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Harmonic generation in the free-electron laser. Theory of the quasiperiodic wiggler

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Free-electron lasers, except for those using helical wigglers, are predicted in most cases to generate higher harmonics of the fundamental optical frequency. The basic equations describing this process are derived and applied in particular to the linearly polarized wiggler.

INTRODUCTION

The possibility of using nonlinear optical processes to generate coherent optical harmonics of the fundamental frequency in the free-electron laser (FEL) has been discussed by us at a number of conferences,[1] but has not been previously published in detail. Several other workers have also described mechanisms of harmonic generation in the FEL.[2-4] A number of FEL devices and undulator experiments[5] now being constructed use linearly polarized wigglers. These devices are expected to produce significant amounts of radiation at higher harmonics. The present paper provides the theoretical framework for understanding such harmonic generation. Numerical solutions of the basic equations derived here will be published elsewhere.

I. DERIVATION OF THE BASIC EQUATIONS

The analysis presented in this paper is based on the use of multiple-scaling perturbation theory and is very similar to that which we used previously to treat the helical wiggler.[6] The starting point of the analysis is the Maxwell-Boltzmann description of an FEL with an arbitrary magnet geometry, as presented in Sec. II of Ref. 6. We then specialize to the case of a quasiperiodic wiggler, introducing separate "fast" and "slow" variables to describe periodic oscillations and slow modulation (if any) of the complex wiggler amplitude. Similarly, we introduce fast and slow variables to describe the temporal behavior of the light and electron distributions on the scale of optical oscillations and on the scale of picosecond pulse envelopes. We next use multiple-scaling perturbation theory to obtain the slowly varying Maxwell and Boltzmann equations. Finally, these equations are transformed to the equivalent coupled Maxwell and single-particle equations, which are more amenable to numerical analysis.

We assume that the electrons have a narrow ultrarelativistic distribution of energies[7] centered at $mc^2\gamma_0$ and introduce a detuning parameter μ, defined by

$$\gamma=\gamma_0(1+\gamma_0^2\mu/k_s), \tag{1}$$

to describe deviations of electron energy from the central value. The constant k_s is to be defined later. Let $\hat{A}_q$ and $\hat{E}_s$ be the on-axis complex components of the wiggler vector potential and the laser electric field with respect to circular polarization basic vectors

$$\vec{\mathbf{A}}_{\text{static}}(z)=2^{-1/2}\hat{e}_-\hat{A}_q(z)+\text{c.c.},$$
$$\vec{\mathbf{E}}_{\text{laser}}(z,t)=2^{-1/2}\hat{e}_-\hat{E}_s(z,t)+\text{c.c.}$$

Also define the mass shift Δ, which is generally a function of position, according to

$$\Delta(z)=1+e^2|\hat{A}_q|^2/m^2c^2. \tag{2}$$

We choose as independent variables the coordinates

$$\tau=t-z/c, \tag{3}$$

$$\xi=\int_0^z\Delta(z')dz', \tag{4}$$

where τ is the retarded time and ξ is a generalized position coordinate. The coordinate ξ is a useful means for taking into account variable mass shift. Although the wiggler in the absence of laser radiation cannot change electron energy (if one neglects incoherent emission), more or less of the electron energy gets transferred to transverse motion as Δ becomes larger or smaller. The z component of electron velocity gets correspondingly smaller or larger. However, in terms of the (ξ,τ) coordinates the electron motion appears uniform, aside from effects induced by the lasing process itself.

If one neglects diffraction, the Maxwell-Boltzmann equations take the form

$$\frac{\partial \hat{E}_s}{\partial \xi}=D\frac{\hat{A}_q}{\Delta}\int d\mu\, h, \tag{5}$$

$$\frac{\partial h}{\partial \xi}+\frac{1}{c}\left[\frac{1}{2\gamma_0^2}-\frac{\mu}{k_s}\right]\frac{\partial h}{\partial \tau}=\kappa\left[\hat{E}_s\frac{\hat{A}_q^*}{\Delta}+\text{c.c.}\right]\frac{\partial h}{\partial \mu}. \tag{6}$$

Here $h(\xi,\tau,\mu)$ is the Boltzmann distribution function, and $D=e^2F\gamma_0^2/2\epsilon_0 k_s$ and $\kappa=e^2k_s/2m^2c^3\gamma_0^4$ are constants defined in Ref. 6. Equations (5) and (6) are the same as Eqs. (21) and (17) of Ref. 6. F is the filling factor (ratio of electron to optical beam areas), which for the time being we assume to be constant. Note that the wiggler field appears only through the ratio $\hat{A}_q/\Delta$. At this stage $\hat{A}_q$ is completely arbitrary.

