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Hydrodynamic theory of biaxial nematics

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Biaxial nematics are usually regarded as characterized by a symmetric, traceless tensor having three different eigenvalues. It is argued that this definition is too restrictive and that a natural extension would embrace any system that breaks all three rotational symmetries while preserving translational invariance. That is, biaxial nematics need not have orthorhombic symmetry, but may be triclinic, hexagonal, cubic, or even isotropic. A nonlinear hydrodynamic theory is presented which emphasizes the fundamental similarities between the different biaxial nematics and clarifies the changes in the static and dynamic behavior as the discrete symmetries vary. The Goldstone modes of any biaxial nematics are identified as two pairs of orbital waves with a complex, and one orbital diffusion with a purely imaginary, dispersion relation. If the longitudinal and transverse variables decouple, it is the longitudinal rotation angle that diffuses.

I. INTRODUCTION

Recently, a biaxial nematic phase in potassium-laurato-1-decanol—water mixture was observed and reported by Yu and Saupe.[1] This confirms the long-held belief of its existence and opens a wide field of new and interesting phenomena for future study.[2,3] The purpose of this paper is to present the hydrodynamic theory of biaxial nematics, defined as a system that breaks all three rotational symmetries but none of the translational ones. Depending on the discrete symmetries, these systems can vary widely. Usually, biaxial nematics are visualized as consisting of oriented ellipsoids with three different axes, or equivalently, of oriented bricks.[2] This paper refers to this type of liquid crystals as "orthorhombic nematics." "Biaxial nematics," on the other hand, will be reserved as a general name for any system that breaks all three rotational symmetries,[3] in order to distinguish such a system from the ordinary, uniaxial nematics breaking only two. In addition to the biaxial nematics having either orthorhombic or other familiar symmetries such as triclinic, hexagonal, or cubic ones we may also encounter those with a symmetry group forbidden in the lattice. For instance, there is no reason to exclude a fivefold or sevenfold symmetry, or more exotically, the point group of an icosahedron. This last case will turn out to be an especially simple, yet complete model system that illustrates nicely the generally complex collective behavior of biaxial nematics.

Any systems that spontaneously break the same continuous symmetries obey hydrodynamic equations of identical structure.[4] They have the same variables and are characterized by equal number of propagating and diffusive modes. In comparison, the discrete symmetries are less consequential. They determine the number of independent elastic and transport coefficients and are thus relevant to the question whether certain modes are coupled, or others are degenerate. The dynamics of solids[4,5] serves as a well-known example here. Any solid has, in addition to the conserved quantities, the displacement vector as three extra hydrodynamic variables to account for the broken translational symmetries. And irrespective of the crystal point group, the collective modes are three pairs of elastic waves, a heat and a defect diffusion.[6] However, due to the difference in the discrete symmetries, glass has two independent elastic constants, while a triclinic crystal has 21.

The situation is completely analogous in biaxial nematics. Here, the additional hydrodynamic variables are the generator of the three infinitesimal rotational angles $d\vec{\theta}$ and, as will be shown below, the collective modes are a pair of sound, two pairs of orbital wave, a heat and an orbital diffusion. If the longitudinal and transversal variables decouple, as they do for certain directions of the wave vector, it is the longitudinal angle (denoting the rotation around the wave vector) that diffuses, while the transversal angles and velocities join to form the modes of orbital wave.

One point needs to be clarified in this context. As is becoming customary in the literature of uniaxial nematics[2,4] and superfluid ^{3}He,[7,8] orbital wave denotes those collective modes that have the form

$$\omega/q^2 = \pm\alpha - i\beta ,$$

where α and β are simple positive functions of various elastic and transport coefficients. In the uniaxial nematic substances known to us, the real part α is rather small and the orbital wave decomposes into a slow mode of orbital diffusion and a fast one of velocity diffusion.[4] However, there is no general symmetry argument to enforce such a behavior. And in little damped systems, uniaxial or biaxial, α may well be much larger than β, rendering the orbital wave propagating. In contrast, orbital diffusion, generally of the form

$$\omega/q^2 = -i\beta ,$$

is always strictly evanescent, independent of material properties.

