Analytic scattering length for potential scattering

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A formal, iterative perturbation series is developed for the inverse of the scattering length, in powers of the coupling constant λ. An analysis of the series is carried out for two examples of screened Coulomb potentials. It allows us to isolate the nearby singularities and hence obtain the analytic continuations for the scattering length, which are valid over a large range of values of λ.

I. INTRODUCTION

The scattering length is an interesting parameter in the collision of particles. It is important in the description of low-energy scattering. Apart from giving the total cross section at zero energy,

$$\sigma(0)=4\pi a^2, \tag{1}$$

a being the s-wave scattering length,

$$a=\lim_{q\to 0}\left(\frac{1}{q}\tan\delta_0(q)\right), \tag{2}$$

it gives information about the bound states of the system. (The scattering length is often defined to be equal to $-a$.) For example, if the potential is attractive, a negative scattering length implies the existence of bound states.[1] It plays a particularly significant role in the scattering of electrons by the screened Coulomb potential of rare gases, where it is related[2] to the Ramsauer-Townsend effect.

For a given screened Coulomb potential one can evaluate the scattering length directly by numerically evaluating $\delta_0(q)$ and taking the appropriate limit (2). Alternatively, one may calculate it from the equation,[3]

$$\frac{da(r)}{dr}=-2\lambda V(r)[r+a(r)]^2, \qquad a(0)=0, \quad a(\infty)=a, \tag{3}$$

or by using the Kohn-Hulthen variational procedure.[4] However, these calculations do not elucidate the structure of the scattering length, a structure which is rich in infinities and zeros when there are many bound states. It would appear that a knowledge of the structure of the scattering length as a function of the coupling constant can provide not only a better understanding of its behavior, but also a practical means of calculating the scattering length for potentials which are not highly singular.

We first develop a formal, iterative perturbation series for a^{-1} in powers of the coupling constant λ, by using the Noyes[5] form of the integral equation for the scattering matrix T. It is preferable to consider the series for a^{-1} rather than for a, since in most cases the series for λa^{-1} has a larger radius of convergence. We analyze in detail the specific case of the scattering length for some screened Coulomb potentials which can be written in the form

$$\lambda V(r)=-\frac{Z}{r}f\left(\frac{r}{r_0}\right), \tag{4}$$

where r_0 is the screening parameter for which the screening becomes significant when $r>r_0$. It is shown by a scale transformation that the scattering length for these potentials is a function essentially of an effective coupling constant λ,

$$\lambda=Zr_0. \tag{5}$$

Then a study of the high-order coefficients in the perturbation series in λ allows us to isolate the nearest poles of a^{-1} and hence obtain an analytic continuation for a^{-1}, which is valid over a fairly large range of values of λ.

As a possible application of these considerations we discuss the scattering lengths for the collision of electrons with rare gases. The screening parameter r_0 may in some cases be determined by requiring that

$$\lambda V(r)\underset{r\to\infty}{\longrightarrow}-\frac{\alpha/2}{r^4}, \tag{6}$$

where α is the polarizability of the atom. We find that the scattering lengths calculated for a typical potential are in good agreement with the experimental values for He and Ne.

II. GENERAL CONSIDERATIONS

Consider a collision process described by the Hamiltonian

$$H=\tfrac{1}{2}p^2+\lambda V(r). \tag{7}$$

We first write an integral equation for the scattering matrix which will allow us to obtain an iterative perturbation series for the inverse of the scattering length as a function of λ.

