Large-scale properties of unstable systems governed by the Kuramoto-Sivashinksi equation

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(Received 3 December 1980)

The dynamic renormalization-group method is developed for investigation of correlations generated by the Kuramoto equation with random initial conditions. It is shown that elimination of modes from domain $\Lambda < k < \infty$ generates the random force and "viscosity" which is positive in the $d=1$ and negative in the $d \geq 2$ cases. The stable fixed point is found in the $d=1$ system while the theory is asymptotically free when $d=3$. No fixed point exists in the $d=2$ case which explains the patterns formation obtained from computer simulations.

It has recently been understood that a wide class of collective phenomena associated with instabilities (chemical turbulence,[1-3] self-turbulization of flames in combustion,[4,5] processes in certain biological systems[6]) can be described by the nonlinear equation

$$\frac{\partial\Theta(\vec{r},t)}{\partial t} = -(\kappa_0\nabla^2 + F_0\nabla^4)\Theta(\vec{r},t) + [\nabla\Theta(\vec{r},t)]^2 \ . \tag{1}$$

The dynamic variable $\Theta(\vec{r},t)$ has different meanings depending on the context. Equation (1) is remarkable not only because it represents an extremely wide class of phenomena close to the bifurcation points. Computer experiments revealed that (1) with *deterministic* initial values $\Theta(\vec{r},0)$ show a turbulencelike irregular motion in a one-dimensional case.[1,2] That is why the processes described by (1) is often classified as self-turbulization.[4,5] It was shown further that in the two-dimensional case Eq. (1) generates different patterns, hexagonal cells, for example.[7] No data on the three-dimensional systems, governed by (1), have been published in the literature.

Analytic investigation of (1) is difficult, although there have been some attempts to attack the problem using the methods borrowed from the theory of hydrodynamic turbulence based on the different kinds of closures.[8] The principle result achieved by these methods was that in the $d=1$ case,

$$\langle\Theta^2(k)\rangle \propto \frac{1}{k^2} \quad \text{when } k \to 0 \ , \tag{2}$$

which correlated with the data obtained from computer experiments.

The aim of this paper is to develop a dynamic renormalization-group approach suitable for (1) plus initial conditions and to investigate correlations generated by this equation of motion in the systems of higher spacial dimensionalities $d \geq 1$.

As it is widely accepted, let us introduce the new variable

$$\vec{v} = \nabla\Theta \quad (\mathrm{rot}\,\vec{v} = 0) \ . \tag{3}$$

Substituting (3) into (1) we readily obtain the Fourier-transformed equation of motion

$$-i\omega v_l(\vec{k},\omega) = (\kappa_0 k^2 - F_0 k^4) v_l(\vec{k},\omega) + i\lambda_0 k_i \int v_l(\vec{q},\Omega) v_l(\vec{k},\vec{q},\omega-\Omega)\, d^d q\, d\Omega \ , \tag{4}$$

where the bookkeeping parameter $\lambda_0 = 1$ is introduced for convenience.

Although at first sight Eq. (4) resembles the Navier-Stokes (NS) equation, we can notice the very deep differences between them. First of all, (1) or (4) are unstable against large-wavelength perturbations due to the sign of the term proportional to k^2. Nonlinear interaction transfers these small-k excitations into the region of large wave vectors where they are dissipated. The second difference is that instead of $\mathrm{div}\,\vec{v} = 0$ in the NS case, here we have (3). As it will be shown below, this is a very important factor, i.e., determining the different properties of (4) in cases of different dimensionalities. The third point is more subtle, though it is of utmost importance. The NS equation is the result of some approximations made on the equations describing real interacting particles. It is clear that it is valid only for the scales $1 >> a$—the mean distance between the particles. We can state that there is natural ultraviolet cutoff ($k << 1/a$) in the theory based on NS. The microscopic scales ($k \approx 1/a$) enter the description through the bare viscosity ν_0 and the random noise which must always be written on the right-hand side of NS. Unlike the NS equation, Eq. (4) is the result of a mathematical model and thus it is defined on the

domain $0 < k < \infty$.

To begin development of the dynamic renormalization-group method for (4) we should introduce the ultraviolet cutoff. This must be done with great caution. One can, in principle, apply the Zwanzig-Mori formalism[9] to project the space of variables of (4) onto a smaller space $0 < k < \Lambda$. We, however, will be using a different approach which is identical to one developed in Ref. 9.