Since the field $\hat{E}_s$ oscillates at optical frequencies, Eqs. (5) and (6) are not readily solvable as they stand. Moreover, one must introduce a periodic structure to the wiggler in order to determine the frequency of operation of the laser and concentrate the gain at this frequency. To this end we define a quasiperiodic wiggler according to

$$\hat{A}_q/\Delta=A_q(\xi,\beta), \tag{7}$$

$$\beta=k_s\xi/2\gamma_0^2. \tag{8}$$

The complex function $A_q(\xi,\beta)$ is assumed to be slowly varying in ξ and to be periodic in β with period 2π and zero mean. Thus, β is an angle describing the phase of the wiggler. Equations (7) and (8) also define k_s. An FEL with such a wiggler will produce radiation at frequency $\omega_s=ck_s$ and, in general, at multiples of ω_s.

Implementation of the multiple-scaling perturbation technique depends on identifying fast and slow scaled variables appropriate to the problem. We use the same slow variables as in Ref. 6, scaling the coordinates with respect to the effective magnet length

$$L'=\int_0^L \Delta(z)dz \tag{9}$$

to get

$$\bar{\xi}=\xi/L',\quad \bar{\mu}=L'\mu,\quad \bar{\tau}=2c\gamma_0^2\tau/L'.$$

The time unit $L'/2c\gamma_0^2$ is typically on the order of picoseconds and is therefore an appropriate one for describing short-pulse propagation. As fast variables we choose the optical phase

$$\phi=\omega_s\tau=\bar{\tau}/\epsilon, \tag{10}$$

and the phase of the bunching potential

$$\theta=\beta-\phi-(\bar{\xi}-\bar{\tau})/\epsilon, \tag{11}$$

where

$$\epsilon=2\gamma_0^2/k_sL'=1/(2\pi N) \tag{12}$$

serves as the small expansion parameter for the theory and N is the number of periods in the wiggler. We regard $\hat{E}_s$, A_q, and h as depending independently on the fast and slow variables. Using Eqs. (10) and (11) and the chain rule, we thus make the replacements

$$\frac{\partial}{\partial \bar{\xi}}\to\frac{\partial}{\partial \bar{\xi}}+\frac{1}{\epsilon}\frac{\partial}{\partial \theta}, \tag{13}$$

$$\frac{\partial}{\partial \bar{\tau}}\to\frac{\partial}{\partial \bar{\tau}}+\frac{1}{\epsilon}\frac{\partial}{\partial \phi}-\frac{1}{\epsilon}\frac{\partial}{\partial \theta}. \tag{14}$$

Equations (5) and (6) now take the form

$$\frac{\partial \hat{E}_s}{\partial \bar{\xi}}+\frac{1}{\epsilon}\frac{\partial \hat{E}_s}{\partial \theta}=DA_q(\xi,\theta+\phi)\int d\bar{\mu}\, h(\bar{\xi},\bar{\tau},\bar{\mu},\theta,\phi), \tag{15}$$

$$\frac{\partial h}{\partial \bar{\xi}}+\frac{\partial h}{\partial \bar{\tau}}+\frac{1}{\epsilon}\frac{\partial h}{\partial \phi}-\bar{\mu}\left[\epsilon\frac{\partial h}{\partial \bar{\tau}}+\frac{\partial h}{\partial \phi}-\frac{\partial h}{\partial \theta}\right]=\kappa L'^2[\hat{E}_s(\bar{\xi},\bar{\tau},\theta,\phi)A_q^*(\bar{\xi},\theta+\phi)+\text{c.c.}]\frac{\partial h}{\partial \bar{\mu}}. \tag{16}$$

Following the same procedure as in Ref. 6, we assume perturbation expansions

$$\hat{E}_s=E^{(0)}+\epsilon E^{(1)}+\cdots, \tag{17}$$

$$h=h^{(0)}+\epsilon h^{(1)}+\cdots. \tag{18}$$

We introduce these expansions into Eqs. (15) and (16) and solve order by order. However, if ϵ is sufficiently small, we are really only interested in $E^{(0)}$ and $h^{(0)}$. The equations of order $1/\epsilon$ are the leading terms. These are

$$\partial E^{(0)}/\partial\theta=0, \tag{19}$$

$$\partial h^{(0)}/\partial\phi=0. \tag{20}$$

These equations say that the laser field does not have rapid oscillations if evaluated on a trajectory moving at the speed of light ($\phi=$const), and that the electron distribution does not have rapid oscillations if evaluated on a trajectory moving at the speed of the bunching potential ($\theta=$const).