A. An integral equation

The Lippmann-Schwinger equation for the partial-wave projection of the T matrix is

$$T_l(p,q)=V_l(p,q)+\frac{2}{\pi}\int_0^\infty k^2dk\frac{V_l(p,k)T_l(k,q)}{E-\frac{1}{2}k^2+i\eta}, \quad (8)$$

where

$$T_l(q,q)=-\frac{e^{i\delta_l}\sin\delta_l}{2q}, \quad (9)$$

$$V_l(p,q)=\lambda\int_0^\infty j_l(pr)V(r)j_l(qr)r^2dr, \quad (10)$$

$$E=\tfrac{1}{2}q^2, \quad (11)$$

with $j_l(r)$ being the spherical Bessel functions of the first kind. For developing a perturbation series for the scattering length, it is convenient to rewrite the integral equation in the Noyes[5] form. For that one writes

$$T_l(p,q)=T_l(q,q)f_l(p,q), \quad (12)$$

with

$$f_l(q,q)=1. \quad (13)$$

On substituting this in Eq. (8), we get

$$T_l(q,q)=\frac{V_l(q,q)}{1-\frac{2}{\pi}\int_0^\infty k^2dk\frac{V_l(q,k)f_l(k,q)}{E-\frac{1}{2}k^2+i\eta}}, \quad (14)$$

$$f_l(k,q)=\frac{V_l(k,q)}{V_l(q,q)}+\frac{2}{\pi}\int_0^\infty k'^2dk'\frac{1}{E-\frac{1}{2}k'^2+i\eta}\times\left(V_l(k,k')-\frac{V_l(k,q)}{V_l(q,q)}V_l(q,k')\right)f_l(k',q). \quad (15)$$

The quantity which is of primary interest to us is the function in the denominator of Eq. (14), which we designate as $D_l(q)$. One can rewrite this function in a form which is more useful for computations by going over to the coordinate space. We now define $u(r)$, $F(r)$, $g(r_1,r_2)$, and β as follows:

$$u(r)=V(r)[j_l(qr)]^2, \quad (16)$$

$$f_l(k,q)=\beta^{-1}\int_0^\infty j_l(kr)V(r)F(r)j_l(qr)r^2dr, \quad (17)$$

$$g(r_1,r_2)=\frac{2}{\pi}\int_0^\infty\frac{k^2dk}{E-\frac{1}{2}k^2+i\eta}\left(\frac{j_l(kr_1)}{j_l(qr_1)}\right)\left(\frac{j_l(kr_2)}{j_l(qr_2)}\right), \quad (18)$$

and

$$\beta=\int_0^\infty u(r)r^2dr, \quad (19)$$

where the dependence of $u(r)$, $F(r)$, $g(r_1,r_2)$, and β on l and q is suppressed. The function $g(r_1,r_2)$ is related to the partial-wave Green's function,[6] and is given by

$$g(r_1,r_2)=2q\left(\frac{n_l(qr_>)}{j_l(qr_>)}-i\right), \quad (20)$$

with $r_>$ being the larger of the variables r_1 and r_2, and $n_l(r)$ being the spherical Bessel function of the second kind. In terms of the above quantities, one can write

$$D_l(q)=1-\frac{\lambda}{\beta}\int_0^\infty dr_1dr_2r_1^2u(r_1)g(r_1,r_2)r_2^2u(r_2)F(r_2), \quad (21)$$

where the function $F(r_2)$ satisfies the integral equation

$$F(r_2)=1+\frac{\lambda}{\beta}\int_0^\infty dr_3dr_4r_3^2u(r_3)[g(r_2,r_4)-g(r_3,r_4)]\times r_4^2u(r_4)F(r_4). \quad (22)$$

The advantage of Eqs. (21) and (22) is that

$$q\cot\delta_l=-\frac{\mathrm{Re}D_l(q)}{2\lambda\beta}, \quad (23)$$

so that an iterative solution for $\mathrm{Re}D_l(q)$ essentially gives an expansion in powers of λ for $q\cot\delta_l$. One can write

$$q\cot\delta_l=-\frac{1}{2\lambda\beta}\sum_{n=0}^\infty b_n\lambda^n, \quad (24)$$

where

$$b_0=1,$$

$$b_n=-\frac{1}{\beta}\int_0^\infty dr_1dr_2r_1^2u(r_1)g_r(r_1,r_2)r_2^2u(r_2)F_n(r_2), \quad n>0. \quad (25)$$

