Analytic velocity-dependent potential for bound and scattering states of electrons and atoms

A. E. S. Green, D. E. Rio, and T. Ueda
University of Florida, Gainesville, Florida 32611
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In this work we represent the static potential of an electron interacting with an atom by a sum of two Debye-Hückel or Yukawa functions. We approximate the nonlocal exchange term by an attractive, velocity-dependent potential using the effective-mass approximation with a regularized Yukawa form factor. This is transformed to an approximate local but energy-dependent potential. We also include an imaginary term with a regularized Yukawa form factor in the static potential. We apply this framework to a description of the bound and scattering states of electrons interacting with atomic oxygen and electrons interacting with neon. The results suggest that a small degree of velocity or energy dependence yields improvements with respect to the use of strictly static electron-atom potentials. In applications to atomic physics the formalism which is economical in parameters should serve as a simple parametrization of experimental data and as a means of extrapolation and interpolation of experimental observations.

I. INTRODUCTION

The work of Green, Sellin, and Zachor (GSZ)[1] and a number of subsequent studies[2-4] have shown that all the single-particle energy levels of any light- or middle-weight neutral atom or ion can be generated quite precisely by the use of Schrödinger's equation with an analytical potential of the form

$$V(r) = -\frac{2}{r}\left(\frac{Z-\eta}{H(e^{r/d}-1)+1}+\eta\right). \tag{1}$$

Here H and d are adjustable parameters, $\eta = 0$ for a negative ion, $\eta = 1$ for a neutral atom, $\eta = 2$ for a singly charged positive ion, etc. The work of Darewyche *et al.*[5] has shown that the energy levels of heavy atoms are well described by Eq. (1) when it is used in conjunction with the Dirac equation.

The calculations of Berg, Purcell, and Green (BPG)[6] have shown that Eq. (1) with $\eta = 0$ realistically represents the interaction of incident electrons with a neutral atom. This is illustrated by the differential elastic cross sections calculated for electron scattering by the rare gases Ar, Kr, and Xe as shown in Fig. 1. Here the boxes are relative experimental cross sections which have been normalized to agree with the corresponding area under the theoretical angular distribution. The parameters H and d were taken from the bound-state parameters of GSZ without adjustment. These impressive scattering results, obtained over a decade ago, suggested that simple analytic independent-particle models (AIPM) for atoms can account for large bodies of experimental data and provide a means of interpolation and extrapolation from limited experimental data. In nuclear physics AIPM have played a similar role.[7-9]

The AIPM has already proved useful in studies of electron energy deposition where the lack of comprehensive experimental data has been a major obstacle. Thus, systematic trends of inelastic and ionization cross sections have been calculated using Schrödinger wave functions generated with AIPM.[10,2,3] The primary purpose of the present study is to examine several refinements of AIPM models for atoms, which might improve the generation of differential elastic scattering cross sections. These are particularly needed for Monte Carlo calculation of spatial-yield spectra (SYS).[11] In such calculations there are tactical advantages to the use of a potential which leads to analytic cross sections in the Born approximation. These advantages have been duscussed by Green, Schippnick, Rio, and Ganas[12] (GSRG), who use a sum of two Yukawa or Debye-Hückel functions as the bound-state potential for O^- and the optical potential for the scattering of electrons by atomic oxygen. They also show that a modified phase-shift formula based on the Born approximation for the double Yukawa (DY) function can serve as an accurate analytic representation of the Schrödinger phase shifts when corrected to satisfy Levinson's theorem.[13,14]

In the present work, we develop a unified bound and scattering state model which goes beyond simple static models. In particular, we include a nonlocal or velocity-dependent part to the potential to represent exchange effects and an imaginary component to represent inelastic processes. We show that the improved model provides a reasonable synthesis of the sparse data for electrons interacting with oxygen (O) and a good synthesis of the abundant data for electrons interacting with neon (Ne).

