Rigorously diffusive deterministic map

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The statistical properties of periodic impulse maps may be obtained from the characteristic functions. Series representations for the characteristic functions, the force correlations, and the momentum diffusion coefficient are presented. These results are applied to the sawtooth map for integer values of the perturbation parameter ϵ, in which case the series may be summed explicitly. It is found that the diffusion coefficient has the quasilinear value for $|\epsilon+2|\geq 2$, it vanishes for $\epsilon=-2$ and -1, and it is infinite for $\epsilon=-3$.

I. INTRODUCTION

"Diffusion is a distinctive random process." (See Ref. 1, p. 326). Hence, the idea that a diffusion coefficient can exist for a deterministic system seems paradoxical. Nevertheless, numerical experiments affirm that deterministic motion can be diffusive with the diffusion coefficient given approximately by the quasilinear value when the nonlinear parameter is large (Ref. 1, Sec. 5). Furthermore, analytical calculations of corrections to the quasilinear value of the diffusion coefficient[2–5] have provided formulas which agree very closely with numerical results. These high-order analytical calculations are equivalent[3] to keeping only the first few terms of the representation of the diffusion coefficient D as an infinite series of force correlations.

In Ref. 2 and 5 external noise was added to the system, and the systems under study were not deterministic. In Ref. 3 a formalism was developed for the noiseless, deterministic case in which the existence of D is no longer certain, in part because of the presence of regular accelerating regions[1,5] of phase space. However, it is reasonable to expect that the diffusion coefficient exists for the stochastic region of phase space. The method of Ref. 3 allows the calculation of D for the stochastic region alone. Still, it has not been proven that the diffusion coefficient for the stochastic region of phase space is finite.

In the present paper we present a deterministic system for which the diffusion coefficient is a nonzero, finite number. This system is the sawtooth map

$$p'=p+\epsilon\, s(x), \quad x'=x+p',$$

where $s(x)$ is given by Eq. (18) and is shown in Fig. 1. When $|\epsilon+2|>2$ the sawtooth map is a C system (Ref. 6, pp. 53–55). When $|\epsilon+2|\geq 2$ and ϵ is an integer we show that the momentum diffusion coefficient is given exactly by its quasilinear value $(\pi\epsilon)^2/6$. Letting ϵ be an integer allows summation of the series for the diffusion constant because only one of the g_l [see Eq. (4)] is nonzero.

In Sec. II we develop the characteristic function formalism[3] for periodic impulse maps. We obtain series representations for the characteristic functions, the force correlations, and the diffusion coefficient. These results are applied to the sawtooth map in Sec. III.

Exact statistical results have also been obtained for a deterministic dissipative mapping by Jensen and Oberman,[7] also by the use of a characteristic function formalism.

II. PERIODIC IMPULSE MAPS AND CHARACTERISTIC FUNCTIONS

We consider a map M defined by

$$p_{n+1}=p_n+\epsilon f(x_n), \tag{1a}$$

$$x_{n+1}=x_n+p_{n+1}, \tag{1b}$$

where $f(x)$ is periodic and (without loss of generality) has the period 2π. M is called an impulse map because it corresponds to a particle receiving an impulse (momentum change) at integer values of time, Eq. (1a), while otherwise freely streaming, Eq. (1b). The standard (or Chirikov-Taylor) map[1,2] is an example of such a map with $f(x)=\sin x$. The map M is area preserving for arbitrary f. Impulse maps can be easily written as second-order difference equations

$$x_{n+1}-2x_n+x_{n-1}=\epsilon f(x_n). \tag{2}$$

Periodic impulse maps are actually doubly

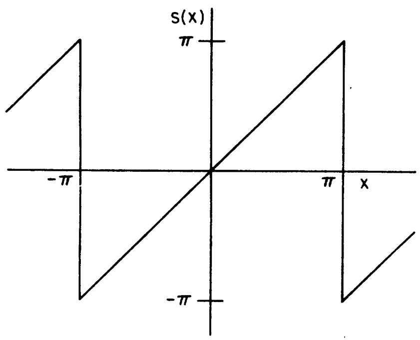

FIG. 1. Plot of the sawtooth function $s(x)$ vs x.

periodic. Consider two initial conditions which differ in momentum and position by multiples of 2π: $x_0'=x_0+2\pi n_0$; $p_0'=p_0+2\pi m_0$ for some integers m_0 and n_0. It is easy to see that Eq. (1) implies that the orbits (x_k',p_k') and (x_k,p_k) always differ by multiples of 2π:

$$x_k'(x_0',p_0')=x_k(x_0,p_0)+2\pi n_k$$

and

$$p_k'(x_0',p_0')=p_k(x_0,p_0)+2\pi m_0.$$

This periodicity allows one to determine many of the properties of M by studying the reduced map, i.e., by letting the phase space be a torus with sides of length 2π. We denote this torus by T.