Here, $g_r(r_1,r_2)$ is the real part of $g(r_1,r_2)$,

$$g_r(r_1,r_2)=2q\frac{n_l(qr_>)}{j_l(qr_>)}, \quad (26)$$

and $F_n(r_2)$ are calculated iteratively:

$$F_1(r_2)=1,$$

$$F_n(r_2)=\frac{1}{\beta}\int_0^\infty dr_3\,dr_4r_3^2u(r_3)[g_r(r_2,r_4)-g_r(r_3,r_4)]\times r_4^2u(r_4)F_{n-1}(r_4), \quad \text{for } n>1. \quad (27)$$

These relations are especially simple in the limit of $q\to 0$ for the s wave, and lead to

$$a^{-1}=-\frac{1}{2\lambda\beta}-\frac{1}{2\beta}\sum_{n=1}^\infty b_n\lambda^{n-1}, \quad (28)$$

where the expansion coefficients should be evaluated for

$$u(r)=V(r), \quad (29)$$

$$g_r(r_1,r_2)=-\frac{2}{r_>}, \quad (30)$$

and

$$\beta = \int_0^\infty dr\, r^2 V(r). \tag{31}$$

Equation (28) is our main result. The coefficients of the expansion series are obtained from Eq. (25) with $F_n(r_2)$ being evaluated from Eq. (27). The analysis of the expansion coefficients b_n for large n then allows us to obtain the analytic structure of a^{-1}, and hence analytically continue it beyond the nearby singularities.

In principle, one may consider the perturbation series for a, which can be obtained, for example, from Eq. (3). However, as λ increases, for most potentials the scattering length first encounters a pole (when a bound state appears) and then goes to zero. Therefore, it is advantageous to consider a^{-1} which has a larger domain of convergence (after separating out the pole at $\lambda = 0$) than a. Of course one could obtain the series for a^{-1} by inverting the series for a. Since we are interested in terms of fairly high order in λ, we prefer to obtain the expansion coefficients directly from Eqs. (25) and (27). While Eq. (27) appears rather formidable, it has an advantage that the integrand, having a difference of two terms, converges rather rapidly.

B. Scale transformations

One can show that for the class of potentials

$$\lambda V(r) = -\frac{Z}{r^n} f\left(\frac{r}{r_0}\right), \tag{32}$$

the scattering length is a function, effectively, of only one variable. To do this, we start with the Schrödinger equation

$$\left[\tfrac{1}{2}p^2 - \frac{Z}{r^n} f\left(\frac{r}{r_0}\right)\right]\psi(r) = E\psi(r) \tag{33}$$

and subject it to a Symanzik transformation[7]

$$\vec{r} \to r_0 \vec{r}, \quad \vec{p} \to \frac{1}{r_0}\vec{p}. \tag{34}$$

The resulting equation is

$$\frac{1}{r_0^2}\left(\tfrac{1}{2}p^2 - \frac{Zr_0^{2-n}}{r^n} f(r)\right)\psi(rr_0) = E\psi(rr_0). \tag{35}$$

Since the phase shifts for Eqs. (33) and (35) are the same, one gets

$$\delta_l(q, Z, r_0) = \delta_l(qr_0, Zr_0^{2-n}, 1), \tag{36}$$

the variables being the momentum, the coupling strength, and the screening parameter. This equation implies that the scattering length in Eq. (2) should satisfy the relation

$$a(Z, r_0) = r_0 a(Zr_0^{2-n}, 1), \tag{37}$$

the variables here being the coupling strength and the screening parameter. [The scaling relation Eq. (37) can also be obtained directly from Eq. (3).] Thus $a(Z, r_0)/r_0$ is a function of only Zr_0^{2-n}, which therefore allows us to treat the scattering length as a function effectively of one variable, i.e., Zr_0^{2-n}. Specifically, if we take $n = 1$ for the screened Coulomb potentials, the scattering lengths for the same value of Zr_0 but different values of r_0, will be linear in r_0. This property greatly simplifies the analysis of the scattering length for the class of screened Coulomb potentials given by Eq. (4).