The biaxial nematodynamics described above will be shared, quite obviously, by any system consisting of identical objects that are devoid of continuous symmetries, if they are orientationally ordered but not positionally.[9] And by altering the structure of the elastic and transport tensors, the different discrete symmetries of these objects also enter the continuous mechanical description of biaxial nematics. However, there is no one-to-one correspondence between the symmetry groups and the sets of hydrodynamic equations. This is because we are dealing with finite rank tensors, and the lower the rank, the less discriminate the tensors become with respect to the different symmetry groups. For instance,[5] a tensor of the rank n does not distinguish a m-fold symmetry axis from a continuous one, if $m > n$. *Hence, a very restricted number of slightly different sets of hydrodynamic equations is sufficient to cover the infinite variety of biaxial nematics.* We may, for instance, encounter "quasi-isotropic nematics", liquids consisting of oriented objects (such as icosahedrons), which, though lacking in any continuous symmetries, are symmetric enough to have their elastic and transport tensors mimic isotropic behavior. (In this sense, hexagonal nematics may be called quasi-uniaxial.) Quasi-isotropic nematics are very simple systems, much simpler even than the ordinary uniaxial nematics. They are characterized by only two elastic and one transport coefficients, in addition to those already present in an isotropic liquid. Yet their collective behavior is biaxially nematic in an essential way, highlighting the relevant features just as the elastic theory of glass does within the group of solid systems. A closer examination of this model nematics, irrespective of whether it in fact exists, is likely to enhance our intuitive understanding of any biaxial nematics.

Next we shall discuss the relation between biaxial nematics and the various superfluid phases of ^{3}He, each of those breaking an array of different continuous symmetries. The B phase, for example, breaks phase symmetry and three relative spin-orbit symmetries.[7,10] It is invariant under a certain combination of spin and orbital space rotation, but not under any other rotation that deviates from this combination—such as one in the orbital space alone. In other words, if a strong spin-orbit coupling is turned on to eliminate the spin density $\vec{S}$ as an independently conserved quantity[11] (i.e., setting $\vec{\omega} = \partial\epsilon/\partial\vec{S} = 0$), and if the superfluid density is set to zero, the B phase, still breaking the orbital rotational symmetry, will behave as a biaxially nematic system, incidentally, as an isotropic one. Not surprisingly, the nematodynamical equations to be presented in the next chapter emerge out of the B-phase dynamics[10] by this prescription. In a much more restricted sense, something similar can be said about the two other superfluid states of ^{3}He: Both the A and A_1 phases "contain" the biaxially nematic case,[12] though its extraction is not easily interpreted in physical terms as in the B phase.

In the next section, we shall first derive the nonlinear biaxial nematodynamic equations, which are valid in the presence of arbitrary nonsingular textures; then we shall discuss the explicit dependence of the elastic and transport tensors on the discrete symmetries. Finally, the complete spectrum of collective modes is obtained for quasi-isotropic nematics generally and for orthorhombic ones with selected directions of the wave vector.

II. THE HYDRODYNAMIC EQUATIONS AND THE COLLECTIVE MODES OF BIAXIAL NEMATICS

We start by investigating the general static properties of biaxial nematics, without specifying the discrete symmetry of the system under investigation. We may take the entropy density s as a function of θ_i, $\nabla_i\theta_j$, and the conserved quantities which are the densities of energy, mass, and momentum, being ϵ, ρ, and $\vec{g}$, respectively. This is customarily expressed in the form

$$d\epsilon = T ds + \mu d\rho + \vec{v}\cdot d\vec{g} + \psi_{ij} d\nabla_i\theta_j + \phi_i d\theta_i \quad . \tag{1}$$