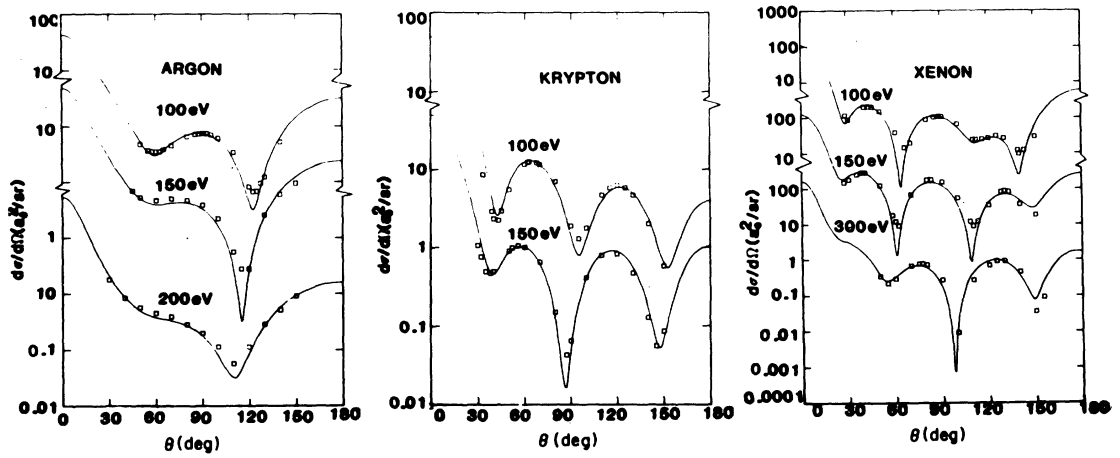

FIG. 1. Composite diagram of theoretical and experimental differential elastic scattering cross sections for Ar, Kr, and Xe. The vertical axis is logarithmic. The energy of the incident electron is listed next to the curve (adapted from Berg *et al.*, Ref. 6).

II. VELOCITY-DEPENDENT BOUND AND SCATTERING STATE MODEL

Following GSRG we assume that the static component of the IPM (independent-particle-model) potential may be represented by the double Yukawa function

$$V(r) = -2Z\left[\frac{1}{H}\, Y_1 + \left(1 - \frac{1}{H}\right) Y_2\right], \tag{2}$$

where

$$Y_1 = r^{-1}e^{-r/d}, \quad Y_2 = r^{-1}e^{-r/s}, \tag{3}$$

and

$$s = d/(H+1). \tag{4}$$

The scale factor relationship, given by Eq. (4), insures that the static DY potential behaves like the GSZ electron-atom potential. The GSZ potential has been demonstrated to be a convenient AIPM which yields eigenvalues and wave functions with precision comparable to those of the best numerical Hartree-Fock IPM potentials.[15,16] It might also be remarked that the Debye-Hückel or Yukawa potential has been considered as a candidate for the electron-atom potential in a number of earlier studies.[17-20] Although a single Yukawa potential does not accurately approximate the electron-atom potential, the two-parameter generalization of the Yukawa function given by Eq. (2), which reduces to a single Y function when $H = 1$, can serve as an accurate model.

In the present work, to provide for nonlocal exchange and correlation effects we assume that the electron-atom system obeys an integro-differential equation of the form[14]

$$\left(E + \frac{\hbar^2\nabla^2}{2m} - V(r)\right)\psi(r) = \int K(\vec{r}, \vec{r}')\psi(\vec{r}')d'\tau', \tag{5}$$

where the term on the right-hand side is the mathematical form of the exchange term. In atomic structure studies Slater,[21] Gaspar,[22] Kohn and Sham,[23] and Slater and Johnson[24] have used a statistical model to reduce the nonlocal exchange term to a static contribution to the electrostatic Hartree potential. This procedure reduces the many-electron problem to a static independent-particle model (IPM), which is quite accurate and has many convenient mathematical features. In scattering studies the nonlocal term may be more important, especially at higher bombarding energies. Here we treat this term by methods used in nuclear physics where such terms are of major importance and hence, have been subjected to intensive analyses.

Following Frahn and Lemmer[25] we assume that the nonlocal term in Eq. (5) may be expressed as

$$K(\vec{r}, \vec{r}') = K(\bar{\vec{r}})\Delta_b(\vec{s}), \tag{6}$$

where

$$\bar{\vec{r}} = \tfrac{1}{2}(\vec{r} + \vec{r}') \quad \vec{s} = \vec{r} - \vec{r}', \tag{7}$$

and

$$\Delta(s) = (\pi b^2)^{-3/2}\exp[-(s^2/b^2)]. \tag{8}$$

By expansion techiques[25,26,9] which neglect terms of the order of ∇^4 or higher we find

$$\int K(\vec{r}, \vec{r}')\psi(\vec{r}')dr' = \bar{K}(\vec{r}, \vec{p})\psi(\vec{r}), \tag{9}$$

where

$$\tilde{K}(\vec{r},\vec{p})\psi(\vec{r}) = K(r)\psi(r) + \frac{b^2}{16}\left[\nabla^2 K(r) + 2\vec{\nabla}K(r)\vec{\nabla} + K(r)\nabla^2\right]\psi(\vec{r}). \tag{10}$$