III. TWO EXAMPLES

From specific applications of the results derived in Sec. II B we consider two examples of screened Coulomb potentials:

$$\lambda V_{\mathrm{I}}(r) = -\frac{Z}{r}\exp\left(\frac{-r}{r_0}\right), \tag{38}$$

$$\lambda V_{\mathrm{II}}(r) = -\frac{Z}{r}\left(1 - \frac{r}{(r^3 + r_0^3)^{1/3}}\right). \tag{39}$$

The first potential is the Yukawa potential which provides a good representation of the static potential for a system of an electron and a neutral atom. The second potential has an asymptotic behavior:

$$\lambda V_{\mathrm{II}} \underset{r\to\infty}{\longrightarrow} -\frac{Zr_0^3}{3r^4}, \tag{40}$$

and hence is a candidate for the interaction between an electron and a polarizable atom.

A. Potential $V_{\mathrm{I}}(r)$

As shown in Sec. II, it is sufficient to consider only

$$\lambda V_{\mathrm{I}}(r) = -\frac{\lambda}{r} e^{-r}, \tag{41}$$

where $\lambda = Zr_0$ is the effective coupling constant. For this potential, one has the exact results

$$\beta = -1, \tag{42}$$

$$b_1 = -1, \tag{43}$$

while the coefficients b_n for $n > 1$ have to be obtained numerically from Eqs. (25) and (27). They are listed in Table I.

The striking thing to be observed in Table I is that after the first few terms, say, for $n \geq 3$, the series approaches a geometric series with the ratio of successive terms approaching the value 0.4489. This strongly suggests that the nearest singularity of the function represented by the series is a pole. We may therefore analytically continue the function beyond the pole by writing

TABLE I. The b_n are the coefficients of expansion $c_n = b_n - A_1(A_2)^{n-1}$ and $d_n = c_n - B_1(B_2)^{n-1}$ for potential $V_I(r)$.

n	b_n	$R_n = \frac{b_{n+1}}{b_n}$	c_n	$R_n' = \frac{c_{n+1}}{c_n}$	d_n	$R_n'' = \frac{d_{n+1}}{d_n}$
1	-1.0000	1.5093×10^{-1}	-7.328×10^{-1}	4.228×10^{-2}	-6.098×10^{-1}	1.65×10^{-2}
2	-1.5093×10^{-1}	3.8358×10^{-1}	-3.098×10^{-2}	1.307×10^{-1}	-1.007×10^{-2}	4.9×10^{-2}
3	-5.7894×10^{-2}	4.2827×10^{-1}	-4.050×10^{-3}	1.54×10^{-1}	-4.95×10^{-4}	
4	-2.4794×10^{-2}	4.4176×10^{-1}	-6.24×10^{-4}	1.65×10^{-1}		
5	-1.0953×10^{-2}	4.4628×10^{-1}	-1.03×10^{-4}	1.70×10^{-1}		
6	-4.8881×10^{-3}	4.4797×10^{-1}	-1.75×10^{-5}			
7	-2.1897×10^{-3}	4.4845×10^{-1}				
8	-9.8197×10^{-4}	4.4872×10^{-1}				
9	-4.4063×10^{-4}	4.4886×10^{-1}				
10	-1.9778×10^{-4}					

$$\sum_{n=1}^{\infty} b_n \lambda^{n-1} = \frac{A_1}{1 - A_2\lambda} + \sum_{n=1}^{\infty} c_n \lambda^{n-1}, \tag{44}$$

where $A_2 \approx 0.4489$, and $\sum c_n \lambda^{n-1}$ has a larger domain of convergence than $|\lambda| < (0.4489)^{-1}$. The quantity A_1 is determined by noting the relation