Although the rotation vector $\vec{\theta}$ is only infinitesimally well defined, it is quite possible, even advantageous, to work with them if due attention is paid to a commutation rule which has become known as the Mermin-Ho relation[13] in the literature of superfluid ^{3}He:

$$(\delta\nabla - \nabla\delta)\vec{\theta} = \delta\vec{\theta}\times\nabla\vec{\theta} \ , \tag{2}$$

where δ and ∇ stand for any first-order differential operator such as ∇_i or $\partial/\partial t$. The equilibrium texture is given by minimizing the energy, leading to

$$\Psi_i = 0 \ , \tag{3}$$

where the molecular field Ψ_i is defined as

$$\Psi_i = \nabla_j\psi_{ji} - \phi_i + \epsilon_{ijk}\psi_{lj}\nabla_l\theta_k \quad .$$

The last term in this expression follows directly from the Mermin-Ho relation, Eq. (2). To be more explicit, we go on to consider the thermodynamic conjugate variables, v_i, μ, T, ψ_{ij}, and ϕ_i that are also functions of the hydrodynamic variables. By virtue of the Galilean invariance, we still have

$$\vec{v} = \vec{g}/\rho \ .$$

Assuming that T and μ continue to be characterized by the three usual susceptibilities

$$\frac{\partial T}{\partial s}, \ \frac{\partial T}{\partial \rho} = \frac{\partial \mu}{\partial s}, \ \frac{\partial \mu}{\partial \rho} \ ,$$

the new part is given by the gradient energy

$$\epsilon_g = \tfrac{1}{2} K_{ijkl}\nabla_i\theta_j\nabla_k\theta_l \tag{4}$$

obtained by expansion to leading orders in $\nabla_i\theta_j$. Note that in nematics less symmetric than orthorhombic ones, terms of the form $\delta\rho\nabla_i\theta_j$ or $\delta s\nabla_i\theta_j$ are not excluded by the symmetry. However, these terms are "self-destructive": They result in nonuniform textures even if the temperature or pressure are changed uniformly. And these equilibrium textures, reminiscent of cholesterics, are no longer biaxially nematic, since they are likely to break the translational symmetries, too. Therefore, we shall take ϵ_g as the only textural contribution to the energy, Eq. (1). In this case, with

$$\psi_{ij} = K_{ijkl}\nabla_k\theta_l \ , \tag{5}$$

$$\phi_i = \tfrac{1}{2}\nabla_j\theta_m\nabla_k\theta_l\frac{\partial K_{jmkl}}{\partial\theta_i} \ , \tag{6}$$

and

$$\partial K_{ijkl}/\partial\theta_m = K_{njkl}\epsilon_{nim} + K_{inke}\epsilon_{njm} + K_{ijnl}\epsilon_{nkm} + K_{ijkn}\epsilon_{nlm} \ , \tag{7}$$

the molecular field is explicitly given as

$$\Psi_i = K_{ijkl}\nabla_j\nabla_k\theta_l + \nabla_k\theta_l\nabla_j\theta_m(\epsilon_{qjm}K_{iqkl} + \epsilon_{qjl}K_{qmik} + \epsilon_{qlm}K_{kqij}\epsilon_{qim}K_{qjkl} + \epsilon_{qij}K_{qmkl}) \quad . \tag{8}$$

We may now proceed to the dynamical part by setting up the equations of motion for the eight hydrodynamic variables. They are

$$\dot{\rho} + \vec{\nabla}\cdot\vec{g} = 0 \ , \tag{9}$$

$$\dot{s} + \nabla_i(s v_i - f_i^R) = R/T \ , \tag{10}$$

$$\dot{g}_i + \nabla_i P - \tfrac{1}{2}(\vec{\nabla}\times\vec{\Psi})_i + \nabla_k(\Pi_{ik}^e - \Pi_{ik}^R) = 0 \ , \tag{11}$$

$$\dot{\theta}_i + (\vec{v}\cdot\vec{\nabla})\theta_i - \tfrac{1}{2}(\vec{\nabla}\times\vec{v})_i - Y_i^R = 0 \ . \tag{12}$$

