ is real. Since $h_0=\phi_0+m_0\phi_1$, the ϕ_k being regular in η, the asymptotic ($t\to\infty$) behavior of $\langle\vec{u}\cdot\vec{u}(t)\rangle$ will be controlled by possible singular behavior of w_1 and m_0 at $\eta=0$. Estimation of these functions is the primary concern of the next section.

B. Solution via comparison equation

As before, we use Langer's method to generate uniform approximations. The approximating potential $V(\xi)$ should contain the singular behavior of v_l at $u=0$ as well as $u\to\infty$. Since $D_\perp(u)/D_\parallel(u) = 1+O(u^2)$, the former is $v_l(u)=l(l+1)/u^2+O(1)$. [$v_l(u,-\eta)$ is displayed in Fig. 5. Note that there is but one turning point when η is small.] Thus, we choose

$$\frac{d^2}{d\xi^2}y(\xi)-\left(\frac{l(l+1)}{\xi^2}-\xi^2\right)y(\xi)=0 \tag{33}$$

as a comparison equation. Its solutions are cylinder functions $y(\xi)=\sqrt{\xi}\,C_\nu(\frac{1}{2}\xi^2)$ with $\nu=\frac{1}{4}(2l+1)$. The mapping of the u axis onto the ξ axis will assign the turning point $u_2(\eta)$ to the fixed point $\xi_l = [l(l+1)]^{1/4}$. For small η there is only one turning point, and its dependence upon l is slight. Thus

$$\begin{aligned}(0\le\xi\le\xi_l)\quad &\int_\xi^{\xi_l} d\xi_1[V(\xi_1)]^{1/2} = \int_u^{u_2(\eta)} du_1[v_l(u_1,-\eta)]^{1/2},\\ (\xi_l\le\xi<\infty)\quad &\int_{\xi_l}^{\xi} d\xi_1\,|V(\xi_1)|^{1/2} = \int_{u_2(\eta)}^{u} du_1\,|v_l(u_1,-\eta)|^{1/2}.\end{aligned} \tag{34}$$

An arbitrary but fixed point in $(0\le u<\infty)$ will correspond to some $\xi(\eta)$ which approaches zero as

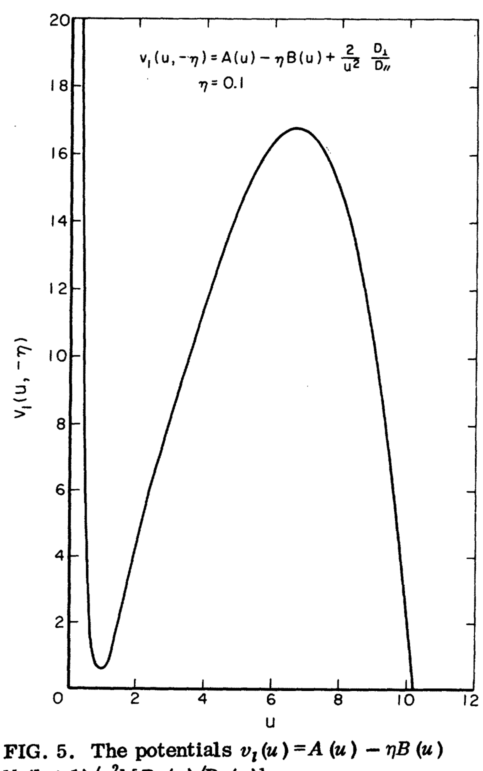

FIG. 5. The potentials $v_l(u)=A(u)-\eta B(u) + [l(l+1)/u^2][D_\perp(u)/D_\parallel(u)]$.

$\eta\to 0$.

We construct (approximate) ϕ_k and w_k from $h(u) = (\xi/\xi_u)^{1/2}C_\nu(\frac{1}{2}\xi^2)$. Of course, $h_0(u)$, the solution well behaved as $u\to 0$, is $(\xi/\xi_u)^{1/2}J_\nu(\frac{1}{2}\xi^2)$. The proper solution for $u\to\infty$, the analytic continuation of the $L^2(a,\infty)$ solution $h_2(u)$, is $(\xi/\xi_u)^{1/2}H_\nu^{(2)}(\frac{1}{2}\xi^2)$. A little additional calculation is needed to extract $m_0(\eta e^{i\pi})$ and $m_2(\eta e^{i\pi})$. We find

$$\begin{aligned}\frac{1}{m_0(\eta e^{i\pi})}&=\frac{d}{du}\ln\left(\frac{\xi}{\xi_u}\right)^{1/2}J_\nu(\tfrac{1}{2}\xi^2)\bigg|_{u=a},\\ \frac{1}{m_2(\eta e^{i\pi})}&=\frac{d}{du}\ln\left(\frac{\xi}{\xi_u}\right)^{1/2}H_\nu^{(2)}(\tfrac{1}{2}\xi^2)\bigg|_{u=a}\end{aligned} \tag{35}$$

[η is contained in the real transformation $\xi = \xi(u,\eta)$]. Clearly, $1/[m_0(\eta e^{i\pi})]$ is real, and along with it we need only compute

$$w_1(\eta)=\left(\operatorname{Im}\frac{1}{m_0-m_2}\right)_{s=\eta e^{i\pi}} = \frac{\pi}{4}\left[\frac{d}{du}\left(\frac{\xi}{\xi_u}\right)^{1/2}J_\nu\right]^2_{u=a}$$

to discuss the behavior of g_l, as $t\to\infty$.

