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Hydrodynamics of biaxial discotics

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The nonlinear hydrodynamic equations for biaxial discotics with two broken translational symmetries (which have been identified experimentally very recently) are presented and the structure of the hydrodynamic excitations is considered. For biaxial discotics with broken rotational symmetries, owing to the existence of two preferred axes, the gradient free energy, representing a generalization of the Frank free energy for nematic liquid crystals, is given and the nonlinear hydrodynamic equations are derived. As a striking result a set of three commutator-type relations reflecting the anholomomity of the hydrodynamic variables characterizing the broken symmetries is found for the first time in the physics of liquid crystals. The influence of a static external magnetic field on both types of biaxial discotics is discussed.

I. INTRODUCTION

Since the discovery of discotic liquid crystals in 1977[1] the chemistry and physics of this first man-made type of liquid crystals proved to be a rapidly growing, fascinating field. Most of the papers published so far deal with uniaxial, hexagonal discotic liquid crystals[1-16] but more recently uniaxial discotic liquid crystals with broken rotational symmetry (thus being quite analogous to nematic liquid crystals) have been identified as well.[17] Very recently experimental evidence for the occurence of biaxial discotic liquid crystals has also been presented[13,18,19] and these biaxial discotic liquid crystals seem to possess two broken translational symmetries (i.e., in two directions the centers of mass of the molecules are arranged regularly). However, due to the disklike shape of the molecules constituting the discotic liquid crystals, one can suppose that biaxial discotic liquid crystals with broken rotational symmetries (the orientations of the molecules are arranged regularly in two directions) will also be produced soon.

In the present paper we study the hydrodynamic equations for both types of biaxial discotics, those with broken translational symmetries as well as those with broken rotational symmetries.

For uniaxial discotics linearized hydrodynamic equations have been studied by Prost and Clark[11,12] and, as it will become obvious in the following, their equations emerge as a special case of our nonlinear hydrodynamics for biaxial discotics with two broken translational symmetries.

The hydrodynamic method used in the present paper has been applied to many systems in condensed-matter physics in the linear as well as in the nonlinear regime. Among those systems are, e.g., simple liquids and paramagnets[20,21] superfluid ^{4}He (Refs. 22 and 23), antiferromagnets and ferromagnets,[24] nematics,[25-32] cholesterics,[25,31-34] crystals,[31,32,35] smectics[31,32] spin glasses,[36] and the various phases of superfluid ^{3}He.[37-52]

The present paper is organized as follows. In Sec. II we discuss biaxial discotics with broken translational symmetries, their nonlinear hydrodynamic equations, the hydrodynamic excitations, and the influence of a static external magnetic field, and in Sec. III we investigate biaxial discotics with broken rotational symmetries, their gradient free energy, and the nonlinear hydrodynamic equations.

II. BIAXIAL DISCOTICS WITH BROKEN TRANSLATIONAL SYMMETRIES

A. Hydrodynamic equations

The type of biaxial discotics we wish to discuss here is that which has been observed in the experiments by Sigaud, Achard, Destrade, and Tinh[18] and by Fugnitto, Strzelecka, Zann, Dubois, and Billard,[19] i.e., that having two broken translational symmetries which are orthogonal to each other. Thus macroscopically this type of liquid crystal behaves like a crystal in two dimensions (the centers of mass of the molecules are long-range ordered) whereas it behaves like a fluid in the third direction.

We denote the two preferred directions characterizing the discotic liquid crystals under investigation by $\vec{n}$ and $\vec{m}$. As in nematics and smectics we assume that $\vec{n}$ and $-\vec{n}$ are indistinguishable, thus the hydrodynamic equations which will be derived in the following have to be invariant under the transformation $\vec{n} \to -\vec{n}$. The same is true for $\vec{m}$ and, therefore, the hydrodynamic equations have to reflect an $\vec{n} \to -\vec{n}$ and a $\vec{m} \to -\vec{m}$ symmetry separately. This has to be contrasted with the case of smectic-C liquid crystals: In this case the hydrodynamic equations are invariant when the replacement $\vec{n} \to -\vec{n}$, $\vec{m} \to -\vec{m}$ is carried out simultaneously. As it will become obvious in the following, these transformation properties allow us to reduce drastically the number of involved phenomenological parameters which will nevertheless remain rather large.

The broken translational symmetries are characterized by the displacement vector $\vec{R}$. However, the translational symmetry is broken only in two orthogonal directions and the third component of $\vec{R}$ is not a hydrodynamic variable. Therefore we must impose the constraint

$$\vec{R}\cdot(\vec{n}\times\vec{m})=0\,. \tag{2.1}$$

To set up the hydrodynamics of the present system we must list the hydrodynamic variables first. The conserved quantities are density ρ, energy density ϵ or entropy density σ, and the density of linear momentum $\vec{g}$. The hydrodynamic variables characterizing the broken symmetries are the components of the displacement vector which satisfy the constraint (2.1) and we arrive at a total of seven hydrodynamic variables which will give rise to the same number of modes if the hydrodynamic equations are discussed in the linearized regime. For the Gibbs relation we have (there is no term proportional to dR_i in the Gibbs relation because the gradient energy does not depend on R_i)

$$d\epsilon=T\,d\sigma+\mu\,d\rho+\vec{v}\cdot d\vec{g}+\phi_{ij}\,d\nabla_j R_i+\psi_{ijk}\,d\nabla_k\nabla_j R_i \tag{2.2}$$

which is the straightforward generalization of Eq. (3.19) of Ref. 32 and the equations of motion read

$$\frac{\partial}{\partial t}\rho+\nabla_i g_i=0\,, \tag{2.3}$$

$$\frac{\partial}{\partial t}g_i+\nabla_j\sigma_{ij}=0\,, \tag{2.4}$$

$$\frac{\partial}{\partial t}\sigma+\nabla_i j_i^{\sigma}=0\,, \tag{2.5}$$

$$\frac{d}{dt}R_i+X_i=0\,, \tag{2.6}$$

where

$$\frac{d}{dt}=\frac{\partial}{\partial t}+v_i\nabla_i\ .$$

For the reversible currents we find, by using general symmetry arguments and the requirement of vanishing entropy production,

$$\sigma_{ij}^{R}=p\delta_{ij}-(n_in_k+m_im_k)\phi_{kj}+\alpha_{kmjl}R_m\nabla_l\phi_{kl}+\phi_{kj}\nabla_iR_k\,, \tag{2.7}$$

$$j_i^{\sigma R}=\sigma v_i\,, \tag{2.8}$$

$$X_i^R=-(n_in_j+m_im_j)v_j+\alpha_{ijkl}R_j\nabla_k v_l\,, \tag{2.9}$$

where the contribution $\sim v_j$ in (2.9) subtracts rigid translations from the dynamic equations of the variables characterizing the broken symmetries, as it should be, because homogeneous translations are not involved in hydrodynamics. For α_{ijkl} we find for a discotic liquid crystal with a fourfold symmetry (tetragonal system)[18,19]

$$\begin{aligned}\alpha_{ijkl}=&\alpha_1(n_in_jm_km_l+m_im_jn_kn_l)\\&+\alpha_2(n_in_jn_kn_l+m_im_jm_km_l)\\&+\alpha_3\delta_{kl}^3(m_im_j+n_in_j)\\&+\alpha_4(m_in_j+n_im_j)(m_kn_l+n_km_l)\,,\end{aligned} \tag{2.10}$$

where

$$\delta_{kl}^3=\delta_{kl}-m_km_l-n_kn_l\,.$$

For a discotic liquid crystal with a twofold symmetry which has also been proposed to explain the experiments[19] we have

$$\begin{aligned}\alpha_{ijkl}=&\alpha_1n_in_jm_km_l+\alpha_2n_in_jn_kn_l+\alpha_3n_in_j\delta_{kl}^3\\&+\alpha_4m_im_jm_km_l+\alpha_5m_im_jn_kn_l\\&+\alpha_6m_im_j\delta_{kl}^3+\alpha_7(m_in_j)(m_kn_l+n_km_l)\\&+\alpha_8(m_jn_i)(m_kn_l+m_ln_k)\,.\end{aligned} \tag{2.11}$$

The terms involving α_7 and α_8 show that α_{ijkl} must be symmetric in the last two indices but not in the first pair, a fact that has already been mentioned in Ref. 32.