All the fluxes with superscript R are contained in the expression for the entropy production

$$R = f_i^R\nabla_i T + \Pi_{ij}^R v_{ij} + Y_i^R\Psi_i \ , \tag{13}$$

and given by expansion in the forces $\nabla_i T$, $v_{ij} = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i)$, and Ψ_i:

$$\begin{bmatrix} f_i^R \\ \Pi_{ij}^R \\ Y_i^R \end{bmatrix} = \begin{bmatrix} \kappa_{ik} & -\beta_{ikl} & \alpha_{ik} \\ \beta_{kij} & \nu_{ijkl} & -\lambda_{ijk} \\ \alpha_{ki} & \lambda_{kli} & \gamma_{ik}^{-1} \end{bmatrix} \begin{bmatrix} \nabla_k T \\ v_{kl} \\ \Psi_k \end{bmatrix} \ . \tag{14}$$

The elements of this (Onsager) matrix, generally referred to as transport coefficients, are phenomenological parameters, wave number and frequency independent. Under the assumption that these elements are even under time inversion, the Onsager reciprocal relation has already been incorporated. If some of them are odd, such as due to a possible (though less likely) existence of a preferred direction that is itself odd under time inversion, an additional minus sign has to be added in front of these coefficients located in the upper half of the matrix. Finally, the nonlinear Erikson[2] stress tensor has the form

$$\Pi^{e}_{ik} = g_i v_k + \psi_{kj} \nabla_i \theta_j \ . \tag{15}$$

With all the fluxes of the hydrodynamic equations of motion, Eqs. (9)–(12), known and given in terms of the hydrodynamic variables, their spatial derivatives, and the phenomenological parameters of elastic and transport coefficients, the hydrodynamic equations of biaxial nematics is closed and the theory complete. So far, only the information of the three broken continuous rotational symmetries has been utilized. Next, we shall investigate the influence of the discrete symmetries. This information enters the static properties only through the elastic tensor K_{ijkl}, while it affects the dynamics also via the transport matrix of Eq. (14). Being expansion coefficients, these two matrices have to reflect the symmetry of the system. This determines their structures, especially the number of independent elements.

Starting again with the static part, we shall first examine the elastic tensor K_{ijkl}. Employing Eq. (2), one can show that

$$\tfrac{1}{2}(K_{ijkl} - K_{kjil}) \nabla_i \theta_j \nabla_k \theta_l$$

yields only surface contribution. With the Maxwell relation

$$K_{ijkl} = K_{klij} \ ,$$

we can, therefore, take the elastic tensor to be invariant under the following permutations of indices:

$$K_{ijkl} = K_{kjil} = K_{klij} = K_{ilkj} \ . \tag{16}$$

(Note the difference to the corresponding relations in crystals.[5]) To account for large angle variations, a set of three orthonormal vectors,

$$e_i^{\alpha} \ \text{with} \ \alpha = 1,2,3$$

can be introduced and attached to the local orientation of the system given by the axes of symmetry, i.e.,

$$\delta \hat{e}^{\alpha} = \delta \vec{\theta} \times \hat{e}^{\alpha} \ . \tag{17}$$

The elastic tensor is then given as

$$K_{ijkl} = K_{\alpha\beta\gamma\delta} e_i^{\alpha} e_j^{\beta} e_k^{\gamma} e_l^{\delta} \ , \tag{18}$$

where $K_{\alpha\beta\gamma\delta}$ displays the same invariance under permutation of indices as K_{ijkl}. The following symmetry considerations are generally valid only for $K_{\alpha\beta\gamma\delta}$. In a uniform texture, of course, $e_i^{\alpha} = \delta_{i\alpha}$, and there is no difference between the elastic tensors with latin and greek indices. Triclinic nematics have no further symmetries, hence all 36 elements given by Eqs. (11), out of a total of 81, are independent. With the respective numbers of independent elements in parenthesis, they are of the type

$$\begin{gathered} K_{\alpha\alpha\alpha\alpha}(3), \ K_{\alpha\beta\alpha\beta}(6), \ K_{\alpha\alpha\beta\beta}(3), \\ K_{\alpha\alpha\alpha\beta}(6), \ K_{\alpha\alpha\beta\alpha}(6), \\ K_{\alpha\beta\gamma\alpha}(6), \ K_{\alpha\beta\alpha\gamma}(3), \ K_{\beta\alpha\gamma\alpha}(3) \ . \end{gathered} \tag{19}$$