To analyze M we use the characteristic function method of Ref. 3. This method relies on the periodicity of f to expand both f and its exponential in a Fourier series:

$$f(x)=\sum_{l=-\infty}^{\infty} f_l e^{ilx}, \quad (3)$$

$$e^{i\epsilon f(x)}=\sum_{l=-\infty}^{\infty} g_l(\epsilon)e^{ilx}. \quad (4)$$

We assume that $f_0=0$ so that the average impulse on a uniform distribution of particles vanishes. This permits the definition of a momentum diffusion coefficient.

The characteristic functions for some measurable region R in the phase space are defined by

$$\chi_k^R(m_0,m_1,\ldots,m_k) \equiv \left\langle \exp\left[i\sum_{j=0}^{k} m_j x_{n+j}(x_0,p_0)\right]\right\rangle_R, \quad (5)$$

where x_{n+j} is a function of the initial conditions (x_0,p_0) through Eq. (1). The angular brackets denote an average which is defined by

$$\langle F(x_0,p_0)\rangle_R \equiv \frac{1}{\mu(R)}\int_R dx_0 dp_0 F(x_0,p_0), \quad (6)$$

where the integrals are over the region R and $\mu(R)$ is the measure of R. We have assumed that R is an invariant region of phase space $[M(R)=R]$ so that χ_k^R is independent of the label n.

Characteristic functions completely determine the statistical properties of M. To see this define the joint probability distributions

$$P_k^R(y_0,y_1\ldots,y_k) \equiv \left\langle \prod_{j=0}^{k} \bar{\delta}(y_j - x_j(x_0,p_0))\right\rangle_R, \quad (7)$$

where $\bar{\delta}$ is the periodic, Dirac δ function:

$$\bar{\delta}(x) \equiv \sum_{n=-\infty}^{\infty} \delta(x-2\pi n). \quad (8)$$

P_k^R is the probability of finding a particle at y_j at time $j\,(j=0,1,\ldots,k)$ given that it was initially in R. It is easy to see that χ_k^R is merely the Fourier transform of P_k^R:

$$\chi_k^R=\int_{-\pi}^{\pi}dy_0\int_{-\pi}^{\pi}dy_1\ldots\int_{-\pi}^{\pi}dy_k \exp\left[i\sum_{j=0}^{k} m_j y_j\right] P_k^R(y_0,\ldots,y_k). \quad (9)$$

As was shown in Ref. 3, the calculation of χ_k^R follows from the recursion relation

$$\chi_k^R(m_0,m_1,\ldots,m_k)=\sum_{l=-\infty}^{\infty} g_l(m_k\epsilon)\chi_{k-1}^R(m_0,m_1,\ldots,m_{k-3},m_{k-2}-m_k,m_{k-1}+2m_k+l). \quad (10)$$

This is obtained by inserting Eqs. (2) and (4) into Eq. (5). The characteristic function χ_1^R is given by the explicit phase-space average

$$\chi_1^R(m_0,m_1)=\langle \exp[i(m_0+m_1)x_0 - i m_0 p_0]\rangle_R. \quad (11)$$

The simplest form for χ_1^R is obtained if R is the torus T, which is an invariant region for the reduced map

$$\chi_1^T(m_0,m_1)=\delta_{m_0,0}\delta_{m_1,0}. \quad (12)$$

If M were ergodic then T would be the unique invariant region. In the general case, however, there may be many invariant regions (e.g., the standard map).

Consider the class of characteristic functions with only the first and last two indices nonzero. The recursion relation (10) maps this class onto itself:

$$\chi_k^R(m_0,0,\ldots,0,m_{k-1},m_k)=\sum_{l=-\infty}^{\infty} g_l(m_k\epsilon)\chi_{k-1}^R(m_0,0,\ldots,0,-m_k,m_{k-1}+2m_k+l). \tag{13}$$

Repeated application of Eq. (13) allows an explicit solution for

$$\chi_k^R(m_0,0,\ldots,0,m_{k-1},m_k)=\sum_{l_1=-\infty}^{\infty}\cdots\sum_{l_{k-1}=-\infty}^{\infty} g_{-m_{k-1}-2m_k+l_1}(m_k\epsilon)$$
$$\times g_{m_k-2l_1+l_2}(l_1\epsilon)g_{l_1-2l_2+l_3}(l_2\epsilon)\cdots$$
$$\times g_{l_{k-3}-2l_{k-2}+l_{k-1}}(l_{k-2}\epsilon)\chi_1^R(m_0-l_{k-2},l_{k-1}). \tag{14}$$

The momentum diffusion coefficient is defined by

$$D^R\equiv\lim_{k\to\infty}\frac{1}{2k}\left\langle[p_k(x_0,p_0)-p_0]^2\right\rangle_R \tag{15}$$

and can be written in terms of the force correlation

$$C_j^R\equiv\langle f(x_i)f(x_{i+j})\rangle_R, \tag{16a}$$

$$D^R=\epsilon^2\left[\tfrac{1}{2}C_0^R+\lim_{k\to\infty}\sum_{j=1}^{k-1}\left[1-\frac{j}{k}\right]C_j^R\right], \tag{16b}$$

$$=\epsilon^2\left[\tfrac{1}{2}C_0^R+\sum_{j=1}^{\infty}C_j^R\right], \tag{16c}$$

where the last expression is valid when the C_j^R falls off rapidly enough with j. Inserting Eqs. (3) and (5) into Eq. (16a) we find

$$C_0^R=\langle f^2(x_0)\rangle_R,$$
$$C_j^R=\sum_{m,n}f_mf_n\chi_j^r(m,0,\ldots,0,n). \tag{17}$$

The term C_0^R yields the "quasilinear" value for D^R. Note that Eq. (14) combined with Eq. (17) yields an explicit solution for C_j^R.