The choice of $K(r)$ must be made with care. If it were taken to be proportional to $V(r)$ as given by Eq. (2), it would cause singularity problems in the radial Schrödinger equation. The Wood-Saxon or the Green-Wyatt functions used in nuclear physics[9,26] are constant near the origin and hence, do not lead to this singularity difficulty. We can avoid it in atomic physics by taking $K(r)$ as a *regularized* combination of Yukawa functions[27]

$$K(r) = -K_0(Y_1 - Y_2) = -K_0\xi(r), \tag{11}$$

a form factor which insures that $K(r)$ is proportional to the static potential $V(r)$ at large distances. The negative sign before K_0 is chosen since theoretical studies of the exchange term[21-24] indicate that it makes an attractive contribution in the complex atom. Inserting Eqs. (4) and (10) into Eq. (5) we find that the Schrödinger equation becomes

$$\left(E + \frac{\hbar^2\nabla^2}{2m}\right)\psi(\vec{r}) = \left(V_s(r) - \frac{K_0 b^2}{16}\left[\nabla^2\xi(r) + 2\nabla\xi(r)\nabla + \xi(r)\nabla^2\right]\right)\psi(\vec{r}), \tag{12}$$

where

$$V_s(r) = V(r) + K(r) = -S_1 Y_1 - S_2 Y_2, \tag{13}$$

$$S_1 = \frac{2Z}{H} + K_0, \quad S_2 = 2Z - S_1. \tag{14}$$

We may combine the momentum-dependent terms in Eq. (12) with the kinetic-energy operator by defining the so-called effective mass

$$M(r) = \frac{m}{1+\phi}, \tag{15}$$

where $\phi = -\delta\xi(r)$ with

$$\delta = K_0\beta^2 a_0^2 m/2\hbar^2 = \beta^2 K_0/4\mathfrak{R} \tag{16}$$

and $\beta = b/a_0$. Then the Schrödinger equation becomes (see Ref. 9, p. 170)

$$-\frac{\hbar^2}{8}\left(\nabla^2\frac{1}{M(r)} + 2\nabla\frac{1}{M(r)}\nabla + \frac{1}{M(r)}\nabla^2\right)\psi(\vec{r}) + V_s(r)\psi(\vec{r}) = E\psi(\vec{r}). \tag{17}$$

Wyatt, Wells, and Green,[26] after reducing this equation to its radial form, solved for bound and scattering wave functions using the Runge-Kutta method. A more frequently used approach in the more recent nuclear literature is to introduce the transformation of Green[28]

$$\psi(r) = \chi(r)(1+\phi)^{-1/2}, \tag{18}$$

which changes the Schrödinger equation to

$$-\nabla^2\chi + V_e(r, E)\chi = E\chi, \tag{19}$$

where

$$V_e(r, E) = \frac{V_s(r) + \frac{1}{4}\nabla^2\phi}{1+\phi} - \frac{1}{4}\left(\frac{\phi'}{1+\phi}\right)^2 + E\frac{\phi}{1+\phi} \tag{20}$$

is now a local but energy-dependent potential. In Eqs. (19) and (20) all lengths are in Bohr radii (a_0) and energies in rydbergs ($\hbar^2/2ma_0^2$). If we go to the static limit of Eq. (20) $\phi = -\delta\xi(r) = -\delta(Y_1 - Y_2)$ vanishes and the effective potential then becomes simply Eqs. (13) and (14). Thus the static part of the potential in Eq. (20) is just the DY potential given by Eq. (2) with extra terms representing a static exchange effect. These terms can readily be absorbed into the primary static terms by a readjustment in the parameter H.

For small but nonvanishing δ the energy dependence of $V_e(r, E)$ in Eq. (20) occurs through the term $E\phi/(1+\phi) \approx -\delta E(Y_1 - Y_2)$. Accordingly, for small degrees of velocity dependence we would expect to be able to approximate $V_e(r, E)$ in linear form by

$$V_e(r, E) = -C_1(E)Y_1 - C_2(E)Y_2, \tag{21}$$

where

$$C_1(E) = S_1 + \delta E, \tag{22}$$

$$C_2(E) = 2Z - S_1 - \delta E. \tag{23}$$

At this point there are four possible adjustable parameters in the velocity-dependent optical potential: d, s, S_1, and δ.

