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Relativistically strong-coupled transverse-longitudinal waves in an electron-ion plasma

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A method for obtaining exact periodic solutions of relativistically strong-coupled transverse-longitudinal waves in an electron-ion plasma is presented. Suitable choice of initial conditions yields wave solutions whose longitudinal component has twice the frequency of the transverse. In addition, the significance of the plasma stream velocity in the nonlinear theory is discussed.

This paper makes a contribution to the study of relativistically strong electromagnetic waves in a cold, infinite, uniform, unmagnetized plasma. In linear theory, an electromagnetic wave can propagate in an unmagnetized plasma with purely transverse linear polarization. In nonlinear theory, however, purely transverse linearly polarized waves cannot exist, in general, as a coupled longitudinal field normally appears. The coupling may arise from the $\vec{v}\times\vec{B}$ Lorentz force, for instance. Most solutions of strong-coupled transverse-longitudinal waves were obtained by approximate analytical[1,2] or numerical[3] methods; hence, the results have limited range of validity. In a recent paper, Chian and Clemmow[4] demonstrated the existence of exact (relativistic, nonlinear) periodic solutions of coupled transverse-longitudinal waves, for which the longitudinal component had twice the frequency of the transverse. The stability of these waves has also been investigated.[5]

Previous works on strong-coupled transverse-longitudinal waves were mostly based on the simplified model of an electron plasma by assuming the positive ions to form a stationary background. Recent studies[2,6,7] indicate that ion dynamics plays an important role in the properties of strong waves. The aim of this paper is to extend the work of Chian and Clemmow[4] to include full ion dynamics.

In the laboratory frame S', the wave propagates with a velocity $c\hat{z}/n$ [where $0 \leq n < 1$ and $\hat{z}=(0,0,1)$], and the solutions depend on the space and time coordinates only through the combination $t'-nz'/c$.

In nonlinear theory, it is convenient to define the plasma stream velocity as the ratio of the average particle flux to the average number density.[8] It follows from Maxwell's equations that the current and charge densities of an electron-ion plasma ($q_e=-e$, $q_i=e$) are zero on average, i.e., $\langle N_e\vec{v}_e'\rangle = \langle N_i\vec{v}_i'\rangle$ and $\langle N_e'\rangle = \langle N_i'\rangle$ (where the angular bracket signifies averaging over a period). Thus, the stream velocities of electrons and ions are equal:

$$\langle N_e'\vec{v}_e'\rangle/\langle N_e'\rangle = \langle N_i'\vec{v}_i'\rangle/\langle N_i'\rangle \equiv \vec{V}_s'. \tag{1}$$

The stationary plasma corresponds to the particular case in which $\langle N_{e,i}'\vec{v}_{e,i}'\rangle=0$. From the continuity equations, the number densities are given by[8]

$$N_{e,i}' = \frac{1-nV_{sz}'/c}{1-nv_{ez,iz}'/c}\langle N_{e,i}'\rangle. \tag{2}$$

Two important relations for the particle velocities can be obtained from the transverse and longitudinal components of the relativistic equations of motion,[2] namely,

$$\vec{u}_{i\perp}' + \mu\vec{u}_{e\perp}' = \vec{D}_\perp', \tag{3}$$

$$(u_{iz}'-n\gamma_i') + \mu(u_{ez}'-n\gamma_e') = D_z', \tag{4}$$

where $\vec{u}'=\gamma'\vec{v}'/c$, $\gamma'=(1+u'^2)^{1/2}$, $\mu=m_e/m_i$, $\perp=(x,y)$, and $D_\perp'$ and D_z' are constants.

The analysis is henceforth referred to a frame S which has velocity $nc\hat{z}$ relative to S'. In S (with quantities referred to this frame unprimed) there is no spatial dependence, the number densities are constant, and the behavior of the wave field is governed by[9]

$$\frac{d^2\vec{u}_e}{d\tau^2} + \frac{\vec{u}_e}{\gamma_e} - \frac{\vec{u}_i}{\gamma_i} = 0, \tag{5}$$

$$\vec{u}_i + \mu\vec{u}_e = \vec{D}, \tag{6}$$

where $\tau=\omega_{pe}t$, $\omega_{pe}^2=4\pi N_e e^2/m_e$, and $\vec{D}$ is a constant vector. The wave magnetic field is transformed away in S, and the electric field is given by $\vec{E}=-(m_e\omega_{pe}c/e)d\vec{u}_e/d\tau$.