The equations of order ϵ^0 are

$$\frac{\partial E^{(0)}}{\partial\bar{\xi}}+\frac{\partial E^{(1)}}{\partial\theta}=DA_q(\bar{\xi},\theta+\phi)\int d\bar{\mu}\,h^{(0)}(\bar{\xi},\bar{\tau},\bar{\mu},\theta), \tag{21}$$

$$\frac{\partial h^{(0)}}{\partial\bar{\xi}}+\frac{\partial h^{(0)}}{\partial\bar{\tau}}+\frac{\partial h^{(1)}}{\partial\phi}+\bar{\mu}\frac{\partial h^{(0)}}{\partial\theta}=\kappa L'^2[E^{(0)}(\bar{\xi},\bar{\tau},\phi)A_q^*(\bar{\xi},\theta+\phi)+\text{c.c.}]\frac{\partial h^{(0)}}{\partial\bar{\mu}}. \tag{22}$$

If we regard Eq. (21) as being of the form

$$\frac{\partial E^{(1)}}{\partial\theta}=\mathcal{A}, \tag{23}$$

where $\mathcal{A}$ represents the driving term which is a specified function of θ, it is clear that the average of the driving term over θ must be zero in order to prevent secular growth in $E^{(1)}$. A similar argument applies to Eq. (22), except that the angle to be averaged over in that case is ϕ. Imposing the condition of no secular growth yields the desired slowly varying Maxwell and Boltzmann equations:

$$\frac{\partial\hat{E}_s}{\partial\xi}=D\frac{1}{2\pi}\int_0^{2\pi}d\theta\,A_q(\xi,\theta+\phi)\int d\mu\,h(\xi,\tau,\mu,\theta), \tag{24}$$

$$\frac{\partial h}{\partial\xi}+\frac{1}{2\gamma_0^2c}\frac{\partial h}{\partial\tau}+\mu\frac{\partial h}{\partial\theta}=\kappa\frac{1}{2\pi}\int_0^{2\pi}d\phi[\hat{E}_s(\xi,\tau,\phi)A_q^*(\xi,\theta+\phi)+\text{c.c.}]\frac{\partial h}{\partial\mu}. \tag{25}$$

Here we have dropped the superscripts for simplicity and restored the units of the independent variables. In the analysis below we again omit the factors of $1/(2\gamma_0^2c)$.

Equations (24) and (25) can be written in a somewhat more illuminating form by introducing Fourier series expansions of the laser and wiggler fields:

$$\hat{E}_s(\xi,\tau,\phi)=\sum_{n\neq0}E_n(\xi,\tau)e^{-in\phi}, \tag{26}$$

$$A_q(\xi,\beta)=\sum_{n\neq0}A_n(\xi)e^{-in\beta}. \tag{27}$$

The Maxwell equation separates into a set of equations for the various harmonic amplitudes:

$$\frac{\partial E_n}{\partial\xi}=DA_n(\xi)\frac{1}{2\pi}\int_0^{2\pi}d\theta\,e^{-in\theta}\int du\,h(\xi,\tau,\mu,\theta). \tag{28}$$

The Boltzmann equation becomes

$$\frac{\partial h}{\partial\xi}+\frac{\partial h}{\partial\tau}+\mu\frac{\partial h}{\partial\theta}=\kappa\sum_{n\neq0}[E_n(\xi,\tau)A_n^*(\xi)e^{in\theta}+\text{c.c.}]\frac{\partial h}{\partial\mu}. \tag{29}$$

We see that the ponderomotive force, while periodic, is, in general, not simply sinusoidal but has higher harmonic content. An exception is the helical wiggler, for which only the $n=1$ term contributes. We see from Eq. (28) that, if $A_n\neq0$, radiation at frequency $n\omega_s$ will be generated even if not initially present. All that is required is that the electron bunching have higher harmonic content, which is invariably the case if the FEL is saturated. On the other hand, if all radiation is in the small-signal regime, it is easy to show[8] that the various harmonics become decoupled and evolve independently.