For a uniaxial discotic liquid crystal with hexagonal symmetry, we find for α_{ijkl} (as has already been mentioned above, the complete *linearized* hydrodynamic equations for this type of discotics have been given by Prost and Clark,[11,12])

$$\alpha_{ijkl}=n_in_j(\alpha_1n_kn_l+\alpha_2\delta_{kl})\,. \tag{2.12}$$

As has to be expected from symmetry considerations α_{ijkl} contains fewer phenomenological parameters for uniaxial discotics than for the various types of biaxial ones.

The dissipative currents which have to satisfy only the constraint of positivity of the entropy production take the form

$$\begin{aligned}X_i^D=&(\zeta_1n_in_j+\zeta_2m_im_j)\nabla_k\phi_{jk}\\&+(\xi_1n_in_j+\xi_2m_im_j)\nabla_jT\,,\end{aligned} \tag{2.13}$$

$$\begin{aligned}j_i^{\sigma D}=&(\kappa_1n_in_j+\kappa_2m_im_j+\kappa_3\delta_{ij}^3)\nabla_jT\\&+(\xi_1n_in_j+\xi_2m_jm_i)\nabla_k\phi_{jk}\,,\end{aligned} \tag{2.14}$$

$$\sigma_{ij}^D=\nu_{ijkl}A_{kl}\,, \tag{2.15}$$

where

$$A_{kl}=\tfrac{1}{2}(\nabla_kv_l+\nabla_lv_k)\,.$$

For biaxial discotics with fourfold symmetry

$$\zeta_1=\zeta_2\,, \tag{2.16}$$

$$\xi_1 = \xi_2 \,, \tag{2.17}$$

and

$$\nu_{ijkl} = [\nu_1(m_i m_j m_k m_l + n_i n_j n_k n_l) + \nu_2 \delta^3_{ij}\delta^3_{kl} + \nu_3 \delta^3_{kl}(m_i m_j + n_i n_j) + \nu_4 m_i m_j n_k n_l + \nu_5 m_i n_j m_k n_l + \nu_6(m_i m_k \delta^3_{jl} + n_i n_k \delta^3_{jl})] \,. \tag{2.18}$$

Equations (2.16) and (2.17) are also valid in uniaxial discotics.[11,12] In the case of biaxial discotics with a twofold symmetry $\xi_1 \neq \xi_2$ and $\xi_1 \neq \xi_2$, in general, and the viscosity tensor reads

$$\nu_{ijkl} = \nu_1 m_i m_j m_k m_l + \nu_2 n_i n_j n_k n_l + \nu_3 \delta^3_{ij}\delta^3_{kl} + \nu_4 m_i m_j n_k n_l + \nu_5 m_i m_j \delta^3_{kl} + \nu_6 n_i n_j \delta^3_{kl} + \nu_7 m_i n_j n_l m_k + \nu_8 m_i m_k \delta^3_{jl} + \nu_9 n_i n_k \delta^3_{jl} \tag{2.19}$$

and thus contains nine independent viscosities, a fact which complicates the discussion of the normal modes considerably. To close the system of hydrodynamic equations we need the equations of state linking intensive and extensive quantities. We begin with the discussion of the gradient free energy and obtain for the part quadratic in the variables (i.e., entering the linear hydrodynamic equations) in lowest possible order in the gradients

$$F^g = \tfrac{1}{2}\chi_{ijkl}(\nabla_j R_i)(\nabla_l R_k) + \rho\gamma^\rho_{ij}\nabla_j R_i + \sigma\gamma^\sigma_{ij}\nabla_j R_i + \overline{K}_1(\delta^3_{ij} n_k \nabla_i \nabla_j R_k)^2 + \overline{K}_2(\delta^3_{ij} m_k \nabla_i \nabla_j R_k)^2 \,, \tag{2.20}$$

where

$$\chi_{ijkl} = \chi_1 n_i n_j n_k n_l + \chi_2 m_i m_j m_k m_l + \chi_3(n_i n_k m_l m_j + m_i m_k n_l n_j + n_j n_k m_l m_i + m_j m_k n_l n_i) + \chi_4(n_j m_l n_i m_k + n_l m_j n_k m_i) \tag{2.21}$$

and

$$\gamma^{(\rho,\sigma)}_{ij} = \gamma^{(\rho,\sigma)}_1 n_i n_j + \gamma^{(\rho,\sigma)}_2 m_i m_j$$

or more explicitly (we choose $\hat{n}^0 \| \hat{e}_z$ and $\hat{m}^0 \| \hat{e}_y$)

$$F^g = \tfrac{1}{2}\chi_1(\nabla_z R_z)^2 + \tfrac{1}{2}\chi_2(\nabla_y R_y)^2 + \tfrac{1}{2}\chi_3(\nabla_y R_z + \nabla_z R_y)^2 + \chi_4(\nabla_y R_y)(\nabla_z R_z) + \overline{K}_1(\nabla^2_x R_z)^2 + \overline{K}_2(\nabla^2_x R_y)^2 + \gamma^{1\rho}\rho(\nabla_y R_y) + \gamma^{2\rho}\rho(\nabla_z R_z) + \gamma^{1\sigma}\sigma(\nabla_y R_y) + \gamma^{2\sigma}\sigma(\nabla_z R_z) \,. \tag{2.22}$$

A corresponding expression for smectic A taking into account higher-order gradient terms due to rotations has been discussed extensively by the present authors in Ref. 32. Equations (2.20)–(2.22) hold for biaxial discotics with a twofold symmetry. For a fourfold symmetry

$$\chi_1 = \chi_2, \quad \gamma^\rho_1 = \gamma^\rho_2, \quad \gamma^\sigma_1 = \gamma^\sigma_2, \quad \overline{K}_1 = \overline{K}_2 \,. \tag{2.23}$$

Nonlinearities will be considered only up to cubic terms in $\nabla_j R_i$; thereby we disregard cubic terms of those $\nabla_j R_i$, which are already present quadratically, i.e., $\nabla_z R_z$, $\nabla_y R_y$ and $\nabla_y R_z + \nabla_z R_y$, and obtain

$$\begin{aligned} F^g = {} & \overline{K}_1 (n_k \delta^3_{ij}\nabla_i\nabla_j R_k)^2 + \overline{K}_2(\delta^3_{ij} m_k \nabla_i \nabla_j R_k)^2 \\ & + (\rho\gamma^{1\rho} + \sigma\gamma^{1\sigma})\{m_i m_j \nabla_i R_j - \delta^3_{ij} m_k m_l (\nabla_i R_k)(\nabla_j R_l) - \tfrac{1}{4}[m_k n_l(\nabla_k R_l - \nabla_l R_k)]^2\} \\ & + (\rho\gamma^{2\rho} + \sigma\gamma^{2\sigma})\{n_i n_j \nabla_i R_j - \delta^3_{ij} n_k n_l (\nabla_i R_k)(\nabla_j R_l) - \tfrac{1}{4}[m_k n_l(\nabla_k R_l - \nabla_l R_k)]^2\} \\ & + \tfrac{1}{2}\chi_1 n_p n_g \nabla_p R_g \{n_i n_j \nabla_i R_j - \delta^3_{kl} n_i n_j (\nabla_k R_i)(\nabla_l R_j) - \tfrac{1}{4}[m_k n_l(\nabla_k R_l - \nabla_l R_k)]^2\} \\ & + \tfrac{1}{2}\chi_2 m_p m_g \nabla_p R_g \{m_i m_j \nabla_i R_j - \delta^3_{kl} m_i m_j (\nabla_k R_i)(\nabla_l R_j) - \tfrac{1}{4}[m_k n_l(\nabla_k R_l - \nabla_l R_k)]^2\} \\ & + \tfrac{1}{2}\chi_3 m_p n_g (\nabla_p R_g + \nabla_g R_p)[m_i n_j(\nabla_i R_j + \nabla_j R_i) - m_i n_j \delta^3_{kl}(\nabla_k R_i)(\nabla_l R_j) \\ & \qquad - \tfrac{1}{2}(n_i m_j)(\nabla_i R_j - \nabla_j R_i)(n_k n_l \nabla_k R_l - m_k m_l \nabla_k R_l)] \\ & + \tfrac{1}{2}\chi_4\{2 m_i m_j n_k n_l (\nabla_i R_j)(\nabla_k R_l) - m_i m_j n_k n_l \delta^3_{gp}[(\nabla_k R_l)(\nabla_g R_i)(\nabla_p R_j) + (\nabla_i R_j)(\nabla_g R_k)(\nabla_p R_l)] \\ & \quad - \tfrac{1}{4}(m_i m_j + n_i n_j)(\nabla_i R_j)(m_k n_l[\nabla_k R_l - \nabla_l R_k])^2\} \,. \end{aligned} \tag{2.24}$$

