PHYSICAL REVIEW A

GENERAL PHYSICS

THIRD SERIES, VOLUME 24, NUMBER 6 DECEMBER 1981

Ermakov systems and quantum-mechanical superposition laws

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(Received 3 August 1981)

Ermakov systems are pairs of coupled, time-dependent, nonlinear dynamical equations possessing a joint constant of the motion called an Ermakov invariant. The invariant provides a link between the two equations and leads to a superposition law between solutions to the Ermakov pair. Extensive studies of Ermakov systems in classical mechanics have been carried out. Here we present a detailed study of Ermakov systems from a quantum point of view, and prove that the solution to the Schrödinger equation for a general Ermakov system can be reduced to the solution of a time-independent Schrödinger equation involving the Ermakov invariant. We thereby arrive at a quantum-mechanical superposition law analogous to the classical superposition law.

I. INTRODUCTION

We report on exact solutions to the Schrödinger equation for certain nonlinear, time-dependent oscillators. These systems, called Ermakov systems, have been studied extensively in classical mechanics,[1–17] since their introduction in Ref. 1. The Ermakov systems we study in this paper are described by the Hamiltonian[14]

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega^2(t)q^2+\frac{1}{x^2}f(q/x)\ , \qquad (1)$$

where q,p are the canonical coordinates, $\omega^2(t)$ is an arbitrary function of time, f is an arbitrary function of its argument, and x is an auxiliary function satisfying the equation

$$\ddot{x}+\omega^2(t)x=k/x^3\ , \qquad (2)$$

where k is an arbitrary constant which could be zero. The equation of motion for q follows from H and is ($p=\dot{q}$)

$$\ddot{q}+\omega^2(t)q=\frac{1}{x^3}f'(q/x)\ , \qquad (3)$$

where $f'=df/d(q/x)$. The form of (2) and (3) is easily recognized as an Ermakov pair with the Ermakov invariant[1]

$$I=\tfrac{1}{2}(xp-q\dot{x})^2+\frac{k}{2}(q/x)^2+f(q/x)\ . \qquad (4)$$

I is constant if x is any solution to (2) and q is any solution to (3). The system (2), (3), and (4) is an Ermakov system.

In the quantum theory of (1), q and p becomes quantum-mechanical operators $p=(\hbar/i)(\partial/\partial q)$; the auxiliary function x remains a c number. The invariant I, (4), is a constant Hermitian operator

$$\frac{dI}{dt}=\frac{1}{i\hbar}[I,H]+\frac{\partial I}{\partial t}=0\ . \qquad (5)$$

Apparently, Lewis[18] and Lewis and Riesenfeld[19] first used constant operators I to solve time-dependent quantum-mechanical problems. In particular, in Ref. 19 they solved the problem for the time-dependent harmonic oscillator $f=0$ and presented the general outline of how to utilize the constant operator I to solve other quantum problems. The important point to note here is that H varies in time $H=H(t)$ in a way that depends upon the arbitrary frequency $\omega(t)$. Thus, the eigenvalues of H will change in time and cannot be determined until $\omega(t)$ is specified. On the other hand, I satisfying (5) has constant eigenvalues which we write as λ_n

$$I\psi_n(q,t)=\lambda_n\psi_n(q,t),\quad \lambda_n=\text{const}\ . \qquad (6)$$

Here, $\psi_n(q,t)$ denotes an eigenfunctions of I which will, in general, be time-dependent. Lewis and

Riesenfeld[19] showed that the general solution to the Schrödinger equation for (1)

$$H\psi=\left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2}+\tfrac{1}{2}\omega^2(t)q^2+\frac{1}{x^2}f(q/x)\right]\psi(q,t)$$
$$=i\hbar\frac{\partial\psi}{\partial t}\ , \qquad (7)$$

can be written in the form

$$\psi(q,t)=\sum c_n e^{i\alpha_n(t)}\psi_n(q,t)\ , \qquad (8)$$

where c_n are constants, $\psi_n(q,t)$ are the eigenfunctions of I defined by (6) and the phase functions $\alpha_n(t)$ are found from the equation

$$\hbar\frac{d\alpha_n(t)}{dt}=\left\langle\psi_n\left|i\hbar\frac{\partial}{\partial t}-H\right|\psi_n\right\rangle\ , \qquad (9)$$

where we have arranged the states $\psi_n(q,t)$ to be orthonormal

$$\langle\psi_n\mid\psi_{n'}\rangle=\delta_{nn'}\ . \qquad (10)$$

For simplicity we assume I to have a discrete spectrum. The time-dependent phases $\alpha_n(t)$ are chosen to satisfy (9) so that $e^{i\alpha_n(t)}\psi_n(q,t)$ satisfies the Schrödinger equation (7) for every n. The general solution to the Schrödinger equation is then the linear combination of these elementary solutions (8). Lewis and Riesenfeld[19] solved explicitly for λ_n, $\alpha_n(t)$ for the time-dependent harmonic oscillator.