Equations (28) and (29) can be reformulated in terms of single-particle coordinates $\hat{\theta}(\xi,\tau_0,\mu_0,\theta_0)$ and $\hat{\mu}(\xi,\tau_0,\mu_0,\theta_0)$ in the same way as in Ref. 6. Here $\tau_0=\tau-\xi$ is retarded time at the speed of the electrons and μ_0, θ_0 are the initial energy detunings and phase angles at $\xi=0$. The single-particle equations corresponding to the Boltzmann equation (29) are

$$\frac{d\hat{\theta}(\xi,\tau_0,\mu_0,\theta_0)}{d\xi}=\hat{\mu}(\xi,\tau_0,\mu_0,\theta_0), \tag{30}$$

$$\frac{d\hat{\mu}(\xi,\tau_0,\mu_0,\theta_0)}{d\xi}=-\kappa\sum_{n\neq 0}\{E_n(\xi,\tau_0+\xi)A_n^*(\xi)\exp[in\hat{\theta}(\xi,\tau_0,\mu_0,\theta_0)]+\text{c.c.}\}. \tag{31}$$

Using these equations, it is easily shown that Eq. (29) has the formal solution

$$h(\xi,\tau,\mu,\theta)=\int d\mu_0\int_0^{2\pi}d\theta_0 h(0,\tau_0,\mu_0,\theta_0)\delta(\mu-\hat{\mu}(\xi,\tau_0,\mu_0,\theta_0))\delta(\theta-\hat{\theta}(\xi,\tau_0,\mu_0,\theta_0)),$$

which is the same as Eq. (44) of Ref. 6. Inserting this equation into the right-hand side of Eq. (28) and carrying out the integrations over μ and θ, we obtain

$$\frac{\partial E_n(\xi,\tau)}{\partial\xi}=DA_n(\xi)\int d\mu_0\frac{1}{2\pi}\int_0^{2\pi}d\theta_0 h(0,\tau_0,\mu_0,\theta_0)\exp[-in\hat{\theta}(\xi,\tau_0,\mu_0,\theta_0)].$$

If we assume that the electrons entering the wiggler are unbunched, then $h(0,\tau_0,\mu_0,\theta_0)$ is independent of θ_0 and is proportional to the current $\mathscr{I}(\tau_0,\mu_0)$ per unit μ_0 entering the wiggler:

$$Dh(0,\tau_0,\mu_0,\theta_0)=(\alpha/|\sigma|^2)\mathscr{I}(\tau_0,\mu_0).$$

Here $\alpha=e/2mc\gamma_0$ and $|\sigma|^2/\epsilon_0 c$ is the optical mode area. We can take into account the fact that the mode area may be different for the different harmonics by including an index n on σ_n. Moreover, by regarding σ_n as a complex function of ξ, we can account approximately for diffractive beam spreading and phase shift. The power in the nth harmonic with a given circular polarization is $|\sigma_n E_n|^2$, whereas $|\sigma_{-n}E_{-n}|^2$ gives the power at the same frequency, but with opposite circular polarization. With these generalizations, the Maxwell equation takes the form

$$\sigma_n^*\frac{\partial[\sigma_n E_n(\xi,\tau)]}{\partial\xi}=\alpha A_n(\xi)\int d\mu_0\mathscr{I}(\tau-\xi,\mu_0)\frac{1}{2\pi}\int_0^{2\pi}d\theta_0\exp[-in\hat{\theta}(\xi,\tau-\xi,\mu_0,\theta_0)]. \tag{32}$$

Equations (30)–(32) are the basic equations for the FEL with a quasiperiodic wiggler. For further discussion of the method by which we include diffraction in the Maxwell equation (32), see Appendix A of Ref. 6. For the case of a strictly periodic wiggler, where A_n is constant, one can show by an argument along the lines given there that the appropriate function σ_n for operation in the fundamental Gaussian mode is given by

$$\sigma_n(\xi)=(\pi\epsilon_0 c/2|n|)^{-1/2}w_0 \times[1+i\,\text{sgn}(n)(2\xi-L')/\bar{\Delta}w_0^2k_s], \tag{33}$$

where w_0 is the beam waist at the fundamental frequency and $\bar{\Delta}$ is the average of the mass shift over a magnet period:

$$\bar{\Delta}=\lambda_q^{-1}\int_0^{\lambda_q}\Delta(z)dz. \tag{34}$$

Equation (33) assumes that the beam waist is located in the middle of the wiggler. Equation (33) should be a good approximation for low-loss resonators, provided that $|n|^{-1/2}w_0$ is large compared to the cross-sectional dimension of the electron beam.