Upon transforming (6) to S' the transverse component yields (3), with $D_\perp = D_\perp'$, while the longitudinal component yields (4), with $D_z=(1-n^2)^{-1/2}D_z'\equiv D$. Periodic solutions are sought for which[2] $\vec{D}_\perp=0$ (corresponding to the assumption that $\vec{V}_{s\perp}'=0$ in the equilibrium state), so $\vec{u}_{i\perp}=-\mu\vec{u}_{e\perp}$. With the notation $\vec{u}_e=(\xi,\eta,\zeta)$, the case examined here is that in which $\xi\equiv 0$ and correspondingly $E_x=0$. The equations to be considered are, therefore,

$$\frac{d^2\eta}{d\tau^2}+\left(\frac{1}{\gamma_e}+\frac{\mu}{\gamma_i}\right)\eta=0,\quad \frac{d^2\zeta}{d\tau^2}+\left(\frac{1}{\gamma_e}+\frac{\mu}{\gamma_i}\right)\zeta=\frac{D}{\gamma_i}, \tag{7}$$

with

$$\gamma_e=(1+\eta^2+\zeta^2)^{1/2}$$

and

$$\gamma_i=[1+\mu^2\eta^2+(D-\mu\zeta)^2]^{1/2}.$$

A first integral of (7) is

$$\frac{1}{2}\left[\left(\frac{d\eta}{d\tau}\right)^2+\left(\frac{d\zeta}{d\tau}\right)^2\right]=W-\gamma_e-\frac{\gamma_i}{\mu}, \tag{8}$$

where W is a constant that determines the amplitude of oscillations, whose value must exceed $[D^2+(1+\mu)^2]/\mu$ in order for a solution to exist.

The linearization of (7) yields two uncoupled equations, solutions of which represent two independent harmonic oscillations. For a stationary electron-ion plasma, the angular frequencies of the oscillations, when transformed to S' gives, for the η oscillation,

$$\omega'^2=(1-n^2)^{-1}(1+\mu)\omega_{pe}'^2, \tag{9}$$

and for the ζ oscillation,

$$\omega'^2=(1+\mu)\omega_{pe}'^2. \tag{10}$$

These are the well-known linear dispersion relations for purely transverse linearly polarized and purely longitudinal waves, respectively.

Any exact solution of (7) can be traced continuously to an associated linear solution by bringing the value of W down to its minimum value, $[D^2+(1+\mu)^2]/\mu$. Exact solutions of purely longitudinal waves are easily obtainable from the ζ equation in (7), with $\eta=0$.[6,9,10] On the other hand, exact solutions of purely transverse linearly polarized waves only exist in two special cases[9]: (i) $D=\zeta=0$ and (ii) $\mu=1$ and $\zeta=D/2$, which are not of practical interest. In general, the associated exact solutions of (9) cannot be purely transverse, but necessarily have a longitudinal component. The present aim is to show the existence of exact periodic solutions of the two coupled equations (7).

A solution of (7) specifies a path in the (η,ζ) plane, and (8) shows that the path is confined within the domain bounded by $\gamma_e+\gamma_i/\mu=W$, which gives an equation for an ellipse

$$\eta^2+\left(1-\frac{D^2}{\mu^2W^2}\right)\left(\zeta-\frac{D}{2\mu}\,\frac{W^2+1-(1+D^2)/\mu^2}{W^2-D^2/\mu^2}\right)^2=\frac{\{W^2-[(1+\mu)^2+D^2]/\mu^2\}\{W^2-[(1-\mu)^2+D^2]/\mu^2\}}{4(W^2-D^2/\mu^2)}. \tag{11}$$

The path satisfies a second-order differential equation[9]

$$\frac{d^2\zeta}{d\eta^2}=\frac{1}{2}\left[1+\left(\frac{d\zeta}{d\eta}\right)^2\right]\frac{\dfrac{D}{\gamma_i}+\left(\dfrac{1}{\gamma_e}+\dfrac{\mu}{\gamma_i}\right)\left(\dfrac{\eta d\zeta}{d\eta}-\zeta\right)}{W-\gamma_e-\gamma_i/\mu}. \tag{12}$$

If initial ($\tau=0$) values are denoted by a suffix zero, the specification of η_0, ζ_0, $(d\zeta/d\eta)_0$, D and W determines $\eta(\tau)$ and $\zeta(\tau)$. For a given point (η_0,ζ_0) on the ellipse, the slope of the path normal to the ellipse is

$$\left(\frac{d\zeta}{d\eta}\right)_0=\left[\zeta_0-\left(\frac{1}{\gamma_{e0}}+\frac{\mu}{\gamma_{i0}}\right)^{-1}\frac{D}{\gamma_{i0}}\right]\eta_0^{-1}. \tag{13}$$

From any initial point (η_0,ζ_0) inside the bounding ellipse, there is a unique path corresponding to any initial value of $(d\zeta/d\eta)_0$. Usually, this path never reaches the ellipse. As the boundary is approached, the denominator in (12) tends to zero and hence the curvature of the path tends to infinity. Exceptionally, however, a path may meet the ellipse by approaching it ultimately along the normal path to the ellipse. In this case, (12) and (13) show that the numerator in (12) also vanishes and the curvature remains finite.