Now we are prepared to write down the equations of state

$$v_i = \frac{1}{\rho} g_i \,, \tag{2.25}$$

$$\delta T = T C_v^{-1}\delta\sigma + \gamma\delta\rho + \gamma^\sigma_{ij}\frac{\partial F^g}{\partial \nabla_i R_j} \,, \tag{2.26}$$

$$\delta\mu = \lambda\delta\rho + \gamma\delta\sigma + \gamma^\rho_{ij}\frac{\partial F^g}{\partial \nabla_i R_j} \,, \tag{2.27}$$

$$\phi_{ij} = \frac{\partial F^g}{\partial \nabla_i R_j} \,, \tag{2.28}$$

$$\psi_{ijk} = \frac{\partial F^g}{\partial \nabla_i \nabla_j R_k} \,. \tag{2.29}$$

B. Influence of an external magnetic field

Let us briefly discuss the orientation of the biaxial discotic liquid crystal in an external magnetic field. For the magnetization we have $(\hat{n}\|\hat{e}_z, \hat{m}\|\hat{e}_y)$

$$M_i = \chi_{ij}H_j = \chi_x\delta^3_{ij}H_j + \chi_y m_i m_j H_j + \chi_z n_i n_j H_j, \tag{2.30}$$

$$M_i = (\chi_z - \chi_x)n_i n_j H_j + (\chi_y - \chi_x)m_i m_j H_j + \chi_x H_i. \tag{2.31}$$

The orientating contributions to the free energy due to the external magnetic fields are

$$F^H = -\tfrac{1}{2}\chi_a^{(1)}(\vec{n}\cdot\vec{H})^2 - \tfrac{1}{2}\chi_a^{(2)}(\vec{m}\cdot\vec{H})^2, \tag{2.32}$$

where

$$\chi_a^{(1)} = \chi_z - \chi_x \text{ and } \chi_a^{(2)} = \chi_y - \chi_x.$$

Minimizing F^H with the respect to the possible orientations of $\vec{n}$ and $\vec{m}$ leads to three cases,

$$\begin{aligned}
&(1)\quad \chi_a^{(2)}<0,\ \chi_a^{(1)}<0\quad \vec{H}\perp\vec{n},\ \vec{H}\perp\vec{m}\\
&(2)\quad \chi_a^{(2)}>0,\ \chi_a^{(1)}<\chi_a^{(2)}\quad \vec{H}\,\|\,\vec{m}\\
&(3a)\quad \chi_a^{(2)}>0,\ \chi_a^{(1)}>\chi_a^{(2)}\quad \vec{H}\,\|\,\vec{n}\\
&(3b)\quad \chi_a^{(2)}<0,\ \chi_a^{(1)}>0\quad \vec{H}\,\|\,\vec{n},
\end{aligned} \tag{2.33}$$

i.e., the molecules always orient parallel to one of their principle axes. The results of Eqs. (2.33) are summarized in Fig. 1.

By the connection for linear changes of $\vec{n}$ and $\vec{m}$ with $\nabla_i R_j$ $(\delta n_i = -n_k\nabla_i R_k,\ \delta m_i = -m_k\nabla_i R_k)$ the orientational energy (2.32) leads to terms quadratic in $(\nabla_x R_z)$, $(\nabla_x R_y)$, and $(\nabla_y R_z - \nabla_z R_y)$ not present in (2.22). These terms, however, will vanish for $H\to 0$ like H^2.

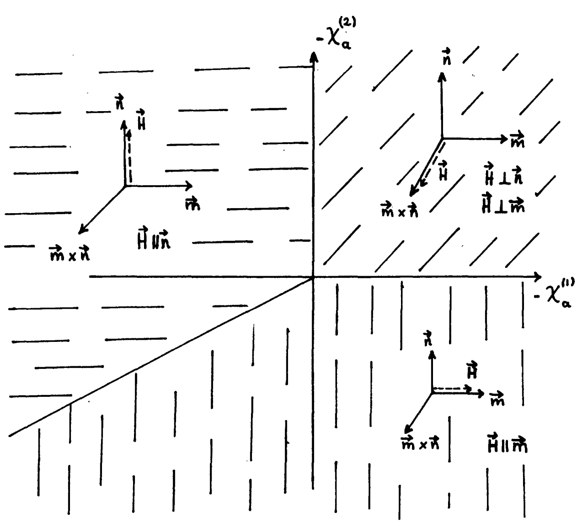

FIG. 1. Orientation of a biaxial discotic liquid crystal in a static external magnetic field: $-\chi_a^{(2)} = \chi_x - \chi_y$, $-\chi_a^{(1)} = \chi_x - \chi_z$.

C. Normal-mode structure

Finally we discuss the normal modes of biaxial discotics with broken translational symmetries. Because we have seven hydrodynamic variables we have to expect seven modes, propagating or diffusive. In the most general case (a wave vector with all three components different from zero) we find a bicubic equation for the reversible motion and one mode which is purely diffusive (it is the mode which is mainly due to heat conduction). The number of propagating modes in the most general case depends on the invariants of Cardani's equation. We find that in any case there is at least one pair of propagating modes corresponding to first sound in an ordinary liquid. For a very special set of k values we find two pairs of propagating modes and for most k values there exist three pairs of propagating modes. The spectrum can be discussed explicitly for $k_y=0$ or $k_z=0$. In these cases one has, as in smectic A, a biquadratic equation and in addition one equation for a third sound. In Figs. 2–4 we have plotted the spectrum qualitatively for

(Fig. 2) $k_y=0$, k_x, k_z arbitrary,

(Fig. 3) $k_z=0$, k_x, k_y arbitrary,

(Fig. 4) $k_x=0$, k_y, k_z arbitrary.

It seems worthwhile mentioning that our cases $k_y=0$ and $k_z=0$ are in close correspondence with the results of J. Prost and N. A. Clark for the case of uniaxial discotics. This has to be expected because in all three cases ($k_y=0$, $k_z=0$, hexagon-

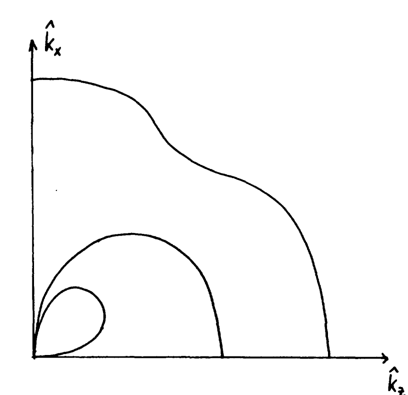

FIG. 2. Speed of propagating modes in a biaxial discotic liquid crystal as a function of direction of propagation for $k_y=0$ (schematic plot).