To allow for absorptive effects we add $iW(r, E)$ to $V_e(r, E)$ in Eq. (21), where we choose

$$W(r, E) = -W_0(E)(Y_1 - Y_2). \tag{24}$$

This regularized Yukawa function should provide a reasonable average transition density function to characterize the inelastic processes which are simulated by the imaginary term. We have no simple basis for determining the form of $W_0(E)$ other than to expect that $W(r, E) = 0$ for E below the threshold (T) of the first excited state of the atom. Additionally, we might expect $W_0(E)$ to

asymptotically go over to the form of high-energy inelastic cross sections, i.e., $W_0(E) \to E^{-1}\ln E$. A suitable form which satisfies these constraints and has sufficient flexibility for fitting data from high to the threshold energy T is[29]

$$W_0(E) = \omega \frac{T}{E} \ln\left[\alpha\left(\frac{E}{T}-1\right)^{\nu}+1\right], \tag{25}$$

where ω, α, and ν are adjusted parameters.

At this point, we have developed all the phenomenological machinery needed for our work, therefore, let us now turn our attention to some simple applications.

III. APPLICATION TO THE e-O SYSTEM

Because of experimental difficulties the electron-oxygen scattering system has been the subject of many theoretical investigations[30-34] but only a few experimental studies[35-37] of recent vintage. The bound-state negative ion O^- has been subjected to considerable examination as summarized recently by Hotop and Lineberger.[38] In addition, an accurate Hartree-Fock description of O^- is available.[39] The magnitude of the eigenvalues obtained in this Hartree-Fock model are listed in row 1 of Table I.

Independent-particle models of the bound states of O^- have been studied recently by Ganas, Talman, and Green.[40] The magnitudes of the eigenvalues obtained by the optimum method are given in the second row of Table I. The eigenvalues produced by use of the analytic GSZ potential[1] adjusted to exactly give the electron affinity (0.1075) are given in row 3. The fourth row gives the results of the static double Yukawa model of GSRG.[12] The fifth row gives the results of this work, the velocity-dependent potential, which has been adjusted to give perfect agreement to the electron affinity and good agreement to the Hartree-Fock 1s and 2s states. Our objective now is to examine to what extent the model defined by the three adjusted parameters (d, H, and δ) change the predications of the purely static GSRG model.

TABLE I. Magnitudes of eigenvalues (Ry) for O^- ($Z=8$, electron affinity = 0.1075).

	Model	Ref.	1s	2s	2p	d	H
1	HF	42	40.396	1.6265	0.2585		
2	OPT	22	37.259	1.3416	0.2540		
3	GSZ	22	36.757	1.1408	0.1075	0.8860	1.9607
4	DY	23	36.698	1.1391	0.1071	1.0562	2.2700
5	VDP	TW[a]	40.270	1.2915	0.1075	1.0862	2.2491

[a] Denotes this work with the velocity-dependent parameter $\beta = 0.1554$ and $\delta = 0.039\,79$ where K_0 has been replaced by S_1 for calculation purposes. $S_1 = 6.5866$, $S_2 = 9.4133$, $d = 1.086$, and $s = 0.3167$.

Figure 2 illustrates the velocity-dependent potentials as defined by Eq. (20) for this work. The curves labeled 1s and 2p show the effective potentials acting in these bound states. For practical purposes the latter may be viewed as the potential at zero energy. The other two curves illustrate sample positive-energy effective potentials. Also shown on this diagram are the optimum (OPT) potential and the static exchange potential in the OPT model of Talman.[15,40]

The numerical calculations for e-O were carried out with the parameters listed in Table I. These correspond to a weak energy dependence as compared to that involved in nuclear physics where the effective mass, a reduced mass, goes to about $0.7m$ near the center of a nucleus.[26] Here the form factor $\xi(r)$ approaches the value 2.224 as $r \to 0$ so that the effective mass, an enhanced mass, goes to about $1.1m_0$ near the atomic center.

The elastic cross section and angular distribution data available for e-O scattering are very sparse. Figure 3 illustrates the resultant values for the differential elastic scattering cross section obtained using the analytic velocity-dependent potential (AVDP) model, as compared with measurements of scattering of 15 eV electrons by O. One

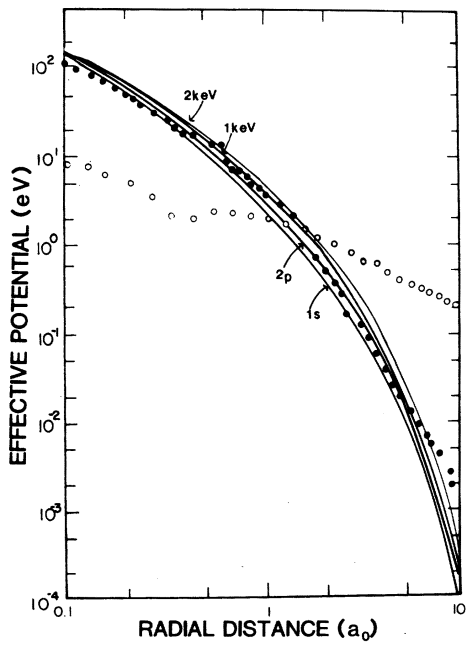

FIG. 2. Curves representing $V_e(r,E)$ given by Eq. (2) for $E(1s)$, $E(2p)$, $E = 1$ and 2 keV. The solid points denote the potential for O^- given by the optimum method of Talman and the open circles show the corresponding static exchange potential.