II. LINEARLY POLARIZED WIGGLER

The linearly polarized wiggler, which is being used in several FEL devices under construction,[2] offers several advantages over the helical wiggler. In particular, it can be constructed of permanent magnets. The wiggler geometry can be varied relatively easily by varying the positions of the magnets. There is also considerable freedom in adjusting the harmonic content of the wiggler by using repeated sequences of magnets of different strengths and shapes. However, even in the case where the wiggler field does not have higher harmonic content when viewed as a function of z, such content is nevertheless effectively created by the mass shift.

Such a field (with the magnetic field polarized in the y direction and the vector potential polarized in the x direction) is described by the function

$$\hat{A}_q(z)=2^{1/2}A_l\cos(k_l z), \tag{35}$$

where A_l is the constant rms amplitude. From Eq. (2) we see that the mass shift Δ_l in such a wiggler is a rapidly varying function of position,

$$\Delta_l = 1+(2A_l^2/m^2c^2)\cos^2(k_l z)$$
$$=\bar{\Delta}_l[1+2\delta_l\cos(2k_l z)].$$

Here $\bar{\Delta}_l=1+A_l^2/m^2c^2$ is the average mass shift and

$$\delta_l \equiv \tfrac{1}{2}(1-1/\bar{\Delta}_l) \tag{36}$$

can take on values between zero and one half.

Carrying out the integration in Eq. (4), we obtain the generalized position coordinate

$$\xi=\bar{\Delta}_l[z+(\delta_l/k_l)\sin(2k_l z)]. \tag{37}$$

The generalized position coordinate ξ provides a mechanism for treating oscillations in the z component of electron velocity associated with periodic transfer of electron energy between longitudinal and transverse motion in the wiggler. Even though the amplitude of such oscillations may easily be a thousand times the laser bandwidth, there is no broadening in the emitted radiation at the fundamental frequency. This is analogous to the unbroadened radiation from a Mössbauer nucleus undergoing large, but periodic, oscillations in a crystal lattice.

The evaluation of the wiggler harmonic coefficients A_n may be carried out by inverting Eq. (27):

$$A_n=\frac{1}{2\pi}\int_0^{2\pi} d\beta\, A_q(\beta)e^{in\beta}. \tag{38}$$

Using Eqs. (4), (7), and (8), this becomes

$$A_n=(2\pi\bar{\Delta}_l)^{-1}\int_0^{2\pi} d\psi \hat{A}_q \exp(ink_l\xi/\bar{\Delta}_l), \tag{39}$$

where $\psi=k_l z$ and we have used the fact that $k_s\bar{\Delta}_l/2\gamma_0^2=k_l$. Inserting Eqs. (35) and (37) into (39), we obtain

$$A_n=2^{1/2}A_l(2\pi\bar{\Delta}_l)^{-1}$$
$$\times\int_0^{2\pi} d\psi \cos\psi \exp[in(\psi+\delta_l\sin 2\psi)]. \tag{40}$$

It is evident that $A_{-n}=A_n^*$, and it is easy to check that it is consistent with the basic Eqs. (30)–(32) to assume $E_{-n}=E_n^*$. This implies that $\vec{E}_{\text{laser}}$ is polarized in the x direction. Note that the total optical power at frequency $n\omega_s$ is $2\,|\,\sigma_n E_n\,|^2$.

The integral (40) is zero for even n, and for odd n has the value[9]

$$A_n=(A_l/2^{1/2}\bar{\Delta}_l)(-1)^{(n-1)/2}$$
$$\times[J_{(n-1)/2}(n\delta_l)-J_{(n+1)/2}(n\delta_l)] \tag{41}$$

and is thus, in fact, real. The harmonic coefficients as functions of δ_l are shown in Fig. 1. Clearly excitation of the harmonics requires an increasingly strong wiggler field as n increases.