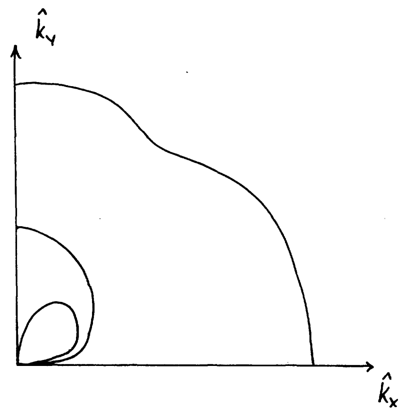

FIG. 3. Speed of propagating modes in a biaxial discotic liquid crystal as a function of direction of propagation for $k_z = 0$ (schematic plot).

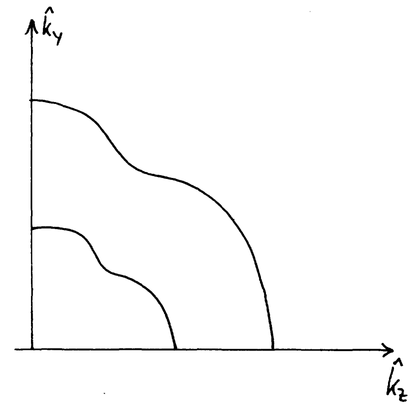

FIG. 4. Speed of propagating modes in a biaxial discotic liquid crystal as a function of direction of propagation for $k_x = 0$ (schematic plot).

al discotic) the mode spectrum clearly separates into two subspaces.

For $k_y = 0$ we obtain

$$\omega^2 = \chi_3 k_z^2 , \qquad (2.34)$$

$$\omega^4 - \omega^2\left(\frac{\partial p}{\partial \rho} k^2 + (\chi_1 + \gamma^{2\rho})k_z^2\right) + k_x^2 k_z^2 \frac{\partial p}{\partial \rho}\chi_1 = 0 , \qquad (2.35)$$

i.e., we get three propagating modes if k_z and k_x are different from zero. For $k_y = k_z = 0$ we have only one pair of propagating modes corresponding to first sound $[\omega^2 = (\partial p/\partial \rho)k^2]$ and for $k_y = k_x = 0$ we get two pairs of propagating modes $[\omega^2 = \chi_3 k_z^2$ and $\omega^2 = (\partial p/\partial \rho + \chi_1 + \gamma^{2\rho})k_z^2]$. All these results are summarized schematically in Fig. 2.

Setting $k_z = 0$ we arrive at

$$\omega^4 - \omega^2\left(\frac{\partial p}{\partial \rho} k^2 + (\chi_2 + \gamma^{1\rho})k_y^2\right) + k_y^2 k_x^2 \frac{\partial p}{\partial \rho}\chi_2 = 0 \qquad (2.36)$$

and

$$\omega^2 = \chi_3 k_y^2 .$$

The discussion for the case $k_y = 0$ can be carried over to the quite analogous case $k_z = 0$ and after similar considerations we obtain for $k_z = 0$ the schematic plot shown in Fig. 3.

The third special case which can easily be discussed is $k_x = 0$ and the corresponding results read

$$\begin{aligned}\omega^4 - \omega^2\left(k^2\frac{\partial p}{\partial \rho} + \gamma^{1\rho}k_y^2 + \gamma^{2\rho}k_z^2 + \hat{\chi}_1^2 + \hat{\chi}_2^2\right) &+ \Big[\hat{\chi}_1^2\hat{\chi}_2^2 - (\chi_3+\chi_4)k_y^2k_z^2\left(2\frac{\partial p}{\partial \rho} + \gamma^{1\rho} + \gamma^{2\rho}\right) \\ &- (\chi_3+\chi_4)k_y^2k_z^2 \\ &+ k_z^2\hat{\chi}_2^2\left(\frac{\partial p}{\partial \rho} + \gamma^{2\rho}\right) + k_y^2\hat{\chi}_1^2\left(\frac{\partial p}{\partial \rho} + \gamma^{1\rho}\right)\Big] = 0 \end{aligned} \qquad (2.37)$$

and

$$\omega^2 = 0 , \qquad (2.38)$$

where

$$\begin{aligned}\hat{\chi}_2^2 &= \chi_2 k_y^2 + \chi_3 k_z^2 , \\ \hat{\chi}_1^2 &= \chi_1 k_z^2 + \chi_3 k_y^2 .\end{aligned} \qquad (2.39)$$

Thus we have two pairs of propagating modes (even if we set in addition $k_z = 0$ or $k_y = 0$) and one pair which is purely diffusive (in discussing the mode structure we have, of course, always one diffusive mode corresponding to energy dissipation). Therefore we get for $k_x = 0$ the result which is displayed qualitatively in Fig. 4.

In the most general case in which all components of $\vec{k}$ are different from zero we obtain the following bicubic equation for ω,

$$\begin{aligned}\omega^6 - \omega^4\left(k^2\frac{\partial p}{\partial \rho} + \gamma^{1\rho}k_y^2 + \gamma^{2\rho}k_z^2 + \hat{\chi}_1^2 + \hat{\chi}_2^2\right) + \omega^2\Big[k_x^2\frac{\partial p}{\partial \rho}(\hat{\chi}_2^2 + \hat{\chi}_1^2) + \hat{\chi}_1^2\hat{\chi}_2^2 - (\chi_3+\chi_4)k_y^2k_z^2\left(2\frac{\partial p}{\partial \rho} + \gamma^{1\rho} + \gamma^{2\rho}\right) \\ - (\chi_3+\chi_4)^2k_y^2k_z^2 + k_z^2\hat{\chi}_2^2\left(\frac{\partial p}{\partial \rho} + \gamma^{2\rho}\right) + \hat{\chi}_1^2k_y^2\left(\frac{\partial p}{\partial \rho} + \gamma^{1\rho}\right)\Big] \\ - \frac{\partial p}{\partial \rho}k_x^2[\hat{\chi}_1^2\hat{\chi}_2^2 - (\chi_3+\chi_4)^2k_y^2k_z^2] = 0 . \end{aligned} \qquad (2.40)$$

As can be easily checked, Eq. (2.40) contains as special cases all results presented so far concerning the mode structure. If one is interested in a configuration where all components of the wave vector are different from zero it is straightforward though somewhat tedious to evaluate the corresponding plot and therefore we leave things there before detailed measurements have been carried out in this regime.

III. BIAXIAL DISCOTICS WITH BROKEN ROTATIONAL SYMMETRIES

A. Motivation

As has been briefly indicated in the Introduction the study of discotic liquid crystals is a rapidly growing field. Among the types of discotic liquid crystals already synthesized are uniaxial hexagonal (broken translational symmetries) and nematiclike phases (broken rotational symmetry because of one preferred axis). Very recently biaxial discotics with two broken translational symmetries have also been identified.[19]

As is well known attempts have been made to find biaxial nematics for many years[53,54] because the underlying fundamental group is non-Abelian leading to interesting properties of the defects of such a phase [e.g., a 180° disclination can catalyze the topological decay of a 360° one (Mermin[55])]. However, up to now the search for biaxial nematics seems to have remained unsuccessful. Possibly this is due to the fact that nematic liquid crystals consist of long, threadlike molecules; it is difficult to imagine that such entities can show two different broken rotational symmetries on a macroscopic scale. We think that it is much more probable that discoticlike liquid crystals show a phase with two different broken rotational symmetries because the basic entities are objects rather extended in two dimensions. (After the manuscript had been submitted for publication the first observation of a biaxial nematic phase was reported by Yu and Saupe for a lyotropic system.) Therefore it seems to be quite suggestive that ellipsoidal, disc-shaped molecules can show the following sequence of phases by lowering the temperature (see also Refs. 53 and 54):

(i) Ordinary isotropic fluid without any macroscopic order;

(ii) Nematiclike phase, where rotational symmetry is spontaneously broken by the existence of one preferred direction, i.e., where the axes normal to the discs are oriented parallel to each other;

(iii) Biaxial nematiclike phase, where rotational symmetry is spontaneously broken by two preferred perpendicular axes, i.e., where, in addition to the case (ii), also the ellipsoidal axes of the molecules are long-ranged oriented;

(iv) Smecticlike phases, where long-ranged positional order occurs. The hydrodynamic description of case (ii) is identical to that of ordinary (threadlike) nematics; case (iv) is treated by Prost and Clark[11,12] and in Sec. II.