$$b_n \underset{n\to\infty}{\longrightarrow} A_1(A_2)^{n-1}, \tag{45}$$

which implies that $A_1 \approx -0.2672$, so that we can identify the location of the nearest pole and its residue. A similar analysis can be carried out for the function represented by $\sum c_n \lambda^{n-1}$ whose coefficients are obtained from

$$c_n = b_n - A_1(A_2)^{n-1} \tag{46}$$

and are given in Table I. Since c_n for large n are not very accurate, having small differences between two nearly equal numbers, they are given only for $n = 1, \ldots, 6$. It is again observed that the ratio of successive values of c_n approaches a constant value of about 0.170 indicating that the nearest singularity of the function represented by $\sum c_n \lambda^{n-1}$ is again a pole, thus allowing us to write

$$\sum c_n \lambda^{n-1} = \frac{B_1}{1 - B_2\lambda} + \sum d_n \lambda^{n-1}, \tag{47}$$

where $B_2 \approx 0.170$ and $\sum d_n \lambda^{n-1}$ has a larger domain of convergence than that of $\sum c_n \lambda^{n-1}$. Then, the quantity B_1 is determined by the condition

$$c_n \underset{n\to\infty}{\longrightarrow} B_1(B_2)^{n-1}, \tag{48}$$

which gives the result that $B_1 \approx -0.123$. We can now similarly proceed with $\sum d_n \lambda^{n-1}$ but with much less accuracy. We finally obtain

$$\sum_{n=1}^{\infty} b_n \lambda^{n-1} = -\frac{0.2672}{1 - 0.4489\lambda} - \frac{0.123}{1 - 0.170\lambda} - \frac{0.206}{1 - 0.049\lambda} - 0.4038, \tag{49}$$

and the scattering length from Eqs. (28) and (42) is

$$a = \frac{2\lambda}{1 + \lambda \sum_{n=1}^{\infty} b_n \lambda^{n-1}}. \tag{50}$$

The position of the third pole is not accurately determined so that Eqs. (49) and (50) are expected to give a good analytic continuation of the scattering length for $\lambda < 7$, i.e., just beyond the second pole in $\sum_{n=1}^{\infty} b_n \lambda^{n-1}$. The predictions for a, for some typical values of λ, are given in Table II, along with the values obtained by numerically solving the Schrödinger equation using Numerov's method. (The numerical value given for $\lambda = 0.5$ is from the summation of the perturbation series.)

One can also calculate the values of λ or Zr_0 at which the first two bound states appear by requiring that

$$1 + \lambda \sum_{n=1}^{\infty} b_n \lambda^{n-1} = 0 \tag{51}$$

under which condition the scattering length becomes infinite. Using Eq. (49), we find that the first bound state appears at the critical value

$$\lambda_1 \approx 0.840, \tag{52}$$

TABLE II. Predictions for the scattering lengths a_p for potential $V_I(r)$, along the numerical values a_n.

λ	a_p	a_n
0.5	2.207	2.207
1.5	-2.122	-2.128
2.5	1.15	1.11
3.5	-9.87	-10.47
4.5	-2.83	-2.95
5.5	-0.90	-1.03
6.5	2.64	2.91

and the second bound state appears at

$$\lambda_2 \approx 3.21 . \tag{53}$$

B. Potential $V_{II}(r)$

In this case, the arguments of Sec. II imply that it is adequate to consider

$$\lambda V_{II}(r) = -\frac{\lambda}{r}\left(1 - \frac{r}{(r^3+1)^{1/3}}\right),$$

where $\lambda = Zr_0$ is the effective coupling constant. For this potential, one has

$$\beta = -\tfrac{1}{2} , \tag{54}$$

while the coefficients b_n are obtained numerically from Eqs. (25) and (27) and are listed in Table III. We now proceed along the same lines as we did for $V_I(r)$ to obtain

$$\sum_{n=1}^{\infty} b_n \lambda^{n-1} = -\frac{0.258\,009}{1-0.345\,207\lambda} - \frac{0.1279}{1-0.130\lambda} - \frac{0.1711}{1-0.035\lambda} - 0.0372 \tag{55}$$