The symmetry group of orthorhombic nematics includes three mutually perpendicular mirror planes, leaving only 12 elements, all with an even number of same indices, nonvanishing:

$$K_{\alpha\alpha\alpha\alpha}(3), \ K_{\alpha\beta\alpha\beta}(6), \ \text{and} \ K_{\alpha\alpha\beta\beta}(3) \ . \tag{20}$$

Hexagonal nematics are characterized by six elastic coefficients:

$$\begin{gathered} K_{1111} = K_{2222} = K_{1212} + 2K_{1122} \ , \\ K_{1212} = K_{2121}, \ K_{3131} = K_{3232} \ , \\ K_{1133} = K_{2233}, \ K_{1313} = K_{2323}, \ \text{and} \ K_{3333} \ . \end{gathered} \tag{21}$$

Of these, the last three groups have to be set to zero to retrieve the Franck energy,[2] because uniaxial nematics do not resist a local rotation around the director. Cubic or quasi-isotropic nematics do not differ from their solid counterparts,[5] they are characterized by the same three or two elastic coefficients, respectively.

Next the dynamic part, i.e., the elements of the Onsager matrix, Eq. (14), will be examined. First we note that the invariance of the viscosity tensor, ν_{ijkl}, under permutations of indices is the same as that of the elastic tensor of crystals, enabling us to copy the corresponding results.[5] Thus triclinic, orthorhombic, hexagonal, cubic, or quasi-isotropic nematics have 21, 9, 5, 3, or 2 independent viscosities, respectively. In orthorhombic or more symmetric nematics both α_{ik} and β_{kij} vanish, while $\kappa_{\alpha\beta}$ and $(\gamma^{-1})_{\alpha\beta}$ are diagonal in the local frame given by the symmetry axes. (By virtue of the Onsager relation, the latter two tensors are symmetric and can always be diagonalized — though by no means necessarily in the same coordinates for less symmetric nematics.) For hexagonal nematics two eigenvalues of the tensor κ, for cubic and quasi-isotropic ones all three, are equal. The same is true for γ^{-1}. Finally, there is the reactive transport tensor λ which in orthorhombic nematics has the form

$$\lambda_{jki}=\Sigma_\alpha\lambda_\alpha(\epsilon_{pki}e_j^\alpha+\epsilon_{pji}e_k^\alpha)e_p^\alpha \;, \tag{22}$$

although only two coefficients are independent. This is because the vanishing term

$$\lambda\Sigma_\alpha(\epsilon_{pki}e_j^\alpha+\epsilon_{pji}e_k^\alpha)e_p^\alpha$$

with an arbitrary λ can be added to Eq. (22). Adopting the symmetric convention of zero trace, we have $\Sigma_\alpha\lambda_\alpha=0$, in orthorhombic nematics,

$$\lambda_1=\lambda_2=-\tfrac{1}{2}\lambda_3 \tag{23}$$

in hexagonal nematics, and

$$\lambda_1=\lambda_2=\lambda_3=0 \tag{24}$$

in cubic or isotropic ones. The last equation is quite consistent with an amusing result by Volovik,[14] who has shown that the corresponding coefficient in uniaxial nematics is 1 for sticks and -1 for discs, so it should be zero for cubes or balls.