In the following we study the hydrodynamic behavior of case (iii) (hereafter called biaxial nematic discotics) and we hope that the results presented here are a further stimulation for the synthesis of such a phase.

B. Hydrodynamic variables describing biaxial nematic discotics

By the existence of two preferred axes, rotational symmetry is spontaneously broken with respect to all (i.e., three) directions. Denoting the preferred axes by the unit vectors $\vec{n}$ and $\vec{m}$, their deviations δn_i and δm_i ($n_i \delta n_i = 0 = m_i \delta m_i$) constitute four variables, three of which are hydrodynamic. The nonhydrodynamic, i.e., rapid variable is eliminated by the constraint

$$\vec{n} \cdot \vec{m} = 0 . \tag{3.1}$$

Thus we have three components of $\vec{n}$ and $\vec{m}$ which we have to keep in our list of hydrodynamic variables, namely $(\vec{n} \times \vec{m}) \cdot \delta\vec{n}$, $(\vec{n} \times \vec{m}) \cdot \delta\vec{m}$, and $\vec{m} \cdot \delta\vec{n} (= -\vec{n} \cdot \delta\vec{m})$. If we choose in a local frame for the equilibrium value $\hat{n}^0 \parallel \hat{e}_z$ and $\hat{m}^0 \parallel \hat{e}_y$ the corresponding hydrodynamic variables characterizing the broken symmetries are δn_x, δn_y, and δm_x [$\delta m_z = -\delta n_y$ via Eq. (3.1)].

As in the case of biaxial discotics with broken translational symmetries we have as conserved hydrodynamic variables the density ρ, the entropy density σ, and the density of linear momentum $\vec{g}$.

Therefore we find for the Gibbs relation

$$d\epsilon = T\, d\sigma + \mu\, d\rho + \vec{v} \cdot d\vec{g} + \phi^n_{ij} d(\nabla_j n_i) + \phi^m_{ij} d(\nabla_j m_i) + h^n_i dn_i + h^m_i dm_i , \tag{3.2}$$

where ϕ^n_{ij}, ϕ^m_{ij} have to vanish in the homogeneous limit $k \to 0$ like k and h^n_i, h^m_i like k^2 for $k \to 0$. Higher-order gradient terms $\sim d\, \nabla_j \nabla_k n_i, d\nabla_j \nabla_k m_i$ can be neglected in biaxial discotics with broken rotational symmetries (contrary to the case of biaxial discotics with broken translational symmetries, cf. Sec. II A).

The three hydrodynamic variables $(\vec{n} \times \vec{m}) \cdot \delta\vec{n}$, $(\vec{n} \times \vec{m}) \cdot \delta\vec{m}$, and $\vec{m} \cdot \delta\vec{n}$ describe rotations and can be interpreted as components of the "vector" $\delta\vec{\Theta} = (\vec{m} \cdot \delta\vec{n}, (\vec{n} \times \vec{m}) \cdot \delta\vec{n}, (\vec{m} \times \vec{n}) \cdot \delta\vec{m})$. Since finite rotations do not commute, our hydrodynamic variables have to satisfy the relations (these relations are derived in the Appendix)

$$(\delta_1\delta_2-\delta_2\delta_1)\vec{\Theta}=(\delta_1\vec{\Theta})\times(\delta_2\vec{\Theta})$$

or explicitly,

$$(\delta_1\delta_2-\delta_2\delta_1)n_i=\delta^3_{ik}m_l(-\delta_1 m_k\delta_2 n_l+\delta_1 n_l\delta_2 m_k)$$
$$+\delta^3_{kl}m_i(-\delta_1 n_k\delta_2 m_l+\delta_1 m_l\delta_2 n_k) \quad (3.3)$$

and

$$(\delta_1\delta_2-\delta_2\delta_1)m_i=\delta^3_{ik}m_l(-\delta_1 n_l\delta_2 n_k+\delta_1 n_k\delta_2 n_l)\ ,$$

where δ_1 and δ_2 stand for any first-order differential operator (like ∂_t or ∇_k) and where δn_i or δm_i have to be interpreted as the appropriate hydrodynamic variables $\vec{m}\cdot\delta\vec{n}$, $(\vec{n}\times\vec{m})\cdot\delta\vec{n}$, or $(\vec{m}\times\vec{n})\cdot\delta\vec{m}$. Thus, when dealing with Eq. (3.2) one has to keep in mind that $d\nabla_j n_i\neq\nabla_j dn_i$, because of Eq. (3.3).

Equation (3.3) represent the first occurrence of such relations in the physics of liquid crystals. Of course they hold equally for biaxial nematics. In the case of the superfluid phases of ^{3}He similar relations have emerged, e.g., in ^{3}He-A as has been pointed out by Mermin and Ho.[56] It seems worth noticing, however, that the relations which hold for the superfluid phases of ^{3}He connect quantities (phase deviations and rotation angles between real and spin space) which are different from those studied in the present work. Like in nematics and in discotics with broken translational symmetry (Sec. II) the state of the system can be equally well described by $-\vec{n}$ instead of $\vec{n}$ or by $-\vec{m}$ instead of $\vec{m}$. Therefore, the hydrodynamic equations must be invariant under

$$\vec{n}\to-\vec{n}\ \text{and}\ \vec{m}\to-\vec{m} \quad (3.4)$$

separately. This property reduces the number of phenomenological parameters considerably.

C. Gradient free energy for biaxial nematic discotics

In this section we consider in detail the gradient free energy of biaxial nematic discotic liquid crystals. The present investigation represents a generalization of the Frank free energy of nematics. First there exist no contributions to the gradient free energy which contain only one gradient and which are invariant under $\vec{n}\to-\vec{n}$, $\vec{m}\to-\vec{m}$, and parity. Such terms would become possible only if a pseudoscalar in real space would exist as, e.g., in cholesterics. Thus we must consider terms which are quadratic in the gradients. We proceed in several steps. The gradient free energy can be split into three parts,

$$\text{(i)}\quad F_1^f=K_{ijlm}(\nabla_i n_j)(\nabla_l n_m)\ ,$$
$$\text{(ii)}\quad F_2^f=M_{ijlm}(\nabla_i m_j)(\nabla_l m_m)\ , \quad (3.5)$$

and

$$\text{(iii)}\quad F_3^f=N_{ijlm}(\nabla_i n_j)(\nabla_l m_m)\ .$$

It proves to be useful to divide Eq. (3.5) further,

$$F_1^f=F_1^a+F_1^b+F_1^c\ , \quad (3.6)$$

where F_1^a contains terms quadratic in the strains of n_i, F_1^b comes from terms containing (curl $\vec{n}$) quadratically whereas F_1^c represents the cross terms.

Let us start with F_1^a. From the textbooks of elasticity[57] we obtain for the biaxial system under consideration

$$F_1^a=\tfrac{1}{2}(\lambda_{xxxx}u_{xx}^2+\lambda_{yyyy}u_{yy}^2+\lambda_{zzzz}u_{zz}^2+2\lambda_{xxyy}u_{xx}u_{yy}$$
$$+2\lambda_{xxzz}u_{xx}u_{zz}+2\lambda_{yyzz}u_{yy}u_{zz}+4\lambda_{xzxz}u_{xz}^2$$
$$+4\lambda_{yzyz}u_{yz}^2+4\lambda_{xyxy}u_{xy}^2)\ , \quad (3.7)$$

where $u_{ij}=\tfrac{1}{2}(\partial_i n_j+\partial_j n_i)$. Equation (3.7) contains nine independent coefficients.

