Bifurcations and multistability in two-photon processes

D. F. Walls* and C. V. Kunasz
Joint Institute for Laboratory Astrophysics, University of Colorado and National Bureau of Standards, Boulder, Colorado 80309

P. D. Drummond
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627

P. Zoller
Institute of Physics, University of Innsbruck, Innsbruck, Austria
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We consider an intracavity nonlinear medium interacting with two modes of the electromagnetic field via a two-photon process. The cavity is coherently driven at the frequencies of the two modes. We consider as the intracavity medium (a) a two-photon absorber, and (b) N three-level atoms. These systems display a new class of bifurcations, and, in the case of the two-photon absorber, optical tristability is possible.

There has been considerable interest recently in the behavior of coherently driven optical resonators which have a nonlinear intracavity medium.[1] An intracavity atomic medium interacting with the electromagnetic field by a one-photon transition has been shown both theoretically[2,3] and experimentally[4,5] to give rise to absorptive and dispersive optical bistability. Intracavity media interacting with the electromagnetic field by two-photon transitions have also been studied and shown theoretically[6–9] and experimentally[10] to give rise to optical bistability. We wish to study further such processes and to demonstrate how they may lead to a new class of bifurcations and under certain conditions give rise to multistability. These new types of transitions, not discussed previously, require a multilevel medium and are different from the usual optical bistability.

We shall consider first the situation of two-photon absorption from an effective two-level system in an optical cavity where all atomic and cavity detunings are set equal to zero. The equations relating the dimensionless transmitted field amplitudes x_1,x_2 to the input amplitudes y_1,y_2 are

$$\frac{\partial}{\partial t}x_1 = y_1 - x_1 f(|x_1|,|x_2|) + \beta_1^{-1/2}\eta_1(t) \ ,$$
$$\frac{\partial}{\partial t}x_2 = y_2 - x_2 f(|x_2|,|x_1|) + \beta_2^{-1/2}\eta_2(t) \ , \qquad (1)$$

where

$$f(|x_i|,|x_j|) = 1 + \frac{2C_i|x_j|^2}{1+|x_i|^2|x_j|^2} \ .$$

The cooperativity parameters C_i, as well as x_i,y_i, are scaled as in Ref. 7. Here and below, t is scaled in units of the interferometer relaxation time. Fluctuations are included via the random thermal forces $\eta_i(t)$, which have the correlation properties

$$\langle \eta_i(t)\eta_j^*(t')\rangle = 2\delta_{ij}\delta(t-t') \ , \qquad (2)$$

and $\beta_i^{-1/2}$, which represents the strength of the fluctuations.

The steady-state solutions to Eq. (1) excluding the fluctuating terms are

$$y_1 = x_1\left(1 + \frac{2C|x_2|^2}{1+|x_1|^2|x_2|^2}\right) \ ,$$
$$y_2 = x_2\left(1 + \frac{2C|x_1|^2}{1+|x_1|^2|x_2|^2}\right) \ , \qquad (3)$$

where we have set $C_1 = C_2 = C$. These equations were derived by Agrawal and Flytzanis,[7] who considered the case in which one driving field, y_2, is held fixed and the other driving field, y_1, is varied. We wish to demonstrate the interesting behavior that may occur when both driving fields are varied together. We consider the case $y_1 = y_2 = y$, for which Eqs. (3) yield three distinct state equations in which x_1 and x_2 must be real

$$x_1 = x_2, \quad y = x_1\left(1 + \frac{2Cx_1^2}{1+x_1^4}\right) \ , \qquad (4a)$$

$$x_1x_2 = C-(C^2-1)^{1/2}, \quad y = x_1 + \frac{C-(C^2-1)^{1/2}}{x_1} \ , \qquad (4b)$$

$$x_1x_2 = C+(C^2-1)^{1/2}, \quad y = x_1 + \frac{C+(C^2-1)^{1/2}}{x_1} \ . \qquad (4c)$$