Now we are in a position to calculate the collective modes. Putting the spatial variation in an orthorhombic nematic system along $\hat{e}^3$, we obtain two diffusive modes, one of $\sigma=s/\rho$, and another of θ_3,

$$\rho\dot{\sigma}-\kappa_3T''=0,\;\; \gamma_3\dot{\theta}_3-K_{3333}\theta_3''=0 \;, \tag{25}$$

one pair of sound,

$$\dot{\rho}+\rho v_3'=0,\;\; \rho\dot{v}_3+P'-\nu_{3333}v_3''=0 \;, \tag{26}$$

and two pairs of orbital waves,

$$\dot{\theta}_1-(K_{3131}/\gamma_1)\theta_1''+(\tfrac{1}{2}+\lambda_3-\lambda_2)v_2'=0 \;,$$

$$\rho\dot{v}_2-\nu_{2323}v_2''-(\tfrac{1}{2}+\lambda_3-\lambda_2)K_{3131}\theta_1''=0 \;, \tag{27}$$

where the second pair is obtained by substituting 2 for 1 and vice versa in the indices of Eq. (27) and putting a minus sign in front of the velocity v_1. The two pairs of orbital waves are obviously sustained by the transverse variables. The roots of Eq. (27) are given by

$$\omega/q^2=\pm(r^2-d_-^2)^{1/2}-id_+ \;, \tag{28}$$

where the three abbreviations

$$r=(\tfrac{1}{2}+\lambda_3-\lambda_2)(K_{3131}/\rho)^{1/2}$$

and

$$d_\pm=\tfrac{1}{2}(K_{3131}/\gamma_1\pm\nu_{2323}/\rho)$$

have been introduced. The solutions closely resemble those of orbital waves in uniaxial nematics,[4] they are purely imaginary if $r^2\leq d_-^2$, and propagating if $r^2>>d_\pm^2$. In a weakly damped system, where the dissipative coefficients γ_1^{-1} and ν_{2323} approach zero, $d_\pm$ will be very small, too, while r, consisting of reactive coefficients only, is not effected. In such a system, orbital waves will propagate. In constrast, the diffusion of the longitudinal rotation angle θ_3 is general, irrespective of any inequalities of material dependent parameters.

Through cyclic changes of all the indices of Eqs. (25), (26), and (27), the corresponding results for the spatial variation along the other two symmetry axes can be easily generated, where 1→2, 2→3, and 3→1 lead to the result along $\hat{e}^1$ and 1→3, 2→1, and 3→2 to those along $\hat{e}^2$. The dispersion relation of the modes along these three axes already involve most of the elastic and transport coefficients of orthorhombic nematics: Out of a total of 12 elastic and 17 transport coefficients only three each, K_{1122}, K_{1133}, K_{2233}, and the viscosities with the same subscripts do not contribute. In quasi-isotropic nematics, there are no preferred directions, hence, Eqs. (25) — (27) are generally valid. With λ_α vanishing and $\gamma_3=\gamma_1$, quasi-isotropic nematics are very simple systems, characterized by only two elastic (K_{3333},K_{3131}) and one transport coefficients (γ_3) in addition to those already present in an isotropic liquid. In this system, it is always the longitudinal rotation angle that diffuses. In an orthorhombic system, on the other hand, when the direction of the wave vector is less symmetric, the structure of the spectrum will be more complicated. However, as is easy to see, with α and β of the Onsager matrix vanishing, first sound and heat diffusion remain uncoupled to the other modes, hence, alterations in the spectrum arise from couplings between the longitudinal rotation angle and the transverse variables. And due to time inversion properties,[4] one of the five modes remain diffusive. Under the assumption that the transport coefficients themselves are even under time inversion [cf. remark below Eq. (14)], the only reactive coupling of the angles is to the velocity, hence, this diffusive mode must be an orbital one, involving a linear combination of the three angles only.

We can compare these results with the spectrum of the uniaxial nematics.[4] There, in addition to first sound and heat diffusion, we have two pairs of orbital waves for any direction of the wave vector except when it is perpendicular to the director, for which case the one director motion corresponding to the longitudinal rotation also diffuses. So the qualitative new feature of biaxial nematics is that there is *always* a purely diffusive mode, which we may identify as the Goldstone mode of the uniaxial-biaxial transition.