Taking into account the fact that $\vec{n}$ is a unit vector, i.e.,

$$0=\nabla(n_x^2+n_y^2+n_z^2)=2n_z\nabla n_z\ , \quad (3.8)$$

leading to $u_{zz}=0$,

$$u_{zx}=\tfrac{1}{2}(\text{curl}\ \vec{n})_y\ ,$$
$$u_{zy}=-\tfrac{1}{2}(\text{curl}\ \vec{n})_x\ , \quad (3.9)$$

we obtain from Eqs. (3.9) and (3.7),

$$F_1^a=\lambda_1(\partial_x n_x)^2+\lambda_2(\partial_y n_y)^2+\lambda_3(\partial_x n_x)(\partial_y n_y)$$
$$+\lambda_4(\partial_x n_y+\partial_y n_x)^2+\lambda_5(\partial_z n_x)^2+\lambda_6(\partial_z n_y)^2\ . \quad (3.10)$$

Next we consider F_1^b containing (curl $\vec{n}$) quadratically. There exist only a few new terms that cannot be incorporated into the coefficients of Eq. (3.10). Under the further restriction of the invariance under $\vec{n}\to-\vec{n}$ and $\vec{m}\to-\vec{m}$, we finally get for F_1^b the novel terms

$$F_1^b=\lambda_7(\partial_x n_y-\partial_y n_x)^2+\lambda_8(\partial_x n_y-\partial_y n_x)$$
$$\times(\partial_x n_y+\partial_y n_x)\ . \quad (3.11)$$

Of course its possible to rearrange the terms λ_4, λ_7, and λ_8 to arrive at

$$\tilde{\lambda}_4(\partial_x n_y)^2+\tilde{\lambda}_8(\partial_x n_y)(\partial_y n_x)+\tilde{\lambda}_7(\partial_y n_x)^2\ . \quad (3.12)$$

Concerning F_1^c, it is straightforward to check that there exists no further contribution that survives all constraints, and that cannot be incorporated in $\lambda_1,\ldots,\lambda_8$.

The evaluation of F_2^f is achieved most easily by replacing $\vec{n}$ with $\vec{m}$ and vice versa in F_1^f. However, by the constraint of (3.1), $m_z=-n_y$, and only m_x has to be dealt with as an independent hydrodynamic variable. Thus we obtain for F_2^f:

$$F_2^g = \lambda_9(\partial_x m_x)^2 + \lambda_{10}(\partial_y m_x)^2 + \lambda_{11}(\partial_z m_x)^2 \,. \tag{3.13}$$

Now we turn to the discussion of F_3^g. The constraint (3.4) requires N_{ijlm} to contain an odd number of $\vec{n}$ and an odd number of $\vec{m}$. We find for N_{ijlm},

$$N_{ijlm} = \lambda_{12} m_i n_j \delta^3_{lk} + \lambda_{13} m_i n_l \delta^3_{jk} + \lambda_{14} m_j n_l \delta^3_{ik} + \lambda_{15} \delta^3_{ik} m_l n_j \,. \tag{3.14}$$

Putting together Eqs. (3.10)–(3.14) we obtain for the gradient free energy of biaxial nematic discotics

$$\begin{aligned} F^g = {} & A_1(\delta^3_{ik}\nabla_i n_k)^2 + A_2(m_i m_k \nabla_i n_k)^2 + A_3\delta^3_{ik} m_j m_l (\nabla_k n_i)(\nabla_l n_j) + A_4 \delta^3_{il} m_j m_k (\nabla_i n_j)(\nabla_l n_k) \\ & + A_5 \delta^3_{il} m_j m_k (\nabla_j n_i)(\nabla_k n_l) + A_6 \delta^3_{il} m_j m_k (\nabla_i n_j)(\nabla_k n_l) + A_7 \delta^3_{ik} n_j n_l (\nabla_j n_i)(\nabla_l n_k) \\ & + A_8 n_j n_l m_i m_k (\nabla_j n_i)(\nabla_l n_k) + (A_9 \delta^3_{ik}\delta^3_{jl} + A_{10}\delta^3_{ik} n_j n_l + A_{11}\delta^3_{ik} m_j m_l)(\nabla_j m_i)(\nabla_l m_k) \\ & + \delta^3_{il} n_j m_k [A_{12}(\nabla_j n_k)(\nabla_i m_l) + A_{13}(\nabla_i n_k)(\nabla_j m_l) + A_{14}(\nabla_k n_i)(\nabla_j m_l) + A_{15}(\nabla_j n_i)(\nabla_k m_l)] \,. \end{aligned} \tag{3.15}$$

Equation (3.15) is the generalization of the Frank free energy for uniaxial nematics to a biaxial liquid crystal, either biaxial nematic discotics or biaxial nematics. It contains 15 independent phenomenological parameters. At least three combinations of them can be determined by the Frederiks transitions. However, unlike the case of nematics, it is somewhat difficult to give a simple picture of the distortions contained in Eq. (3.15). Most probably $A_1, \dots, A_8$ are the largest coefficients and of the same order of magnitude whereas A_9, A_{10}, A_{11} assume a value which is supposed to be considerably lower. These predictions may be derived from the sequence of phase transitions described in Sec. III B.

In addition one may suspect that the $A_{12}, \dots, A_{15}$ are the smallest of all contributions because they connect the variations of the different axes. If an external magnetic field is applied, the orientational free energy is given by Eq. (2.32). The discussion of the influence of the magnetic field for discotics with broken translational invariance (Sec. II B) is equally valid for the biaxial nematic discotics.

D. Hydrodynamic equations for biaxial nematic discotics

The conservation laws and quasiconservation laws for the hydrodynamic variables described in Sec. III B are

$$\begin{aligned} & \frac{\partial}{\partial t}\rho + \nabla_i g_i = 0 \,, \\ & \frac{\partial}{\partial t} g_i + \nabla_j \sigma_{ij} = 0 \,, \\ & \frac{\partial}{\partial t}\sigma + \nabla_i j_i^\sigma = 0 \,, \\ & \frac{d}{dt} m_i + X_i^{(m)} = 0 \,, \\ & \frac{d}{dt} n_i + X_i^{(n)} = 0 \,. \end{aligned} \tag{3.16}$$

For the equations of state, which relate the variables to their conjugates defined by Eq. (3.2) we obtain

$$\begin{aligned} & \vec{v} = \frac{1}{\rho}\vec{g} \,, \\ & \delta\mu = \lambda\delta\rho + \gamma\delta\sigma \,, \\ & \delta T = T C_v^{-1}\delta\sigma + \gamma\delta\rho \,, \\ & \phi^m_{ij} = \frac{\partial F^g}{\partial \nabla_j m_i} \,, \\ & \phi^n_{ij} = \frac{\partial F^g}{\partial \nabla_j n_i} \,, \end{aligned} \tag{3.17}$$

and

$$h_i^m = \frac{\partial F^g}{\partial m_i} \,,$$

$$h_i^n = \frac{\partial F^g}{\partial n_i} \,,$$

where the gradient free energy F^g has been studied in Sec. III C. We wish to stress that nowhere in the derivation of the hydrodynamic equations already presented (or still to be discussed, as is the case for reversible and irreversible currents) are $\vec{n}$ and $\vec{m}$ restricted to being constant in space in equilibrium, i.e., the hydrodynamic equations which are presented in this section can be applied to hydrodynamic motions in inhomogeneous textures as well. In addition we wish to point out that ϕ^m_{ij}, h^m_i as well as ϕ^n_{ij}, h^n_i depend on *both* $\vec{n}$ and $\vec{m}$.