The solutions (4b) and (4c) do not exist for $C < 1$, while for $C = 1$ they coalesce. Curve (4a) intersects curve (4b) at $y = 2[C - (C^2-1)^{1/2}]^{1/2}$ and curve (4c) at $y = 2[C + (C^2-1)^{1/2}]^{1/2}$. These three curves are plotted in Fig. 1 for $C = 2$. The stability of the various branches of these curves is best investigated by introducing a probability distribution function $P(x_1,x_2)$. This probability distribution obeys a Fokker-Planck equation which is equivalent to the stochastic differential equations (1). The steady-state solution to this Fokker-Planck equation is

$$P(x_1,x_2) = \exp\beta[\mathrm{Re}(x_1)y_1 + \mathrm{Re}(x_2)y_2 - \tfrac{1}{2}(|x_1|^2 + |x_2|^2) - C\ln(1 + |x_1|^2|x_2|^2)] \quad . \tag{5}$$

Since x_1 and x_2 are complex amplitudes, the above distribution function is four dimensional. A plot of the reduced two-dimensional distribution function obtained by taking $\mathrm{Im}(x_i) = 0$ is presented in Fig. 2 for $C = 2$ and $y_1 = y_2 = 4.5$.

Varying y results in different distributions whose maxima and minima follow the curves in Fig. 1. For $y < 1.02$ there is only one solution corresponding to the symmetric solution $x_1 = x_2$; hence the probability distribution consists of a single peak. At $y = 1.02$ the solution bifurcates, and there are two branches corresponding to $x_1 \neq x_2$ and one branch corresponding to $x_1 = x_2$. The probability distribution in this region displays two maxima at the $x_1 \neq x_2$ solutions, indicating they are stable solutions, and a minimum at $x_1 = x_2$ which is therefore unstable. At $y = 3.86$ the $x_1 = x_2$ curve bifurcates again, and for $y > 3.86$ we have five possible solutions for a given value of y. Figure 2 shows that the probability distribution in this region has three maxima, indicating the curves (4a) and (4b) are stable while (4c) is unstable. In this region we have the possibility of optical tristability. (An entirely different mechanism for optical tristability using the polarization states of light interacting with spin-$\frac{1}{2}$ atoms has recently been suggested by Kitano *et al.*[11]) As y is increased the probability maximum at $x_1 = x_2$ increases until there is just a single peak corresponding to the symmetric branch at high y values.

Hence, as y is increased from zero, the symmetric solution $x_1 = x_2$ undergoes a continuous bifurcation to the solution $x_1 \neq x_2$ corresponding to Eq. (4b). The system remains on this branch as y is increased until fluctuations cause it to jump to the branch $x_1 = x_2$ which has dominant probability for large y values. As y is reduced the solution $x_1 = x_2$ undergoes a discontinuous bifurcation to the $x_1 \neq x_2$ solution [Eq. (4b)]. For increasing values of $C > 1$ the region of tristability diminishes. For example, for $C = 6$, the tristability is observable only for $10 \lesssim y \lesssim 11$.

Let us now consider a three-level λ system. Optical bistability from N three-level atoms driven with a single mode of the radiation field was recently predicted by Walls and Zoller.[12] In the present paper we wish to consider N three-level atoms interacting with two modes of the radiation field. The equations

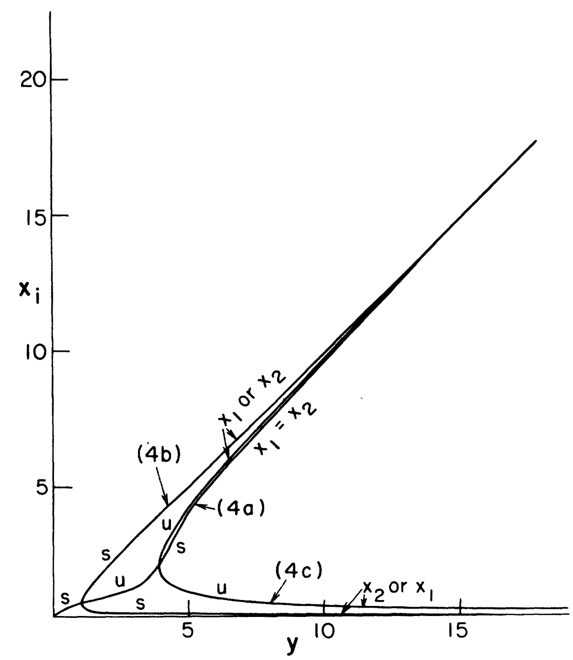

FIG. 1. State equations for two-mode, two-photon processes ($y_1 = y_2 = y$, $C = 2$). Markings s and u denote stable and unstable branches, respectively. For curves (4b) and (4c), if x_1 is plotted above the intersection with (4a), then x_2 is the part of the curve below the intersection and vice versa.