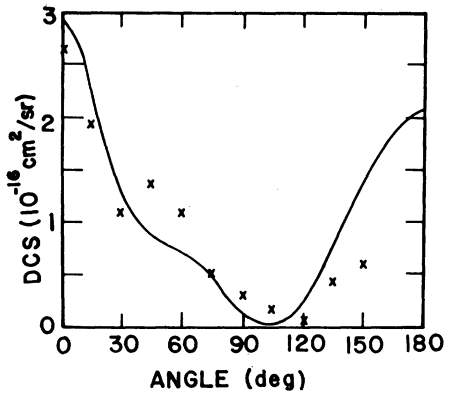

FIG. 3. Experimental differential cross sections for total scattering from oxygen measured by Dehmel *et al.* (crosses) and theoretical cross sections based upon AVDP model (solid curve).

notes that the agreement is reasonable even though we have not included an imaginary term. We could, of course, further improve the fit by including W_0, but in the case of scattering by O the data are simply too sparse to make the effort worthwhile.

Figure 4 shows the angular distributions calculated with the static potentials of GSRG and the velocity-dependent potentials of this work. One sees that the changes are small over this limited range of energies.

IV. APPLICATIONS TO *e*-Ne SCATTERING

In contrast to the *e*-O system, there are no bound states of *e*-Ne but there is an abundance of scattering data. Accordingly, we have concentrated our efforts on fitting scattering data with our velocity-dependent-potential model.

The total elastic and inelastic cross sections were taken from the compilation of de Heer *et al.*[41] At low energies absolute DESC (differential elastic scattering cross section) data were obtained from William and Crowe.[42] From 100–500 eV we used absolute data from William and Crowe, Jensen,[43] and Bromberg.[44] A weighted average based on the assigned absolute error was calculated for data which overlapped. Relative DESCs were not used.

Our optical-model computer code was coupled to a nonlinear least-square (NLLS) code. Initial parameter guesses of C_1, d, and s were used based upon the systematics of static potentials. The parameters were adjusted iteratively to optimize the fit of the calculated DESC to available data at each energy. Using these parameter values we calculated a *well strength parameter*[45]

$$\mathcal{S} = -\int_0^\infty V(r, E) r\, dr = C_1(d - s) + 2Zs$$

$$= (S_1 + \delta E)(d - s) + 2Zs\,. \tag{26}$$

Despite large fluctuations in C_1, d, and s we found that $\mathcal{S}(E)$ conformed quite well with a linear relationship for $C_1(E)$ [Eq. (22)], but with δ having an opposite sign as compared to the O system. The results are illustrated in Fig. 5. Using the $\mathcal{S}(E)$ line as a constraint we found it possible by iteration to arrive at fixed values of S_1, d, s, and δ (see Table II), which gave very good DESC. The values of $C_1(E)$ and $C_2(E)$ given by Eqs. (22) and (23) are also shown on Fig. 5.

During the final parameter searches we required that the inelastic reaction cross section σ_R, produced by the optical model, match available experimental values. This was accomplished

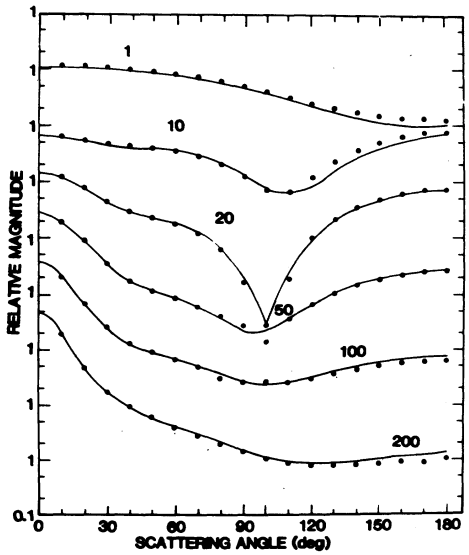

FIG. 4. Relative magnitudes of differential cross sections for velocity-dependent AIPM model (solid curves) and static model of GSRG.