Choosing Λ very large, such as $F_0\Lambda^4 >> \kappa_0\Lambda^2$, we can assume with the good reasoning[8] that in the zeroth approximation

$$\vec{v}_0(\vec{k},t) = \vec{v}_0(\vec{k},0)\exp(-F_0k^4t) \tag{5}$$

for any $k > \Lambda \to \infty$. The initial velocity field is chosen to be Gaussian with the correlator

$$\langle \vec{v}(\vec{k},0)\cdot\vec{v}(\vec{k},0)\rangle = D(k)\delta(\vec{k}+\vec{k}') \quad (k\to\infty) \ . \tag{6}$$

Taking $\vec{v}(\vec{k},0)$ from (6) we, in fact, accepted the assumption that Eq. (4) describes a stochastic process disregarding the actual initial conditions $\vec{v}(\vec{k},-\infty)$, or in other words, we believe that after some time the solution of (4) is always random.

Let us denote the modes corresponding to domains $0 < k < \Lambda$ and $\Lambda < k < \infty$ by $v_i^<(\vec{k},\omega)$ and $v_i^>(\vec{k},\omega)$, respectively. It follows from (4) that

$$\begin{aligned} v_i^<(\vec{k},\omega) = {} & i\lambda_0 k_l G^0(k,\omega)\int v_l^<(\vec{q},\Omega)v_l^<(\vec{k}-\vec{q},\omega-\Omega)\,d\Omega \\ & + i\lambda_0 k_l G^0(k,\omega)\int [\, v_l^<(\vec{q},\Omega)v_l^>(\vec{k}-\vec{q},\omega-\Omega) + v_l^>(\vec{q},\Omega)v_l^<(\vec{k}-\vec{q},\omega-\Omega) \\ & + v_l^>(\vec{q},\Omega)v_l^>(\vec{k}-\vec{q},\omega-\Omega)]\,d\Omega \ . \end{aligned} \tag{7}$$

Expression (5) combined with (6) implies that in the limit $k\to\infty$ or $\omega\to 0$

$$\langle v_0(\vec{k},\omega)v_0(\vec{k}',\omega')\rangle \propto \frac{D(k)}{k^4}\delta(\vec{k}+\vec{k}')\delta(\omega+\omega') \ . \tag{8}$$

Using (8) it is possible to eliminate the modes $v^>$ from the equation of motion (7). To do so, one must express $v^<$ in all the integrals in the brackets of the right-hand side of (7) through Eq. (7) itself. This gives rise to the perturbation expansion in powers of the nonlinear coupling λ_0. After that all the modes $v^>$ should be taken in the zeroth approximation (5). Assuming statistical independence of $v^>$ and $v^<$ we can average according to prescription (8) and, as a consequence, eliminate modes $v^>$ from the problem.

It is a matter of simple algebra to show that in the second order of perturbation expansion the equation of motion (7) becomes, after elimination of fast variables from the domain $\Lambda < q < \infty$ ($k\to 0$; $\omega\to 0$),

$$v_i(\vec{k},\omega) = f_i G_d^0 + i\lambda_0 G_d^0 k_l \int d\Omega\, v_l(\vec{q},\Omega)v_l(\vec{k}-\vec{q},\omega-\Omega) + O(\lambda_0^2k^2), \quad 0 < k < \Lambda \ , \tag{9}$$

where

$$G_d^0 = \frac{1}{-i\omega + \nu_d k^2 + F_0k^4} \tag{10}$$

is the propagator acting on a smaller space $0 < k < \Lambda$. It is important that the new viscosity ν_d depends on the spatial dimensionality:

$$\nu_d = -\kappa_0 + A_d\frac{2-d}{d} \ , \tag{11}$$

with

$$A_d = \frac{1}{2^{d-1}\pi^{d/2}\Gamma(d/2)}\frac{\lambda_0^2}{F_0^2}\int_\Lambda^\infty q^{d-9}D(q)\,dq > 0 \ .$$

Recalling that κ_0 is small at the bifurcation point we can always choose a cutoff such as $\nu_1 > 0$. At the same time $\nu_2 = -\kappa_0 < 0$ and $\nu_3 < -\kappa_0 < 0$. As we shall see below this is a key to understanding why the systems of different dimensionality governed by (1) behave in a drastically different way.