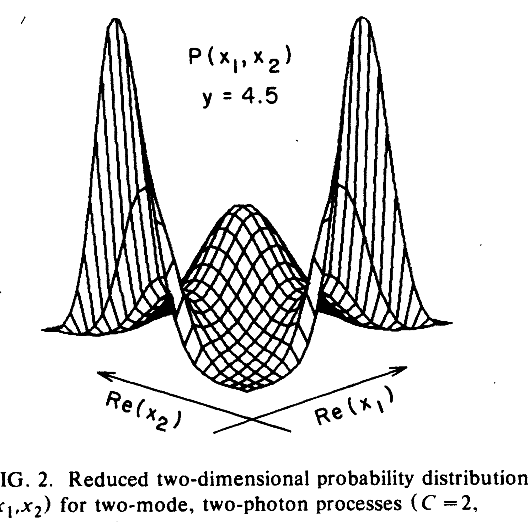

FIG. 2. Reduced two-dimensional probability distribution $P(x_1,x_2)$ for two-mode, two-photon processes ($C = 2$, $y = y_1 = y_2 = 4.5$).

of motion for this system are

$$\dot{x}_1 = y_1 - x_1 + 4\left(\frac{T_1}{T_2}\right)^{1/2} C_1\rho_{31} \ ,$$
$$\dot{x}_2 = y_2 - x_2 + 4\left(\frac{T_1}{T_2}\right)^{1/2} C_2\rho_{32} \ , \qquad (6)$$

where x_i and y_i are dimensionless transmitted and input field amplitudes defined so that $\sqrt{2}|x_i|/T_2$ is the ith transition Rabi frequency and ρ_{ij} are the density matrix elements of a three-level atom for which the decay rates exceed the cavity decay rates. The cooperativity parameters are $C_i = N|g|^2T_2/2\kappa$, where g is the atomic dipole matrix element, κ is the cavity decay rate, N is the number of interacting atoms, and T_2 is the atomic dephasing rate. We shall henceforth write all equations in terms of intensities $Y_i = |y_i|^2$, $X_i = |x_i|^2$. Although we have developed a computer program for finding solutions for the more general two-mode problem, we shall now assume that the detuning of the two modes to the respective atomic-resonance frequencies are equal and opposite (δ and $-\delta$, respectively), the damping is purely radiative ($T_2 = 2T_1$), and that $C_1 = C_2 = C$. Then the state equation is

$$Y_i = X_i\left\{1 + \frac{2CX_{3-i}}{\Pi(X_1,X_2)}\left[2 + \frac{2CX_{3-i}}{\Pi(X_1,X_2)} \times \left(1 + \delta^2 - X_0 + \frac{X_0^2}{4\delta^2}\right)\right]\right\} \ , \qquad (7)$$

$$i = 1,2$$

where $X_0 = (X_1 + X_2)/2$, and

$$\Pi(X_1,X_2) = 3X_1X_2 + 2X_0\left(1 + \delta^2 - X_0 + \frac{X_0^2}{4\delta^2}\right) \ .$$

We have plotted this state equation in Fig. 3 where Y_2 is held fixed at 10 and Y_1 is varied. This shows a bistable behavior in both field intensities X_1 and X_2. Both driving fields are varied together for Fig. 4, in which case there can be up to three solutions (X_1,X_2) for a given value of $Y = Y_1 = Y_2$. No potential solution exists for the steady-state distribution function, but a linearized stability analysis of Eqs. (6) reveals that the closed loop with $X_1 \neq X_2$ is completely stable, while the curve with $X_1 = X_2$ is unstable up to point B and stable for y values beyond point B. Thus as one increases Y from zero, the system assumes the asymmetric solution $X_1 \neq X_2$. Which mode assumes