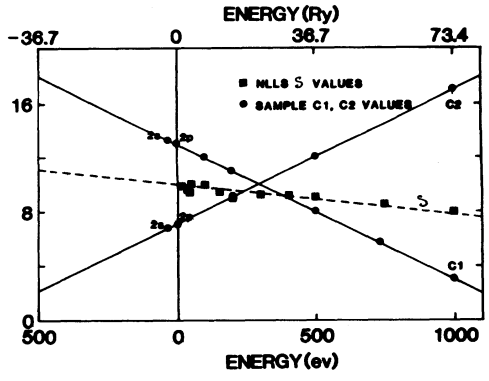

FIG. 5. Graphs of functional dependence of Yukawa coefficients $C_1(E)$ and $C_2(E)$ and associated $\mathcal{S}(E)$ used in modeling neon for both bound and scattering energies.

TABLE II. Parameters for double Yukawa velocity-dependent-potential model ($T=16.85$).

Parameters	Numerical value
S_1	13.00
δ	−0.01
d	0.5956
s	0.3438
ω	21.96
α	0.01
ν	1.788

by allowing $W_0(E)$, the coefficient of the imaginary potential, to establish its own energy dependence once we had set d and s. The calculated and experimental values of the inelastic reaction cross section are given in Table III. The corresponding values of $W_0(E)$ are presented in Fig. 6. A fit of $W_0(E)$ using Eq. (25) with the parameters ω, α, and ν given in Table II is also shown.

The final fits of our model to the e-Ne DESC data are presented in Fig. 7. It is gratifying that the systematics of the experimental data are well represented by this analytical energy dependent IPM. Also shown on this figure are results of the optical model of McCarthy *et al.*[46] Table IV gives a gross comparison of the χ^2 statistic for these two models at energies and angles for which McCarthy *et al.* presented numerical data. From the viewpoint of the intended applications it is gratifying that our simple analytic velocity-dependent-potential model compares favorably with their more elaborate optical model. By local adjustment of parameters one can fit the data almost exactly.

Having set the energy dependence of the scattering potential we may consider the system of the neon atom and incoming electron to approximate a Ne^- ion with its associated bound states.

TABLE III. Comparison of calculated and pseudoexperimental total inelastic reaction cross section (units are a_0^2).

E (eV)	Reaction cross section Calculated	Experimental
30	0.827	0.754
40	1.41	1.35
50	1.83	1.85
100	2.74	2.92
150	2.93	3.05
200	2.92	2.98
300	2.69	2.62
400	2.41	2.30
500	2.15	2.05
750	1.59	1.60[a]
1000	1.23	1.31

[a] Interpolation.

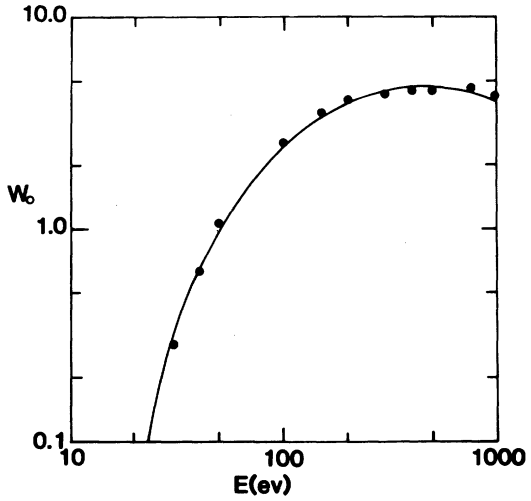

FIG. 6. Values of W_0 used in NILS fit to obtain the inelastic reaction cross section σ_R for neon. The solid line represents the analytic fit based on Eq. (25).

We have calculated the bound-state eigenenergies for the initial Ne^- system by simply using the potential model and associated parameters. These are presented in Fig. 8 along with the corresponding eigenenergies of neighboring negative ions.[40] It is pleasing to see that the bound states fall into a reasonable pattern. It is also gratifying that the $3s$ state does not bind since no negative ion of neon exists.

V. SUMMARY AND CONCLUSION

We have presented a convenient velocity- or energy-dependent analytic independent-particle model which describes the bound states of O^- and gives a reasonable account of differential elastic scattering cross section (DESC) data using a semiempirical IPM potential. We have also adjusted this model to the DESC data for the e-Ne system from 20–1000 eV. In this case we have also required cross section for the same range of energies. The model provides a good synthesis of the large body of data available for this system.