The "stirring force" $\vec{f}$ which has been generated by the elimination of the fast variables is given by

$$\begin{aligned} f_i(\vec{k},t) = {} & -i\lambda_0 k_l \int^> \langle v_{0l}^>(\vec{q},\Omega)v_{0l}^>(\vec{k}-\vec{q},\omega-\Omega)\rangle \\ & \times \exp(-F_0qt)\exp(-F_0/\vec{k}-\vec{q}/t)\,d^dq \end{aligned} \tag{12}$$

$$\left[\int^> f(q)\,dq \equiv \int_\Lambda^\infty f(q)\,dq\right] \ .$$

Using (6) we derive readily that $\langle f_i(\vec{k},t)\rangle = 0$ and thus $\vec{f}$ has the properties of a random force. It can be checked by the simple calculations

$$\langle f_i(\vec{k},\omega)f_j(\vec{k},\omega')\rangle \propto k_ik_j\delta(\vec{k}+\vec{k}')\delta(\omega+\omega') \tag{13}$$

in the limit $k\to 0$, $\omega\to 0$.

The main outcome of the present development so far is that introduction of ultraviolet cutoff by elimination of the modes from the region $\Lambda < k < \infty$ brings about a stirring force into the equation which was free being defined on the entire space $0 < k < \infty$. In addition to this effect we derived that the "viscosity" (coefficient in front of k^2 in the prop-

agator) is positive when $d=1$ and negative for any $d \geq 2$. This is understood easily if we recall that the "kinetic energy" is conserved only in the $d=1$ case.

Equation (9) with the stirring force with correlator (13) can be investigated by the dynamic renormalization-group method developed for different problems of hydrodynamics by Forster, Nelson, and Stephen (FNS).[10]

Being interested in the limit $k \to 0$ we can disregard the term proportional to k^4 in the propagator G_d^0. In principle we could keep it but it is clear that it disappears upon rescaling. Thus Eq. (9) is reduced to the Burger's equation with negative viscosity, however, in $d \geq 2$ cases. From now on the results of FNS can be applied directly.

a. $d=1$. A stable fixed point can be found[10] and the correlator, corresponding to this fixed point, is[10]

$$\frac{\langle v_i(\vec{k},\omega)v_j(\vec{k}',\omega')\rangle}{\delta(\vec{k}+\vec{k}')\delta(\omega+\omega')} \propto \frac{1}{k^{3/2}}F(\omega/k^{3/2}) \qquad (14)$$

and, taking into account definition (3),

$$\langle \Theta^2(k)\rangle \propto \frac{1}{k^{7/2}}\int F(\omega/k^{3/2})\,d\omega \propto \frac{1}{k^2}\ , \qquad (15)$$

which coincides with (2) obtained from computer simulations.

b. $d=2$. There is no fixed point.[10] This means that the system does not possess scale-invariant properties. This manifests the possibility of the pattern's formation, obtained by the computer simulations. We must emphasize that the negative sign of the viscosity which, as we have seen above (see also Ref. 10), is not renormalizable ($d=2$) and does not influence the FNS conclusions.

c. $d=3$. This case differs from the one considered by FNS.[10] It can be readily shown that the negative bare dissipation ν_3 becomes more and more negative upon renormalization. This is evident from expression (11). One can check that all the recursion relations of FNS hold except the first one for the renormalized viscosity which now reads

$$\frac{d\nu_3}{dl} = \nu_3(l)(-2+z+\tfrac{1}{4}K_3\bar{\lambda}^2)\bar{\lambda} \qquad (16)$$

$$(K_3>0,\ \text{Ref. 10})\ .$$

Combining (16) with the results of FNS we obtain recursion relation for the dimensionless coupling parameter $\bar{\lambda}^2=\lambda^2 D_0/|\nu_3|^3$ (all the notations are given in Ref. 10):

$$\frac{d\bar{\lambda}}{dl} = \tfrac{1}{2}\bar{\lambda}(-1-\tfrac{1}{4}K_3\bar{\lambda}^2) \quad (d=3)\ .$$

We see that in this case the theory is asymptotically free in the limit $k \to 0$. The scale-invariant solution is

$$\langle \Theta^2(k)\rangle \propto \frac{1}{k^6}\int \frac{d\omega k^2}{(\omega^2/k^4)+1} \propto \frac{1}{k^2}\ .$$

The systems of higher dimensionalities are of a doubtful interest. It is to be stressed that starting with the positive bare viscosity (like FNS) one obtains the unstable fixed point when $d \geq 3$.[10] Thus the negative value of this coefficient brings about the most important consequences.

I would like to thank I. Procaccia for helpful discussions.

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