and

$$a = \frac{\lambda}{1+\lambda\sum_{n=1}^{\infty} b_n \lambda^{n-1}} \tag{56}$$

analogous to Eqs. (49) and (50) for $V_I(r)$. Here also, the position of the third pole is not accurately determined so that the analytic continuation for the scattering length is expected to be good for $\lambda < 9$, i.e., just beyond the second pole in $\sum_{n=1}^{\infty} b_n \lambda^{n-1}$. The predictions for a, for some typical values of λ, are given in Table IV, along with the values obtained by numerically solving the Schrödinger equation using Numerov's method.

One can use Eqs. (55) and (56) for calculating the values of λ or Zr_0 at which the first two bound states appear, by requiring that

$$1 + \lambda \sum_{n=1}^{\infty} b_n \lambda^{n-1} = 0 , \tag{57}$$

for which the scattering length becomes infinite. From Eq. (55), it is found that the first bound state appears at

$$\lambda_1 \approx 1.226 , \tag{58}$$

while the second bound state appears at

$$\lambda_2 \approx 4.85 . \tag{59}$$

IV. AN APPLICATION

As a practical application of the results obtained, we discuss the scattering lengths for the scattering of an electron by rare gases.[2] The potential for this scattering should have the property

$$\lambda V(r) \underset{r\to 0}{\longrightarrow} -\frac{Z}{r} \tag{60}$$

and

$$\lambda V(r) \underset{r\to\infty}{\longrightarrow} -\frac{\alpha/2}{r^4} , \tag{61}$$

where α is the polarizability of the atom. One may therefore take $\lambda V_{II}(r)$ as a good candidate for simulating such a potential, provided we take

$$r_0 = \left(\frac{3\alpha}{2Z}\right)^{1/3} , \tag{62}$$

which ensures that Eq. (40) has the same asymptotic behavior as Eq. (61). The required scattering length is then deduced by using Eq. (37),

$$a\left(Z, \left(\frac{3\alpha}{2Z}\right)^{1/3}\right) = \left(\frac{3\alpha}{2Z}\right)^{1/3} a\left(Z\left(\frac{3\alpha}{2Z}\right)^{1/3}, 1\right), \tag{63}$$

where the right-hand side can be obtained from Eq. (56), with $\lambda = Z(3\alpha/2Z)^{1/3}$.

TABLE III. The b_n are the coefficients of expansion $c_n = b_n - A_1(A_2)^{n-1}$ and $d_n = c_n - B_1(B_2)^{n-1}$ for potential $V_{II}(r)$.

n	b_n	$R_n = \frac{b_{n+1}}{b_n}$	c_n	$R_n' = \frac{c_{n+1}}{c_n}$	d_n	$R_n'' = \frac{d_{n+1}}{d_n}$
1	$-5.942\,45\times10^{-1}$	$1.878\,96\times10^{-1}$	-3.3624×10^{-1}	6.718×10^{-2}	-2.0834×10^{-1}	2.856×10^{-2}
2	$-1.116\,56\times10^{-1}$	$2.966\,02\times10^{-1}$	-2.2590×10^{-2}	1.050×10^{-1}	-5.9495×10^{-3}	3.5×10^{-2}
3	$-3.311\,75\times10^{-2}$	$3.291\,44\times10^{-1}$	-2.3711×10^{-3}	1.209×10^{-1}	-2.096×10^{-4}	
4	$-1.090\,04\times10^{-2}$	$3.394\,81\times10^{-1}$	-2.8657×10^{-4}	1.274×10^{-1}		
5	$-3.700\,49\times10^{-3}$	$3.430\,84\times10^{-1}$	-3.651×10^{-5}	1.30×10^{-1}		
6	$-1.269\,58\times10^{-3}$	$3.444\,04\times10^{-1}$	-4.749×10^{-6}	1.30×10^{-1}		
7	$-4.372\,49\times10^{-4}$	$3.449\,01\times10^{-1}$	-6.197×10^{-7}			
8	$-1.508\,08\times10^{-4}$	$3.450\,91\times10^{-1}$				
9	$-5.204\,23\times10^{-5}$	$3.451\,64\times10^{-1}$				
10	$-1.796\,31\times10^{-5}$	$3.451\,92\times10^{-1}$				
11	$-6.200\,71\times10^{-6}$	$3.452\,03\times10^{-1}$				
12	$-2.140\,50\times10^{-6}$					