We turn to the relation between ξ and u as $\eta\to 0$. In that limit, $u_2(\eta)\to\infty$ and the rhs of Eq. (34) diverges, while—for fixed u—$\xi(u,\eta)\to 0$. The rhs may be analyzed along the lines of Appendix A, with quite similar results. In the limit,

$$[l(l+1)]^{1/2}\ln\frac{1}{\xi}=\frac{4}{15\eta^2}+O(1),$$

the $O(1)$ term containing the fixed u. Since the Bessel functions will be evaluated for small values, it is no surprise that computation gives

$$\begin{aligned}\omega_1(\eta)&=[B_l(a)+o(1)]\exp\left(-\frac{\gamma_l^2}{2\eta^2}\right),\\ \gamma_l^2&=\frac{16}{15}\frac{\nu}{[l(l+1)]^{1/2}},\\ m_0(\eta e^{i\pi})&=[A_l(a)+O(1)],\end{aligned} \tag{36}$$

the $o(1)$ terms vanishing with η. These results are approximate in that they stem from the first step in the iterative sequence described above [Eq. (25)]. But an analysis of error along those earlier lines show that iteration merely alters the values of the constants A_l and B_l. For the mapping comparison potential $V(\xi)$ has correct behavior at u, $\xi\to 0$ (fixed η) so that the correction $Q_{0,uu}/Q_0$ introduces a term regular as $\xi\to 0$. At large u, including the turning point, the analysis is quite similar to that presented for $l=0$. The corrections are insignificant.

Equation (36) gives some insight into the relaxation of g_l [Eq. (30)] and the autocorrelation function, Eq. (31). The asymptotic evaluation of the inversion integral gives.

$$g_l(u,u_0;t)\sim c_l\frac{e^{-3/2\tau_l^{2/3}}}{\tau_l^{2/3}}\,h_0\!\left(u,-\frac{\gamma_l}{t_l^{1/3}}\right)h_0\!\left(u_0,-\frac{\gamma_l}{t_l^{1/3}}\right),$$

$$\langle\vec{u}\cdot\vec{u}(t)\rangle\sim D\frac{e^{-3/2\tau_1^{2/3}}}{\tau_1^{2/3}},\tag{37}$$

where

$$\tau_l=\epsilon\gamma_l t,\quad \gamma_l^2=\frac{16}{15}\frac{\nu}{[l(l+1)]^{1/2}},$$

and $h_0=\phi_0(u,-\gamma_l/\tau_l^{1/3})+[A_l+o(1)]\phi_1(u,-\gamma_l/\tau_l^{1/3})$, the $o(1)$ referring to the small quantity $\gamma_l/\tau_l^{1/3}$. The dependence of τ_l upon l is quite weak. As we have presented the quantities c_l and D, they may depend upon the (arbitrary) point a. In that case, the dependence will be removed by iteration, as sketched in Eq. (29). There is no point in evaluating these constants here.

V. FINAL COMMENTS

A. Diffusion in two dimensions (2D)

The analysis of Eq. (1) is quite straightforward in a (2D) velocity space. Thus,

$$D_\perp=\tfrac{1}{2}\sqrt{\pi}\,e^{-u^2/2}[I_0(\tfrac{1}{2}u^2)+I_1(\tfrac{1}{2}u^2)],$$

$$D_\parallel=\tfrac{1}{2}\sqrt{\pi}\,e^{-u^2/2}[I_0(\tfrac{1}{2}u^2)-I_1(\tfrac{1}{2}u^2)],$$

The relation $D_\parallel=(\partial/\partial u)(uD_\perp)$ holds, as it did before. As in the case of (3D), $D_\parallel=O(u^{-1})$, for $u\to\infty$. This aspect of the diffusion tensor comes as a surprise. In fact, $D_\parallel=1/2u^3+\cdots$. When the kinetic equation for (2D) is written in the form of Eq. (6b) in Eq. (26), one notes that the dominant large $-u$ behavior is altered only in the replacement of s (or η) by $2s$ (or 2η), and the dominant behavior at small u is altered only by the replacement of $l(l+1)$ $(l=0,1,\ldots)$ by l^2. Thus, except for trivial changes, the relaxation, at long times, of the (2D) equation is quite similar to that of the (3D). The equation describing relaxation via a sequence of weak scatterings in a passive medium does not show the interesting dependence upon dimension that appears in systems of hard spheres and/or hard disks.[12]