III. CONCLUSIONS

There is an infinite variety of biaxial nematics. From the least symmetric triclinic ones over the popular orthorhombic system to the quasi-isotropic nematics possessing the point group of, say, icosahedrons, all obey a few sets of closely related hydrodynamic equations. On the other hand, the number of phenomenological parameters in these systems vary considerably. While triclinic nematics are described by 36 independent elastic and double as many transport coefficients, orthorhombic nematics manage with 12 and 14, and isotropic nematics get along with only 2 and 1, all in addition to those already present in an isotropic liquid. A general hydrodynamic theory is presented that describes all these systems and is valid in the presence of arbitrary nonsingular textures. The equations are solved for special cases to obtain the spectrum of collective modes. They are the usual pair of sound and a heat diffusion, in addition to two pairs of orbital waves and one orbital diffusion. If the longitudinal and transverse variables decouple, as they do generally in isotropic nematics, and also for special directions of the wave vector in orthorhombic ones, it is the longitudinal angle that diffuses, while the transverse angles and velocities join to form the modes of orbital waves.

Note added in proof. I have received two unpublished manuscripts meanwhile, by W. M. Saslow and by H. Brand and H. Pleiner. Both are about the case of orthorhombic nematics. The spirit of all three works is similar though discrepancies exist. They can be seen by comparing the number of independent elastic and transport coefficients. For instance, both seem to need three, instead of two, independent elements in λ_{jki}, cf. Eq. (22). In addition, Brand and Pleiner claim to find 15 rather than 12 different elastic coefficients.

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[3]V. Poénarn and G. Toulouse, J. Phys. (Paris) 8, 887 (1977); N. D. Mermin, Rev. Mod. Phys. 51, 592 (1979). Mermin was probably the first to have used the word of biaxial nematics the way it is employed in this paper. In the group theoretical language of his article biaxial nematics is a system with an order parameter space that can be taken as SO(3)/H, where H is a *discrete* subgroup of SO(3). The ordinary nematics is excluded because H is a continuous subgroup.

[4]P. C. Martin, P. Parodi, and P. S. Pershan, Phys. Rev. A 6, 2402 (1972); F. Jähnig and H. Schmidt, Ann. Phys. (N.Y.) 71, 129 (1972).

[5]L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Pergamon, Oxford, 1970).

[6]One might think of isotropic ferromagnets and antiferromagnets as an exception to this rule: Despite the fact that both share the same broken continuous symmetries, their collective spectrum differs considerably. It is, of course, the "pathological" behavior of ferromagnets to have the conserved magnetization acting simultaneously as the order parameter that spoils the analogy. In superfluid helium or crystals this pathology would correspond to the degeneracy of phase and number density or displacement vector and momentum density, respectively; cf. B. I. Halperin and P. C. Hohenberg, Phys. Rev. 188, 898 (1969).

[7]A. J. Leggett, Rev. Mod. Phys. 47, 331 (1975).

[8]W. F. Brinkman and M. C. Cross, in *Progress in Low Temperature Physics* (North-Holland, Amsterdam, 1978), Vol. 7.

[9]Note that the symmetry of the building blocks is by no means a necessary condition for the possibility of a nematic mesophase of the same symmetry. Even a system of isotropic constituents may — with an appropriate short-range positional arrangement — become, say, hexagonal nematics; cf. A. Zippelius, B. I. Halperin, and D. R. Nelson, Phys. Rev. B 22, 2514 (1980).

[10]M. Liu and M. C. Cross, Phys. Rev. Lett. 41, 250 (1978).

[11]Setting $\vec{\omega}$ to zero is the straightforward way to eliminate the spin density as a hydrodynamic variable, though one should keep in mind that what actually happens is more subtle: A spin-orbit coupling destroys spin conservation and, at the same time, gives rise to an antisymmetric part of the stress tensor, such

that the total angular momentum density $\vec{s}+\vec{r}\times\vec{g}$ remains conserved. For time scales large compared to the spin relaxation time, i.e., in the proper hydrodynamic regime, $\vec{\omega}$ assumes the value of the local vorticity, hence, $\vec{g}$ and $\vec{s}$ combine to yield only one independent hydrodynamic variable, $\vec{g}+\frac{1}{2}\,\vec{\nabla}\times\vec{s}$, for which the stress tensor is again symmetric; cf. Ref. 4.

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