For both substances we have found it helpful to include an energy-dependent term of the form $\delta E(Y_2 - Y_1)$, which is expected to follow from any small nonlocal term in the electron-atom interaction. The value of δ which was assigned on the basis of the bound-state data in O^- came out small and positive. The value of δ assigned on the basis of the e-Ne scattering came out small and negative. This might represent nonlocal exchange effects which conceivably might reverse sign between O^-, which may be viewed as a hole in a complete shell and Ne^- which has a particle beyond a complete shell. Alternatively we might be *mocking up* polarization, relativistic, or cor-

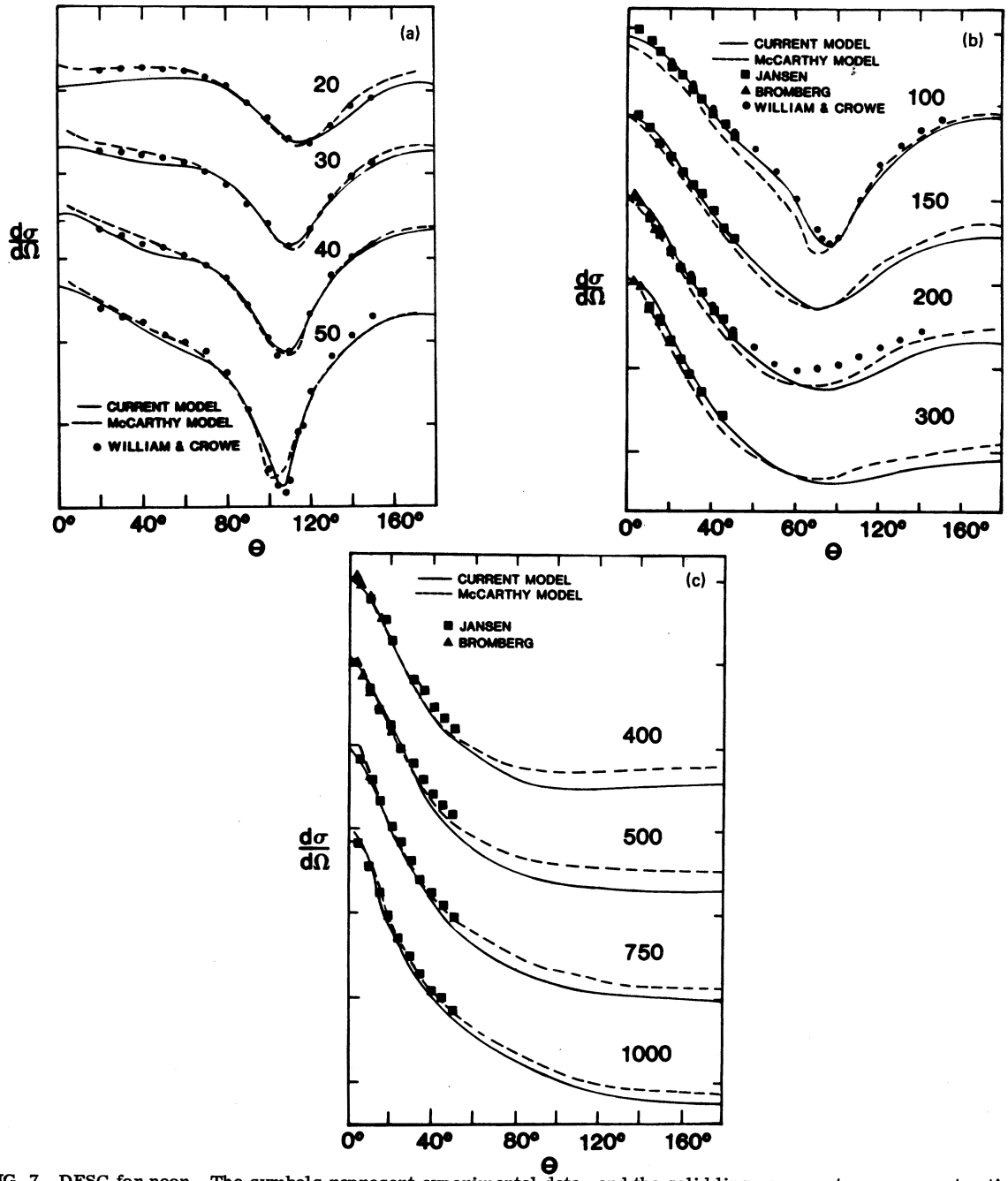

FIG. 7. DESC for neon. The symbols represent experimental data, and the solid line represents our current optical model. The dashed line represents the optical model of McCarthy *et al.*

relation effects which are not explicitly included in the static potential.