Next we turn to the discussion of the reversible currents. We find taking into account general symmetry arguments, the reversible currents of the variables characterizing the broken symmetries

$$\begin{aligned} X_i^{(n)R} = {} & (\alpha_1 \delta^3_{il} + \alpha_2 m_i m_l) n_k (\nabla_l v_k + \nabla_k v_l) \\ & + \tfrac{1}{2}(\alpha_3 \delta^3_{il} + \alpha_4 m_i m_l) n_k (\nabla_l v_k - \nabla_k v_l) \,, \end{aligned} \tag{3.18}$$

$$X_i^{(m)R} = \alpha_5 \delta^3_{il}(\nabla_l v_k + \nabla_k v_l) m_k + \tfrac{1}{2}\alpha_6 \delta^3_{il}(\nabla_l v_k - \nabla_k v_l) m_k , \tag{3.19}$$

and

$$j_i^{\sigma R} = \sigma v_i . \tag{3.20}$$

It should be emphasized that only one component enters in $X_i^{(m)}(X_i^{(m)} \perp \vec{n}, X_i^{(m)} \perp \vec{m})$ because the second one ($\| \vec{n}$) is identical to a component $X_i^{(n)}(X_i^{(n)} \| \vec{m})$ (cf. the discussion on p. 26).

Vanishing entropy production requires counter terms to $X_i^{(n)R}$ and $X_i^{(m)R}$ in σ^R_{ij}:

$$\sigma^R_{ij} = p\delta_{ij} + \phi^n_{kj}\nabla_i n_k + \phi^m_{kj}\nabla_i m_k + \rho v_i v_j + \Sigma'_{ij} , \tag{3.21}$$

where

$$\Sigma'_{ij} = H^n_l(\alpha_1 \delta^3_{lj} + \alpha_2 m_l m_j) n_i + (i \leftrightarrow j) + \tfrac{1}{2} H^n_l(\alpha_3 \delta^3_{lj} + \alpha_4 m_l m_j) n_i - (i \leftrightarrow j) + H^m_l \alpha_5 (\delta^3_{lj} m_i + \delta^3_{li} m_j) + \tfrac{1}{2} H^m_l \alpha_6 (\delta^3_{lj} m_i - \delta^3_{li} m_j) \tag{3.22}$$

and

$$H^n_l = h^n_l - \nabla_j \phi^n_{lj} - \phi^n_{pj}\delta^3_{pk} m_l \nabla_j m_k + \phi^n_{pj} m_p \delta^3_{lk}\nabla_j m_k + \phi^m_{pj}\delta^3_{pk} m_l \nabla_j n_k - \phi^m_{pj}\delta^3_{pl} m_k \nabla_j n_k , \tag{3.23}$$

$$H^m_l = h^m_l - \nabla_j \phi^m_{lj} + \phi^n_{pj}\delta^3_{pl} m_k \nabla_j n_k - \phi^n_{pj}\delta^3_{pk} m_l \nabla_j n_k . \tag{3.24}$$

By deriving Eqs. (3.21)–(3.24) with the help of the Gibbs relation (3.2), the commutator-type relations (3.3) must be taken into account. As is well known, in order to preserve angular momentum, it is always possible to choose the stress tensor symmetric locally[58,31,47] (we do not consider the angular momentum of the molecules due to rotations about their centers of mass, since these are microscopic excitations, not connected with a spontaneously broken symmetry), i.e., the antisymmetric part of σ_{ij} must be zero or a total divergence. From Eqs. (3.21)–(3.24) we have

$$\begin{aligned}\epsilon_{ija}\sigma_{ij} = \epsilon_{ija}\{&\phi^n_{kj}\nabla_k n_i + \phi^m_{kj}\nabla_k m_i + \tfrac{1}{2}\alpha_3 h^n_l n_i \delta^3_{lj} + \tfrac{1}{2}\alpha_4 h^n_l m_l m_j n_i + \tfrac{1}{2}\alpha_6 h^m_l \delta^3_{lj} m_i - \tfrac{1}{2}\alpha_3 \nabla_p \phi^n_{lp} n_i \delta^3_{lj} \\ &- \tfrac{1}{2}\alpha_4 \nabla_p \phi^n_{lp} m_l m_j n_i - \tfrac{1}{2}\alpha_6 \nabla_p \phi^m_{lp}\delta^3_{lj} m_i + [\phi^n_{pt} m_p \delta^3_{lk}\nabla_t m_k - \phi^n_{pt}\delta^3_{pk} m_l \nabla_t m_k + \phi^m_{pt} m_l (\nabla_t n_k)\delta^3_{pk} \\ &\qquad - \phi^m_{pt}\delta^3_{pl} m_k \nabla_t n_k](\tfrac{1}{2}\alpha_3 n_i \delta^3_{lj} + \tfrac{1}{2}\alpha_4 m_l m_j n_i) \\ &+ \tfrac{1}{2}\alpha_6 \delta^3_{lj} m_i (\phi^n_{pt}\delta^3_{pl} m_k \nabla_t n_k - \phi^n_{pt}\delta^3_{pk} m_l \nabla_t n_k)\} .\end{aligned} \tag{3.25}$$

Because the gradient energy is rotational invariant we have the additional identity

$$0 = \delta\epsilon = \phi^n_{ij}\delta\nabla_j n_i + \phi^m_{ij}\delta\nabla_j m_i + h^n_i \delta n_i + h^m_i \delta m_i , \tag{3.26}$$

where

$$\delta\nabla_j n_i = \Omega_{jk}\nabla_k n_i + \nabla_j \Omega_{ik} n_k + \delta^3_{kl} m_i (\delta n_k \nabla_j m_l - \delta m_l \nabla_j n_k) + \delta^3_{ik} m_l (\delta m_k \nabla_j n_l - \delta n_l \nabla_j m_k) \tag{3.27}$$

and

$$\delta\nabla_j m_i = \Omega_{jk}\nabla_k m_i + \nabla_j \Omega_{ik} m_k + \delta^3_{ik} m_l (\delta n_l \nabla_j n_k - \delta n_k \nabla_j n_l)$$

with Ω_{ij} any antisymmetric matrix. Again, the commutator-type relations (3.3) are of special importance while deriving (3.26)–(3.28). If we make use of Eqs. (3.26) and (3.27), the antisymmetric part of the stress tensor (3.25) is found after a straightforward, but lengthy calculation, to be of the required form if

$$\alpha_3 = \alpha_4 = \alpha_6 = 1 . \tag{3.28}$$

As is easily seen, the case of uniaxial discotics with broken rotational symmetry or uniaxial nematics is contained as a special case (cf. also Ref. 32).

By linearizing Eqs. (3.18) and (3.19) we can obtain further insight if we discuss the symmetric contributions involving α_1, α_2, and α_5 in the framework of Mori's projector formalism which was introduced into hydrodynamics by Forster during his study of nematic liquid crystals (cf. Refs. 30 and 50 for a detailed exposition of the method). As usual α_1, α_2, and α_5 pick up a contribution from the frequency matrix which contains the instantaneous noncollisional effects. It seems important to note, however, that α_1, α_2, and α_5 get, to the same order in k in the equations for $\vec{n}$ and $\vec{g}$, a contribution from the memory matrix which contains the noninstantaneous collision-dominated processes.

Thus the memory matrix contributes to the reversible hydrodynamic equations of biaxial nematic discotics. The same is true for uniaxial nematics, as has been discussed by Forster, and for smectic-C liquid crystals. For the latter system this fact seems to have been overlooked so far. These systems (nematic, smectic-C, and biaxial discotics) have broken rotational symmetries in real space in common, whereas all liquid crystals with broken translational symmetries (smectics-A, -B, and -G, biaxial discotics with broken translational symmetries) have no reversible contributions from the memory matrix in their hydrodynamic equations. This fact leads quite naturally to the suggestion that only hydrodynamic systems with broken rotational symmetry in real space possess reversible noninstantaneous contributions in their hydrodynamic equations.