Let us now compare the FEL with a linearly polarized wiggler with the FEL with a helical wiggler. If we suppose that the higher harmonics in the linear FEL are suppressed (for instance, by making the resonator lossy to the higher harmonics), we can ask what, if any, helical wiggler will give the same FEL operation as a given linear wiggler. We require that both FEL's operate with the same ω_s and γ_0, and the same temporal evolution of electron and optical pulses. Operation at the same ω_s requires that

$$k_l/\bar{\Delta}_l=k_h/\Delta_h, \tag{42}$$

where the subscripts h refer to the helical wiggler. On the other hand, the effective length L' of the two wigglers must be the same in order that the

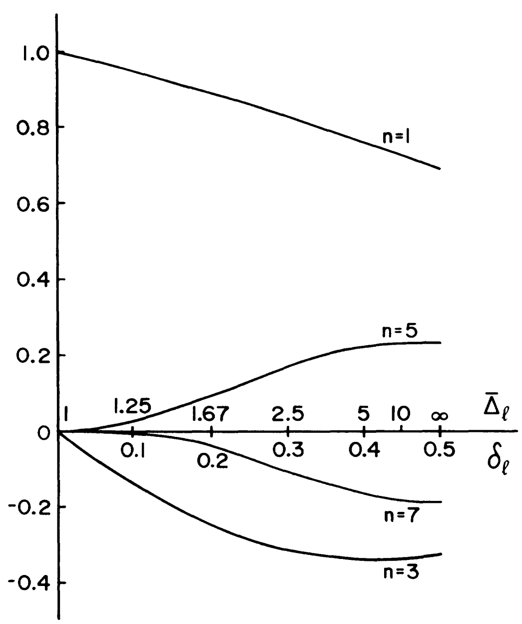

FIG. 1. Harmonic coefficients of the linearly polarized wiggler are shown as a function of δ_l (a measure of wiggler field strength) for the first, third, fifth, and seventh harmonics. The harmonic coefficients A_n are found by multiplying the ordinate by $A_l/2^{1/2}\bar{\Delta}_l$.

electron phase angles evolve to the same amount. Thus,

$$\bar{\Delta}_l L_l = \Delta_h L_h . \tag{43}$$

By multiplying Eqs. (42) and (43), we see that both wigglers must have the same number of periods.

The helical wiggler is described by the function

$$\hat{A}_q(z) = A_h \exp(-ik_h z) . \tag{44}$$

Only the $n=1$ wiggler amplitude in Eq. (27) is nonzero and has the value $A_{1h} = A_h/\Delta_h$. Bearing in mind that the laser power of the helical FEL is $|\sigma_{1h}E_{1h}|^2$, whereas the laser power of the linear FEL is $2|\sigma_{1l}E_{1l}|^2$ because of the equal contributions of $n=1$ and $n=-1$ terms, it is easy to see that the basic equations (30)–(32) for the two FEL's becomes identical provided we choose $2^{1/2}A_{1l}/\sigma_{1l}^* = A_{1h}/\sigma_{1h}^*$. Using Eq. (41), this condition becomes

$$(A_l/\bar{\Delta}_l \sigma_{1l}^*)[J_0(\delta_l) - J_1(\delta_l)] = A_h/\Delta_h \sigma_{1h}^* . \tag{45}$$

Now using Eq. (33), we see that in order for Eq. (45) to hold, we must choose the beam waists according to

$$\bar{\Delta}_l^{1/2} w_{0l} = \Delta_h^{1/2} w_{0h} , \tag{46}$$

and furthermore must choose

$$A_l \bar{\Delta}_l^{-1/2}[J_0(\delta_l) - J_1(\delta_l)] = A_h \Delta_h^{-1/2} . \tag{47}$$

The latter condition may be expressed[10] more simply as

$$\delta_h = \delta_l [J_0(\delta_l) - J_1(\delta_l)]^2 , \tag{48}$$

where δ_h is defined analogously to Eq. (36). Figure 2 shows a plot of Eq. (48). The slope is unity at $\delta_l = 0$ and goes to zero at $\delta_l = 0.5$. We see that,

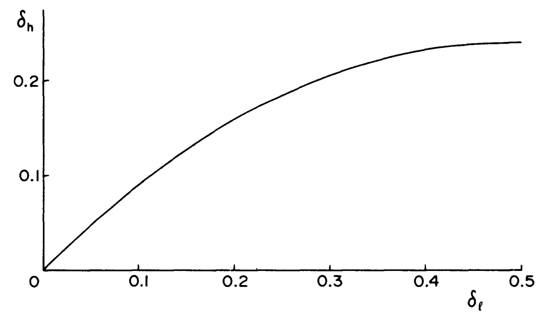

FIG. 2. Curve relating the wiggler amplitudes of linear and helical wigglers giving equivalent FEL operation, as measured in terms of the parameters δ_l and δ_h.