TABLE IV. Predictions for the scattering lengths a_p for potential $V_{II}(r)$, along with the numerical values a_n.

λ	a_p	a_n
1	4.07	4.06
2	−1.38	−1.40
3	0.14	0.1
4	2.35	2.3
5	−18.3	−20
6	−2.34	−2.5
7	0.74	0.8
8	0.31	0.4
9	1.43	2.0

For He, $Z=2$, $\alpha=1.36$, and the corresponding value of λ is

$$\lambda_{He}\approx 2.0. \tag{64}$$

Hence, by using Eq. (56) or Table IV, and Eq. (63), we get

$$a_{He}\approx -1.38, \tag{65}$$

which compares favorably with the observed value[8] of $a_{He}\approx -1.19$. Similarly for Ne, $Z=10$ and $\alpha=2.65$, so that

$$\lambda_{Ne}\approx 7.35. \tag{66}$$

Then using Eqs. (56) and (63), one obtains

$$a_{Ne}\approx -0.26, \tag{67}$$

which is in good agreement with the experimental value[8] of $a_{Ne}\approx -0.24$. However, the agreement in these cases should be regarded as encouraging but inconclusive since we have omitted all the exchange and polarization effects.

For heavier rare gases such as Ar, Kr, and Xe, the effective value of λ is quite large, e.g., $\lambda_{Ar}\approx 18$ so that it is outside the range of validity of Eq. (56). In any case, the structure of these atoms is complicated and it is not realistic to describe it in terms of only its polarizability.

V. DISCUSSION

We have developed a formal, iterative perturbation series for the inverse of the scattering length, in powers of the coupling constant λ. This series is analyzed in detail for a class of screened Coulomb potentials. The large-n behavior of the sequence of the coefficients of the series allows us to isolate the nearby singularities as a function of λ, and hence obtain an analytic continuation for the scattering length which is valid over a fairly large range of values of λ. This technique can, in principle, be used to obtain the analytic continuations of the scattering length as a function of the coupling constant, for other potentials as well.

Our sequential method of the analytic continuation of the scattering length is similar in spirit to the Pade-approximant technique. However, since the region of interest extends into the region of singularities, the major concern of the sequential method is to stabilize the parameters of the nearby singularities, which in the cases considered, are the positions of the poles and their residues. It is likely that if the Pade approximant for the perturbation series converges, it will give for $n\to\infty$, the same result as the sequential method.

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[1]N. Levinson, K. Dan. Vidensk. Selsk., Mat.-Fys. Medd. 25, No. 9 (1947).
[2]T. F. O'Malley, Phys. Rev. 130, 1020 (1963).
[3]F. Calogero, *Variable Phase Approach to Potential Scattering* (Academic, New York, 1967), p. 69.
[4]L. Hulthen, Ark. Mat. Astron. Fys. 35A, No. 25 (1948); W. Kohn, Phys. Rev. 74, 1763 (1948).
[5]P. Noyes, Phys. Rev. Lett. 15, 538 (1965).
[6]T-Y. Wu and T. Ohmura, *Quantum Theory of Scattering* (Prentice-Hall, Englewood Cliffs, N. J., 1962), p. 43.
[7]B. Simon, Ann. Phys. (N.Y.) 58, 79 (1970).
[8]B. H. Bransden, *Atomic Collision Theory* (Benjamin, New York, 1970), p. 117.