It is encumbent upon us to compare this AIPM model with the work of McCarthy *et al.*[46] which treats elastic scattering of electrons from the rare gases by a more theoretical formalism which starts with tables of Hartree-Fock wave functions. These are used to generate a screening potential which together with the potential due to the nuclear charge provides the static potential acting upon the incoming electrons. They add an equivalent energy-dependent local exchange potential based upon the approximation of Furness and McCarthy,[47] Perey and Buck,[48] and a local WKB approximation. They also include a static polarization potential based upon the work of

TABLE IV. Gross comparison of presented optical model and optical model of McCarthy *et al.* χ^2 based on limited set of angular values.

E (eV)	χ^2 values Model	McCarthy
20	0.845	0.142
30	0.537	0.207
40	0.315	0.174
50	0.709	0.336
100	1.19	1.57
150	0.308	0.150
200	0.869	0.234
300	0.934	0.225
400	0.775	0.266
500	0.591	0.599
750	0.301	0.869
1000	0.243	0.935

Temkin and Lamkin[49] and the approximate imaginary optical-model potential of Furness and McCarthy. They achieve very good descriptions of DESC for the rare gases from 20–3000 eV essentially with only the adjustment (at each energy) of the imaginary strength parameter. When a negative ion exists the present analytic velocity-dependent-potential model starts with a very good and very convenient analytic approximation to the Hartree-Fock potential, whose parameters are adjusted to accurately represent the electron affinity of the negative ion (which HF does not achieve[40]), and give reasonable inner bound-state properties. An exchange or nonlocal term is treated by the effective-mass approximation to transform it into velocity- (or momentum-dependent) terms. These are then transformed away to arrive at an effective local but energy-dependent potential. The procedure is different from the Furness-McCarthy approximation and the exchange potential approximations used by Bransden *et al.*[37] While slightly less economical in the parameters adjusted to the scattering data, this AIPM model is much simpler to apply and to communicate the essential results (e.g., the potential parameters). Indeed, the energy-dependent AIPM model is almost as simple to apply as the static GSZ potential which, as shown by Berg *et al.*[6], provides excellent fits to relative data on elastic scattering of electrons by the rare gases without any adjustment of parameters. The extra flexibility afforded by this energy-dependent model makes it possible to do better yet, and to fit the absolute data which is now available with a quality comparable to and sometimes better than that achieved by McCarthy *et al.*[46] Accordingly, we believe the present model can serve a useful role in atomic physics in facilitating the discovery of the nature of interactions by noting the values of the parameters fitted to data and observing consistent trends for varying energies and targets. The fact that we are here using Yukawa potentials which have convenient Born approximation transforms should further facilitate such studies.

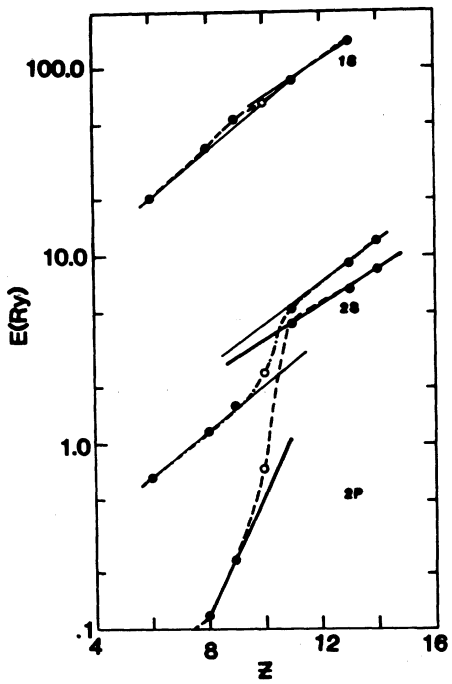

FIG. 8. Bound-state energies for 1s, 2s, and 2p states for various negative ions from Ganas *et al.* along with our estimates for virtual Ne^-.

Finally, it should be noted that the velocity-dependent methodology used here has been used rather extensively in nuclear,[25,26,48] nucleon-nucleon,[50–52] and particle physics.[53–55] In the nuclear case the use of Eqs. (9) and (10) turned out to be much better approximations than originally expected because important velocity- or momentum-dependent potentials actually arise out of the basic nucleon-nucleon interaction,[50–52] in addition to those arising from the nonlocal exchangelike terms. Accordingly, it should be expected that the further development and application of velocity-dependent potentials in the context of atomic physics, a better understood area of physics, might well prove helpful in the understanding of nuclear, nucleon-nucleon, and particle physics.

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