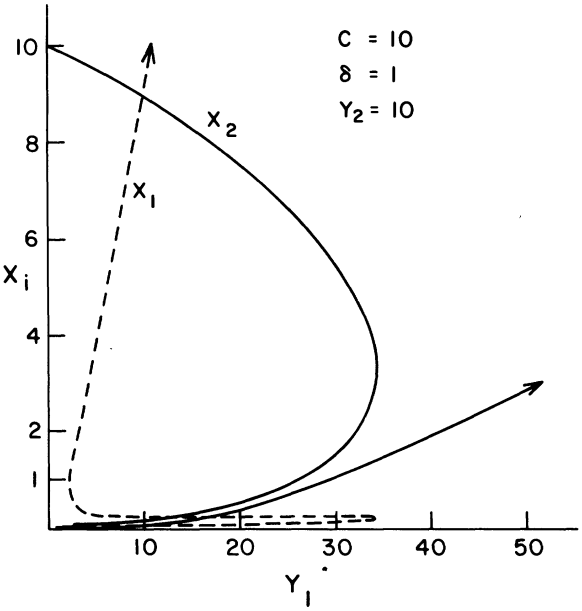

FIG. 3. State equation for three-level atoms interacting with two field modes ($Y_2 = 10$ fixed, $C = 10$, $\delta = 1$).

the high intensity solution and which the low is determined by initial fluctuations and any small asymmetry in Y_1 and Y_2. As Y is increased up to the point B the two solutions coalesce and for larger Y values only the symmetric solution $X_1 = X_2$ remains. Point A is not a bifurcation point since the symmetric curve with $X_1 = X_2$ does not intersect the two asymmetric curves for X_1 and X_2 at a point where $X_1 = X_2$. Thus, if viewed in three-dimensional space, point A is not a curve crossing.

Analytic expressions for these results may be obtained in the limit of small detunings $\delta << X << 1$, where for $Y_1 = Y_2 = Y$ we find two solution curves

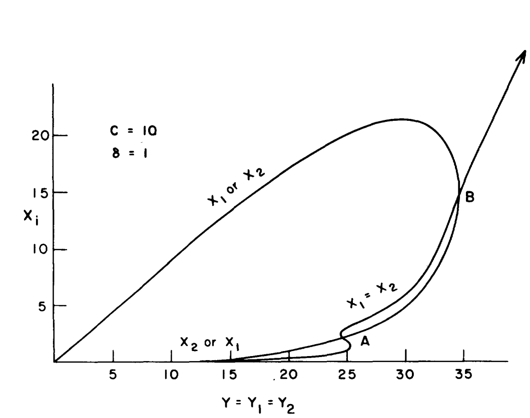

FIG. 4. State equation for three-level atoms interacting with two field modes ($Y = Y_1 = Y_2$, $C = 10$, $\delta = 1$).

parametrized relative to the mean intensity X_0:

$$\bar{X}_1=\bar{X}_2, \quad \bar{Y}=\bar{X}\left[\left(1+\frac{C}{1+\bar{X}_0^2}\right)^2+\left(\frac{C\bar{X}_0}{1+\bar{X}_0^2}\right)^2\right], \tag{8a}$$

$$\bar{X}_{1,2}=\bar{X}_0\left[1\pm\left(1-\frac{1+\bar{X}_0^2}{C^2}\right)^{1/2}\right], \quad \bar{Y}=\bar{X}_1\left[\left(1+\frac{\bar{X}_0}{C\bar{X}_1}\right)^2+\left(\frac{\bar{X}_0^2}{C\bar{X}_1}\right)^2\right], \tag{8b}$$

where

$$\bar{X}_i=\frac{X_i}{2\delta}, \quad \bar{Y}=\frac{Y_i}{2\delta}, \quad \bar{X}_0=\frac{\bar{X}_1+\bar{X}_2}{2} .$$

Equations (8) display behavior similar to that shown in Fig. 4 with a bifurcation point at $\bar{X}_1=\bar{X}_2=(C^2-1)^{1/2}$.

The existence of a new class of bifurcations in two-photon processes, including the possibility of optical tristability, has been demonstrated. All of these phenomena should be accessible to experimental measurement. Further studies in nonlinear optical systems involving more than one mode such as the Raman effect and four-wave mixing should reveal a rich variety of bifurcation phenomena.

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[*]Visitor at JILA, on leave from the Phys. Dept., University of Waikato, Hamilton, New Zealand.

[1]For a review of the work in this field see V. N. Lugovoi, Sov. J. Quantum Electron. 9, 1207 (1980) [Kuant. Elektron. (Moskva) 6, 2053 (1979)], and the references contained therein.

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