More recently Khandekar and Lawande[20,21] solved the quantum problem using the Lewis-Riesenfeld theory for the case when $f=cx^2/q^2$ which yields the Hamiltonian

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega^2(t)q^2+\frac{c}{q^2}\ . \qquad (11)$$

In this paper we reduce the solution of the Schrödinger equation (7) to the solution of a time-independent one-dimensional Schrödinger equation incorporating the results for $f=0$, $f=cx^2/q^2$ just mentioned into a framework valid for *arbitrary* f.

II. SOLUTION TO THE SCHRÖDINGER EQUATION

The central equation of Lewis-Riesenfeld technique is the eigenvalue equation for the invariant

$$I\psi_n(q,t)=\lambda_n\psi_n(q,t)\ , \qquad (12)$$

where

$$I=\tfrac{1}{2}(xp-\dot{x}q)^2+\frac{k}{2}(q/x)^2+f(q/x)\ . \qquad (13)$$

The key point of our analysis is to perform the unitary transformation

$$\psi_n'(q,t)=e^{-i\dot{x}q^2/(2\hbar x)}\psi_n(q,t)$$
$$=U\psi_n\ . \qquad (14)$$

The operator I changes into I':

$$I'=UIU^\dagger\ . \qquad (15)$$

The eigenvalue equation (12) is mapped into

$$I'\psi_n'(q,t)=\lambda_n\psi_n'(q,t)\ , \qquad (16)$$

where we find by straightforward calculation that

$$I'=\frac{\hbar^2}{2}x^2\frac{\partial^2}{\partial q^2}+\frac{k}{2}(q/x)^2+f(q/x)\ . \qquad (17)$$

If we now define a new independent variable $\sigma=q/x$ we can write the eigenvalue equation in the form

$$\left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\sigma^2}+\frac{k}{2}\sigma^2+f(\sigma)\right]\phi_n(\sigma)=\lambda_n\phi_n(\sigma)\ ,$$

or

$$I'\phi_n(\sigma)=\lambda_n\phi_n(\sigma)\ , \qquad (18)$$

where

$$\psi_n'(q,t)=\frac{1}{x^{1/2}}\phi_n(\sigma)=\frac{1}{x^{1/2}}\phi_n(q/x)\ . \qquad (19)$$

The factor $1/x^{1/2}$ is introduced into (19) so that the normalization conditions

$$\int\psi_n'^*(q,t)\psi_n'(q,t)dq=\int\phi_n^*(\sigma)\phi_n(\sigma)d\sigma=1 \qquad (20)$$

hold. The important point is that the transformed eigenvalue problem (18) is an ordinary one-dimensional time-independent Schrödinger equation with potential

$$V(\sigma)=\frac{k}{2}\sigma^2+f(\sigma)\ .$$

We can make use of the extensive knowledge about solutions to this equation to find the eigenvalues λ_n and eigenfunctions $\phi_n(\sigma)$. Once we have the orthonormal eigenfunctions $\phi_n(\sigma)$ we can construct the Lewis-Riesenfeld orthonormal states via

$$\psi_n(q,t)=\frac{1}{x^{1/2}}e^{i\dot{x}q^2/(2\hbar x)}\phi_n(q/x)\ . \qquad (21)$$

There remain the problem of finding the phases $\alpha_n(t)$ which satisfy

$$\hbar\frac{d\alpha_n(t)}{dt}=\left\langle \psi_n \left| i\hbar\frac{\partial}{\partial t}-H \right| \psi_n \right\rangle . \tag{22}$$

Carrying out the unitary transformation U the right-hand side of Eq. (22) becomes

$$\hbar\frac{d\alpha_n}{dt}=\left\langle \psi_n' \left| i\hbar\frac{\partial}{\partial t}-\frac{\dot{x}}{x}\frac{\hbar}{i}q\frac{\partial}{\partial q}+\frac{\dot{x}i\hbar}{2x}-\frac{1}{x^2}I' \right| \psi_n' \right\rangle , \tag{23}$$

where we have used the auxiliary equation (2) to eliminate $\omega^2(t)$ from H. Next substituting

$$\psi_n'(q,t)=\frac{1}{x^{1/2}}\phi_n(q/x) , \tag{24}$$

into (23) we find

$$\hbar\frac{d\alpha_n}{dt}=\left\langle \phi_n \left| -\frac{1}{x^2}I' \right| \phi_n \right\rangle . \tag{25}$$