while every linear FEL has a corresponding helical FEL, a helical FEL with sufficiently large δ_h does not correspond to any linear FEL. Since $\delta_l > \delta_h$, it follows that $\bar{\Delta}_l > \Delta_h$, $A_l > A_h$, and $k_l > k_h$. The magnetic field $B_l = k_l A_l$ required for the linear FEL is therefore greater than the field $B_h = k_h A_h$ of a corresponding helical FEL. Recall that B_l is the rms field, so that the peak field is higher yet by a factor of $\sqrt{2}$. On the other hand, the wiggler length of the linear FEL is less than that of the corresponding helical FEL, which could be advantageous in cases where space is a problem (e.g., storage-ring operation).

Our group has been carrying out numerical calculations of harmonic generation in the linear FEL, both in the cw and short-pulsed regimes. We plan to publish there results in a subsequent paper.

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[1]We have discussed harmonic generation in the FEL at the following conferences: Stanford FEL Workshop, Stanford University, 1979 (unpublished); Conference on the Physics of Quantum Electronics, Telluride, Colorado, 1979 (unpublished); Los Alamos Free-Electron Workshop, Los Alamos, New Mexico, 1980 (unpublished); International Quantum Electronics Conference, Boston, 1980 [abstract published in J. Opt. Soc. Am. 70, 620 (1980)]; International School of Quantum Electronics, Erice, Italy, 1980 (unpublished); ONR Workshop on Free-Electron Lasers, Sun Valley, Idaho, 1981 (proceedings to be published by Addison-Wesley, Reading Mass., in press). W. Becker, currently a member of our group, has discussed quantum aspects of harmonic generation in the FEL in Z. Phys. 42B, 87 (1981).

[2]F. De Martini, in *Free-Electron Generators of Coherent Radiation, Proceedings of the Conference on the Physics of Quantum Electronics, Telluride, Colorado, 1979*, edited by S. F. Jacobs, H. S. Pilloff, M. Sargent, M. O. Scully, and R. Spitzer (Addison-Wesley, Reading, Mass., 1980), Vol. 7, p. 789.

[3]J. M. J. Madey and R. C. Taber, Ref. 2. p. 741.

[4]W. B. Colson, IEEE J. Quantum Electron. QE-17, 1417 (1981). W. B. Colson, Phys. Rev. A 24, 639 (1981).

[5]Linearly polarized wigglers are being used for the FEL and undulator experimental projects headed by C.

Brau at Los Alamos National Laboratory, New Mexico; J. Slater, at Math Sciences Northwest, Bellevue, Washington; G. Neil at TRW, Redondo Beach, California H. Winick at SPEAR, Stanford, California; C. Pelligrini at Brookhaven National Laboratory, Upton, N.Y.; V. Baier at Novosobirsk, USSR; L. Elias at University of California at Santa Barbara; and Y. Farge at Orsay. J. M. J. Madey, at the ONR Workshop on Free-Electron Lasers, Sun Valley, Idaho, 1981 (Addison-Wesley, Reading, Mass, in press), has reported observation of second-harmonic emission from the Stanford helical wiggler. It is not known if this emission is coherent. Such emission is not predicted by our theory, although off-axis incoherent harmonic radiation has been calculated for the helical wiggler by W. B. Colson (Ref. 4).

[6]G. T. Moore and M. O. Scully, Phys. Rev. A 21, 2000 (1980).

[7]Generalizations of the theory can be made to cases where the electrons lose a substantial fraction of their energy by becoming trapped in the wells of the ponderomotive potential of a tapered wiggler and also cases where the electrons are not ultrarelativistic. These results will be published in the proceedings of the 1981 Sun Valley, Idaho FEL Workshop (see Ref. 1).

[8]Equations for the small-signal regime are easily derived along the lines used by R. Bonifacio, P. Meystre, G. T. Moore, and M. O. Scully, Phys. Rev. A 21, 2009 (1980).

[9]We disagree here with the Bessel function expression of J. M. J. Madey and R. C. Taber [Eq. (1) of Ref. 3]. The difference seems to be traceable to sign errors in their Eq. (52).

[10]D. A. G. Deacon pointed out how such Bessel function expressions affect the effective amplitude of the linear wiggler in lectures at the International School of Quantum Electronics, Erice, Italy, 1980 (unpublished).