Previous studies of coupled transverse-longitudinal waves[1-4] indicate that there is a periodic solution whose longitudinal component has twice the frequency of transverse. Such a solution corresponds to a path represented by a single line, symmetric about the ζ axis, that joins the points $(\pm\eta_0,\zeta_0)$ on the bounding ellipse. Computed solutions of (12) starting from various different points on the ellipse with the initial slope (13) indicate that, for an arbitrary pair of values of W and D, there is just one such path. Typical results are shown in Fig. 1. The path in question is the line joining P and P_1. For paths that start above (below) P, it is seen that ζ is still decreasing(increasing) at $\eta=0$. Such paths do not give periodic solutions since they bend round before reaching the boundary of the ellipse in $\eta<0$, and presumably, in general, spiral indefinitely without meeting the boundary again.

The technique of seeking the starting point P that gives a computed path which crosses the ζ axis orthogonally can be applied to a range of values of W and D in order to obtain the general behavior of strong-coupled transverse-longitudinal waves. This has been done for the case of an electron plasma.[4] Upon transformation to the laboratory frame, the solution represents a traveling wave with $\vec{E}'=(0,E_y',E_z')$ and $\vec{B}'=(B_x',0,0)$, where

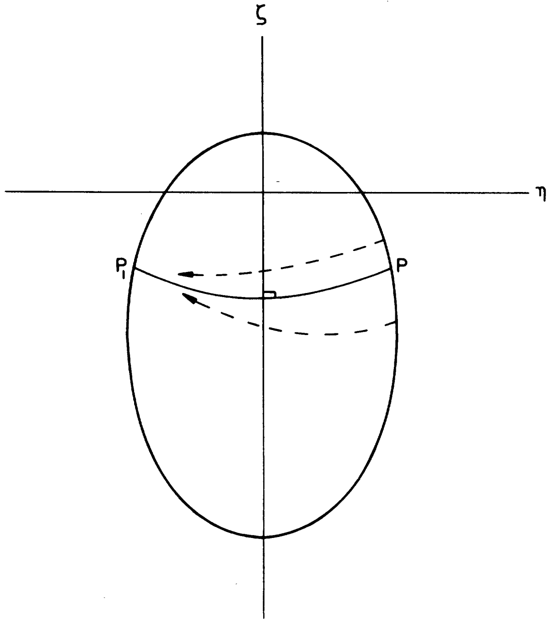

FIG. 1. Paths in the (η, ζ) plane starting from bounding ellipse. Periodic solutions of strong-coupled transverse-longitudinal waves are represented by the line joining P and P_1; nonperiodic solutions are represented by the dashed lines.

$E_y'=(1-n^2)^{-1/2}E_y$, $E_z'=E_z$, and $B_x'=(n/c)(1-n^2)^{-1/2}E_y$. Figure 2 shows an example of the solution.

When discussing a specific solution, it is necessary to decide what value should be assigned to the parameter D. For a stationary electron plasma, (4) shows that $D=-n(1-n^2)^{-1/2}$. In general, however, D cannot be preselected; rather, it should be regarded as an adjustable quantity,[2,10] a proper choice of which will yield a self-consistent solution with prescribed values of wave amplitude, wave velocity, and plasma stream velocity. For small wave amplitudes, the values of D are close to its equilibrium value, which for an electron-ion plasma streaming in the z direction is, from (4), given by

$$D=(1+\mu)(V_{sz}'/c-n)[(1-n^2)(1-V_{sz}'^2/c^2)]^{-1/2}. \qquad (14)$$

In the limit $\eta_0'\gg 1$ and $n\ll 1$, Max[2] found that in order to satisfy the condition $\langle N_i' V_{iz}'\rangle = V_{sz}'=0$, the following equation must be satisfied:

$$1-\tfrac{3}{4}\ln 2+\frac{\alpha}{16}\ln\left[\left(\rho_0^{-2}+\frac{n^2\alpha^2}{4}\right)\left((\mu\rho_0)^{-2}+\frac{n^2\alpha^2}{4}\right)\right]=0, \qquad (15)$$

where $\rho_0=\eta_0$ and $\alpha=-(1-n^2)^{1/2}(n\mu\eta_0)^{-1}D$. Thus, (15) is an implicit relation for D in terms of η_0 and n. In a streaming plasma, D also depends on $\vec{V}_s'$.