B. Physics?

Do our results have anything to do with any systems of physical interest? Plasma physicists certainly use the Fokker-Planck equation with abandon,[13] but not much attention has been paid to its asymptotics. One might expect the equation to govern the relaxation of certain fluctuations in a model plasma at long times (low frequencies). Thus, the computer studies of self-diffusion, carried out by Hansen, McDonald, and Pollack,[14] and the related discussions by Gould and Mazenko[15] and by Baus,[16] may provide a clue. The plasma should, of course, be quite "hot" ($\Gamma\ll 1$) and the autocorrelation function should be examined in an interval of time that is long with respect to the time between collisions, but short with respect to the time required for hydrodynamic feedback to occur. I have not been able to find such a regime in the computer dynamical data. On the other hand, our observations about the Fokker-Planck equation in 2D versus 3D may have some connection with recent remarks of Baus. In Ref. 16, Baus provides arguments to support the computer discovery that self-diffusion exists in the two-dimensional as well as the three-dimensional electron liquid. As we have remarked, analysis of the Fokker-Planck equation leads to the same conclusion.

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APPENDIX A

1. The function $a(\eta)$

We define $a(\eta)$ as

$$a(\eta)=\frac{1}{\pi}\int_{u_1(\eta)}^{u_2(\eta)}du\sqrt{v(u)}\ ,\tag{A1}$$

and in estimating it for small η, we are making a calculation analogous to the determination of the period of a classical motion, as a function of a parameter in the Hamiltonian. We shall use

$$u_1(\eta)=u_{10}+O(1),$$

$$u_2(\eta)=\frac{1}{\eta}+O(\eta^3),$$

$$v(u,\eta)=-\eta u^3+u^2+\frac{3}{4u^2}+\cdots\qquad(u\to\infty)$$

where the ellipsis represent exponentially small terms and

$$v_\infty(u,\eta)\equiv-\eta u^3+u^2,\quad u_{20}=\frac{1}{\eta}<u_2(\eta)$$

to show that

$$a(\eta)=\frac{4}{15}\frac{1}{\pi\eta^2}+a_0+o(1)\ \text{ as }\eta\to 0.\tag{A2}$$

We compare the integrals $a(\eta)$ and

$$a_\infty(\eta) = \frac{1}{\pi} \int_0^{u_{20}} du \sqrt{v_\infty(u)} = \frac{4}{15} \frac{1}{\pi\eta^2} .$$

We write

$$\pi[a(\eta) - a_\infty(\eta)] = -\int_0^{u_1} du \sqrt{v_\infty} + \int_{u_1}^{u_{20}} du(\sqrt{v} - \sqrt{v_\infty}) + \int_{u_{20}}^{u_2} du \sqrt{v(u)} . \tag{A3}$$

The first integral is $O(1)$, the third is $o(1)$, since $u_2 - u_{20}$ is $o(1)$ and the integrand is bounded. The second integral is

$$\int_{u_1}^{u_{20}} d\bar{u} \frac{v(u) - v_\infty(u)}{\sqrt{v(u)} + \sqrt{v_\infty(u)}} ,$$

and its principal contribution comes from the region $u > u_* \gg 1$. Then, it is

$$\tfrac{3}{8} \int_{u_*}^{1/\eta} \frac{du}{u^3} \frac{1}{\sqrt{1-u}} + O(1) .$$

The integral is $\frac{3}{16}(1/u_*^2) + o(1)$, whence the right-hand side of (A3) is $O(1)$ and (A2) holds. Numerical integration gives

$$a_0 = -0.392 .$$

2. The function $t(\eta)$

We see that

$$t_0^{3/2}(\eta) = \tfrac{3}{2} \int_0^{u_1(\eta)} du \, |v(u, \eta)|^{1/2}$$

approached

$$t_{00}^{3/2} = \tfrac{3}{2} \int_0^{u_{10}} du \, |A(u)|^{1/2} \tag{A4}$$

as $\eta \to 0$. $A(u)$ is the function defined in Eq. (6b). Numerical integration gives $t_\infty \cong (\frac{27}{22})^{1/3} \pi^{2/3} = 2.297\ldots$ (more precisely, $t_\infty = 2.306\ldots$).

3. The function $\zeta_0(\eta)$

From Eq. (17) and the definition of $t(\eta)$,

$$\int_1^{\zeta_0} ds (s^2 - 1)^{1/2} = \frac{1}{3} \frac{t_0^{3/2}(\eta)}{a(\eta)} . \tag{A5}$$

As $\eta \to 0$, $\zeta \to 1$. Expanding both sides of the equation gives

$$\zeta_0 - 1 = \frac{t_{00}}{2} (\tfrac{15}{4}\pi)^{2/3} \eta^{4/3} + \cdots .$$

Then, the combination $\psi(\eta) = (\zeta_0^2 - 1) a / t_0$, appearing in Eq. (19) becomes

$$\psi(\eta) \sim [a(\eta)]^{1/3} \sim \left(\frac{4}{15\pi}\right)^{1/3} \frac{1}{\eta^{2/3}}$$

in the limit.

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