Finally using (18) and the normalization of ϕ_n we have

$$\hbar\frac{d\alpha_n}{dt}=-\frac{\lambda_n}{x^2} , \tag{26}$$

with solution

$$\alpha_n(t)=-\frac{\lambda_n}{\hbar}\int^{(t)}\frac{dt}{x^2} . \tag{27}$$

Thus, the phases $\alpha_n(t)$ are determined in terms of the eigenvalues λ_n and the integral of the auxiliary function $\int dt/x^2$.

To summarize the results of this section the exact solution to the Ermakov Schrödinger equation (7) is

$$\psi(q,t)=\sum_n c_n e^{i\alpha_n(t)}\psi_n(q,t) , \tag{28}$$

where,

$$\psi_n(q,t)=\frac{1}{x^{1/2}}e^{i\dot{x}q^2/(2\hbar x)}\phi_n(q/x) , \tag{29}$$

$$\alpha_n=-\frac{\lambda_n}{\hbar}\int\frac{dt}{x^2} , \tag{30}$$

$$c_n=\langle \psi_n(q,0),\psi(q,0)\rangle e^{-i\alpha_n(0)} . \tag{31}$$

λ_n and $\phi_n(\sigma)$ are determined from the one-dimensional Schrödinger equation (18) and x is any solution to the auxiliary equation (2).

III. DISCUSSION

There are two Ermakov systems, of those under consideration in this paper, for which the Schrödinger equation (7) does not depend on the auxiliary variable x. These are (1) $f=0$ the Lewis-Riesenfeld[19] harmonic oscillator problem and (2) $f=cx^2/q^2$, the problem treated by Khandekar and Lawande.[20,21] In all other cases the auxiliary function x appears in the potential energy of the Schrödinger equation (7). It is then to be interpreted as an external field whose time dependence is to be determined from the auxiliary equation (2). In the uncoupled Ermakov systems $f=0$, $f=cx^2/q^2$, x is just an auxiliary variable whose particular form drops out of any calculation of transition matrix elements.[19] However, in the coupled case x is a physical field whose form determines the interaction of the system with the field through the interaction potential $(1/x^2)f(q/x)$. Our solutions, of course, hold for both the coupled and uncoupled Ermakov systems. The division into coupled and uncoupled cases also occurs in the classical case.[1-17]

We have proven in this paper that the solution to the Ermakov Schrödinger equation (7) reduces to solving the one-dimensional Schrödinger equation (18). This has an interesting analog in the classical theory. The classical result is that the general solution to the equation for q, (3), can be written

$$q=xr , \tag{31}$$

where r is the general solution to the autonomous equation

$$\frac{d^2r}{d\tau^2}+kr+\frac{df(r)}{dr}=0 , \tag{32}$$

where the independent variable τ is defined by

$$d\tau=dt/x^2 . \tag{33}$$

In (31) x is any particular solution to (2) in the uncoupled case or the external field solution to (2) in the coupled case. Equation (32) has the energylike first integral

$$I=\tfrac{1}{2}\left[\frac{dr}{d\tau}\right]^2+\frac{k}{2}r^2+f(r) , \tag{34}$$

which can be proven to be equal to the Ermakov invariant I, Eq. (4).[1-17] The law (31) for obtaining the solution to q in terms of the solution to x and r

is called a superposition law.[6,7,11,16] Thus, we see a close connection between superposition in the classical case and the reduction to the one-dimensional Schrödinger equation, (18), in the quantum case. In fact Eq. (28), together with (29) and (30), is a *quantum-mechanical superposition law.* It is interesting that the same variables $\int (dt/x^2)$, q/x appear in both the classical and quantum superposition laws. This correspondence between classical and quantum superposition rules for Ermakov systems could imply a similar relationship for other types of superposition laws, e.g., Bäcklund transformations, etc.

Since the work of Lewis and Riesenfeld there have been other studies of the solution of the time-dependent Schrödinger equation. Extensive work has been done at the Lebedev Institute of Physics in Moscow. Some of this work along with references to other work by this group may be found in Ref. 22. Also work by Burgan[23] is along the same lines.

Part of this work was performed while J. R. Ray was a participant in the NASA-ASEE Summer Faculty Fellowship Program at the Marshall Space Flight Center.

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