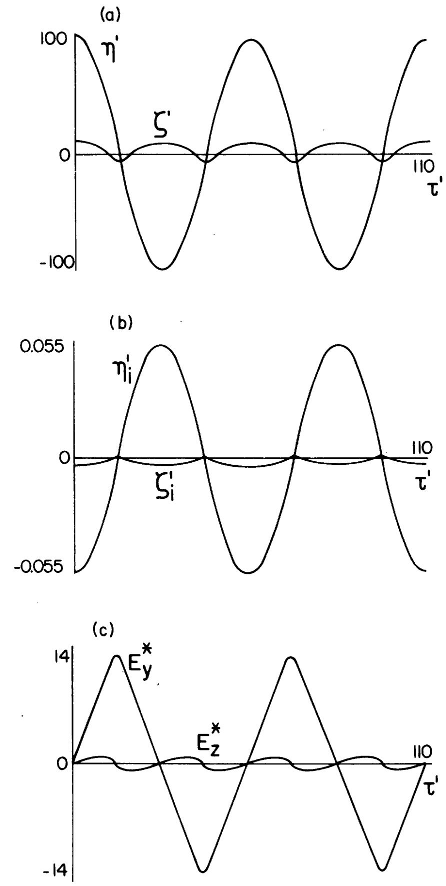

FIG. 2. Variation with $\tau'=\omega_{pe}'(t'-nz'/c)$ of (a) η' and and ζ', (b) η_i' and ζ_i', (c) $E_y^*=(e/m_e\omega_{pe}'c)E_y'$ and $E_z^*=(e/m_e\omega_{pe}'c)E_z'$ for $n=0.1$, $\eta_0'=100$, $V_s'=0$, and $\mu=\frac{1}{1837}$.

Next, the significance of the plasma stream velocity in the nonlinear theory is considered. It is easy to show that $\vec{V}_s'$, as defined by (1), Lorentz transforms like any velocity. Suppose there are two frames S' and S, where S has a velocity $V\hat{z}$ relative to S'. Then, the averaged particle fluxes and number densities in two frames are related by

$$\langle N\vec{v}_\perp\rangle=\langle N'\vec{v}_\perp'\rangle, \quad \langle Nv_z\rangle=\Gamma(\langle N'v_z'\rangle - V\langle N'\rangle), \qquad (16)$$

$$\langle N\rangle=\Gamma(\langle N'\rangle - V\langle N'v_z'\rangle/c^2), \qquad (17)$$

where $\Gamma=(1-V^2/c^2)^{-1/2}$. Note that phase averaging is a Lorentz-invariant operation. Dividing (16) by (17) gives

$$V_{s\perp}=\frac{\langle Nv_\perp\rangle}{\langle N\rangle}=\frac{V_{s\perp}'}{\Gamma(1-VV_{sz}'/c^2)},$$
$$V_{sz}=\frac{\langle Nv_z\rangle}{\langle N\rangle}=\frac{V_{sz}'-V}{1-VV_{sz}'/c^2}, \qquad (18)$$

which indeed satisfies the Lorentz transformation for velocities. This suggests that if in S' the plasma has a certain stream velocity $\vec{V}_s'$, then an observer in S that has a velocity $\vec{V}_s'$ relative to S' will "see" a stationary plasma with $\vec{V}_s=0$.

As an example, the following nonlinear dispersion relation, valid for $\eta_0'\gg 1$ and $n\ll 1$, was obtained by Max[2] for strong-coupled transverse-longitudinal waves in an electron-ion plasma, with $\langle N'v_z'\rangle=0$ in the laboratory frame S':

$$n^2=1-\pi\omega_{pe}'^2(\nu\omega'^2)^{-1}, \qquad (19)$$

where $\nu=eE'_{y\,\max}/m_e\omega'c$. It follows from the above discussions that if the plasma has a stream velocity $V_s'\hat{z}$, for example, in S', (19) can then be Lorentz transformed to give

$$n^2=1-\pi(1-V_s'^2/c^2)^{1/2}\omega_{pe}'^2(\nu\omega'^2)^{-1}, \qquad (20)$$

by noting that ν, ω_{pe}^2/γ_e, and $\omega^2-c^2k^2$ are invariant. Note that the above procedure may be applied to any electromagnetic wave. Therefore, it can be concluded that the nonlinear dispersion relation for a streaming plasma can be obtained from the nonlinear dispersion relation for a stationary plasma by a Lorentz transformation.

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