For the irreversible currents we have

$$g_i^D = 0\,, \tag{3.29}$$

$$\sigma_{ji}^D = \nu_{jikl} A_{kl}\,, \tag{3.30}$$

$$j_i^{\sigma D} = (\kappa_1 \delta_{ij}^3 + \kappa_2 n_i n_j + \kappa_3 m_i m_j) \nabla_j T\,, \tag{3.31}$$

$$X_i^{(m)D} = \zeta_3 \delta_{ik}^3 H_k^m\,, \tag{3.32}$$

$$X_i^{(n)D} = (\zeta_1\, \delta_{ik}^3 + \zeta_2 m_i m_k) H_k^n\,, \tag{3.33}$$

where

$$A_{ij} = (\nabla_i v_j + \nabla_j v_i)\,.$$

The structure of the quantities ν_{ijkl} and κ_{ij} is the same as in biaxial discotics with broken translational symmetries (Sec. II A), i.e., we have nine independent viscosities (compared to five for uniaxial nematics) and three thermal conductivities (two for uniaxial nematics). Contrary to the case of discotics with broken translational symmetries we find three independent dissipation coefficients of the order parameter ζ_1, ζ_2, ζ_3. A cross coupling between the variables characterizing the broken symmetries and entropy density is not possible for biaxial nematic discotics due to the behavior of $\vec{m}$ and $\vec{n}$ under spatial parity which is different from that of $\vec{R}$.

In closing the section on biaxial nematic discotics we briefly discuss the mode structure of the linearized hydrodynamic equations. Compared to uniaxial nematics we have one additional variable characterizing a broken symmetry. If the viscosities and elastic coefficients are of the same order of magnitude as in nematics we expect one pair of propagating modes (first sound) and six diffusing modes (energy dissipation, rotational diffusion, and shear diffusion). Of course the diffusion peaks show an angular dependence reflecting the fact that we consider a biaxial system. If, unexpectedly, the phenomenological parameters connected with the broken symmetries (elastic constants, reversible transport parameters α_i) are large, propagating modes of the form $\omega = \Gamma k^2 + iDk^2$ become possible (Γ and D are highly anisotropic). This can be seen by consideration of the coupled motion of $\vec{n}$, $\vec{m}$ and the density of linear momentum yielding a bicubic equation. Owing to the complicated structure of the gradient free energy and the viscosity tensor the explicit expressions are not very illuminating.

IV. CONCLUSION

We have given the sets of nonlinear hydrodynamic equations for biaxial discotic liquid crystals with broken translational symmetries and broken rotational symmetries, respectively. Among the results which seem to be most easily accessible to experiments are predictions of the various possible orientations which a sample of a biaxial discotic liquid crystal can assume under the influence of a static external magnetic field. Furthermore, it should be possible to detect the propagating modes (depending in number on the orientation of the wave vector) in the case of biaxial discotics with broken translational symmetries.

More interesting from a fundamental point of view, although experimentally more difficult to attack, are the intriguing hydrodynamic properties of a biaxial discotic liquid crystal with broken rotational symmetries. Among these fascinating properties the most important ones seem to be the existence of commutator-type relations for the variables characterizing the broken symmetries (a feature which may be related to the intricate properties of the defects in such media) and the structure of the gradient free energy for these variables.

It is a pleasure to thank N. A. Clark for a stimulating discussion.

APPENDIX

In the following we derive the equations [cf. Eqs. (3.3) of the main text]

$$(\delta_1\delta_2 - \delta_2\delta_1)\vec{\Theta} = (\delta_1\Theta) \times (\delta_2\vec{\Theta})\,, \tag{A1}$$

where $\delta\vec{\Theta}$ is given by

$$\delta\vec{\Theta} = \begin{pmatrix} \delta\Theta_1 \\ \delta\Theta_2 \\ \delta\Theta_3 \end{pmatrix} \equiv \begin{pmatrix} \vec{m}\cdot\delta\vec{n} \\ (\vec{n}\times\vec{m})\cdot\delta\vec{n} \\ (\vec{m}\times\vec{n})\cdot\delta\vec{m} \end{pmatrix}, \tag{A2}$$

and where δ_1 and δ_2 stand for any first-order differential operator. From inspection of the quantities $\delta\Theta_i$ it becomes clear that the $\delta\Theta_i$ are defined only locally and cannot be integrated in a biaxial nematic liquid crystal. Thus, a vector $\vec{\Theta}$ is globally not defined. Although we always deal with $\delta\vec{\Theta}$ (and never with $\vec{\Theta}$), the nonexistence of $\vec{\Theta}$ has the consequence for $\delta\vec{\Theta}$, that $\delta_1(\delta_2\vec{\Theta}) \neq \delta_2(\delta_1\vec{\Theta})$, which is brought about by Eqs. (A1). Let us start with the derivation of the first equation in (A1):

$$(\delta_1\delta_2-\delta_2\delta_1)\Theta_1\equiv\delta_1(\vec{m}\cdot\delta_2\vec{n})-\delta_2(\vec{m}\cdot\delta_1\vec{n}) \tag{A3}$$

$$=(\delta_1\vec{m})\cdot(\delta_2\vec{n})-(\delta_2\vec{m})\cdot(\delta_1\vec{n}) \tag{A4}$$

$$=\delta_{ij}[(\delta_1 m_i)(\delta_2 n_j)-(\delta_2 m_i)(\delta_1 n_j)] \tag{A5}$$

$$=(\delta^3_{ij}+m_i m_j+n_i n_j)[(\delta_1 m_i)(\delta_2 n_j)-(\delta_2 m_i)(\delta_1 n_j)] \tag{A6}$$

$$=\delta^3_{ij}[(\delta_1 m_i)(\delta_2 n_j)-(\delta_2 m_i)(\delta_1 n_j)]=(\delta_2\Theta_3)(\delta_1\Theta_2)-(\delta_1\Theta_3)(\delta_2\Theta_2). \qquad \text{(A7)} \;\; \text{(A8)}$$

Equation (A8) completes the demonstration of the first of the equations given in (3.3) or (A1), respectively. Equation (A7) follows from Eq. (A6) because

$$m_i\delta_k m_i=0=n_i\delta_k n_i\,. \tag{A9}$$

For the dynamic equation for $\delta\Theta_2$ we have

$$(\delta_1\delta_2-\delta_2\delta_1)\Theta_2\equiv\delta_1((\vec{n}\times\vec{m})\cdot\delta_2\vec{n})-\delta_2((\vec{n}\times\vec{m})\cdot\delta_1\vec{n}) \tag{A10}$$

$$=\epsilon_{ijk}(\delta_2 n_i)(\delta_1 n_j)m_k+\epsilon_{ijk}(\delta_2 n_i)(\delta_1 m_k)n_j-\epsilon_{ijk}(\delta_1 n_i)(\delta_2 n_j)m_k-\epsilon_{ijk}(\delta_1 n_i)(\delta_2 m_k)n_j \tag{A11}$$

$$=\epsilon_{ijk}n_j(\delta_2 n_i)\delta_1 m_k)-\epsilon_{ijk}(\delta_1 n_i)n_j(\delta_2 m_k) \tag{A12}$$

$$=(\delta^3_{il}+n_i n_l+m_i m_l)\epsilon_{ljk}n_j[(\delta_2 n_i)(\delta_1 m_k)-(\delta_1 n_i)(\delta_2 m_k)] \tag{A13}$$

$$=m_i m_l\epsilon_{ljk}n_j[(\delta_2 n_i)(\delta_1 m_k)-(\delta_1 n_i)(\delta_2 m_k)] \tag{A14}$$

$$=(\delta_2\Theta_1)(\delta_1\Theta_3)-(\delta_1\Theta_1)(\delta_2\Theta_3)\,. \tag{A15}$$

Equation (A15) completes the demonstration of the second equation of (A1). Equation (A12) holds because the second and fourth terms in (A11) cancel each other, and Eq. (A14) follows from (A13) by taking into account that

$$\epsilon_{ljk}n_i n_k=0 \tag{A16}$$

and by remembering that $\vec{n}$, $\vec{m}$, and $(\vec{n}\times\vec{m})$ always form a triad of orthogonal vectors.

The demonstration of the equation for $\delta\Theta_3$ perfectly parallels that for $\delta\Theta_2$ and thus our derivation of the anholonomity relations for the variables characterizing the broken rotational symmetries of a biaxial nematic is complete. It should be noticed that Eqs. (A1) are not restricted to homogeneous equilibrium textures, but apply to arbitrary nonsingular textures.

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