The characteristic function (5) and correlation functions (16a) are well defined regardless of the properties of the map. The expression for D^R, however, is valid only if the invariant region is chosen so that the series (16) converges. It is generally believed that D^R exists if R is chosen to be a stochastic region of phase space.

III. SAWTOOTH MAP

For the sawtooth map we let $f(x)=s(x)$ where

$$s(x)=x,\ -\pi<x\le\pi,$$
$$s(x+2\pi n)=s(x), \tag{18}$$

is the sawtooth function (Fig. 1). This map is a C system[6] when $|\epsilon+2|>2$ and is therefore ergodic in the torus

$$T=\{x,p,\mid -\pi<x\le\pi,-\pi<p\le\pi\}.$$

The function f_l and $g_l(\epsilon)$ become, by Eqs. (3) and (4),

$$f_l=\begin{cases}(-1)^l\dfrac{i}{l}, & l\neq0\\ 0, & l=0\end{cases}$$
$$g_l(\epsilon)=\frac{\sin[\pi(\epsilon-l)]}{\pi(\epsilon-l)}. \tag{19}$$

We can, therefore, immediately obtain the correlation functions from Eqs. (14) and (16):

$$C_{j+1}=\left[\frac{-1}{\pi}\right]^j\sum_{\substack{l_1,l_2,\ldots,l_j\\ l_1\neq0,l_j\neq0}}\frac{1}{l_1l_j}\prod_{k=1}^{j}\frac{\sin(\pi l_k\alpha)}{l_{k-1}-\alpha l_k+l_{k+1}}, \tag{20}$$

where $\alpha\equiv\epsilon+2$, $l_0=0$, and $l_{j+1}=0$. The first few of the C_J are easily evaluated[4]

$$C_0=\pi^2/3,$$

$$C_1=0,$$

$$C_2=\frac{-2}{\pi\alpha}\sum_{k=1}^{\infty}\frac{\sin\pi k\alpha}{k^3}=\frac{-\pi^2}{6|\alpha|}\hat{\alpha}(\hat{\alpha}^2-1),$$

where $\hat{\alpha}\equiv|\alpha|-(2m+1)$ with the integer m is defined by $2m\leq|\alpha|<2m+2$.

Note that $C_2=0$ whenever α (and, therefore, ϵ) is an integer. In fact, for integer values of ϵ (when $|\alpha|\geq 2$) we will show that $C_j=0$ for all $j>0$. We can obtain this result because precisely one $g_l(\epsilon)$ of Eq. (19) is nonzero for integral ϵ:

$$g_l(\epsilon)=\Delta(l-\epsilon)\equiv\begin{cases}1, & l-\epsilon=0\\ 0, & l-\epsilon\neq 0\end{cases} \tag{21}$$

where Δ is the Kronecker function. Substitution of Eq. (21) into Eq. (14) together with Eq. (12) gives

$$\chi_k^T(m_0,0,\ldots,0,m_{k-1},m_k)=\sum_{l_1,\ldots,l_{k-1}}\Delta(-m_{k-1}-\alpha m_k+l_1)\Delta(m_k-\alpha l_1+l_2)\times\Delta(l_1-\alpha l_2+l_3)\cdots\Delta(l_{k-3}-\alpha l_{k-2}+l_{k-1})\Delta(l_{k-2}-m_0)\Delta(l_{k-1}) \tag{22}$$

for the characteristic function. There are $k-1$ sums in Eq. (22) and $k+1$ Kronecker deltas. This implies that for any m_0 there is only one pair (m_{k-1},m_k) for which χ_k^T is nonzero.

To determine that pair we begin with the last sum and work backwards, finding

$$l_{k-1}=0$$

$$l_{k-2}=m_0, \tag{23a}$$

$$l_{k-(j+1)}-\alpha l_{k-j}+l_{k-(j-1)}=0. \tag{23b}$$

This finite difference equation with the two initial conditions can be solved for $m_k=l_0$ and $m_{k-1}=-l_{-1}$. This process yields

$$m_k=m_0\frac{\gamma_+^{k-1}-\gamma_-^{k-1}}{\gamma_+-\gamma_-},$$

$$m_{k-1}=-m_0\frac{\gamma_+^{k}-\gamma_-^{k}}{\gamma_+-\gamma_-},\quad k>1 \tag{24}$$

where $\gamma_\pm=\frac{1}{2}[\alpha\pm(\alpha^2-4)^{1/2}]$, when $|\alpha|\neq 2$. Note that $\gamma_\pm$ are the eigenvalues of the linearization of the sawtooth map. It is easy to see that when $|\alpha|>2$ both m_{k-1} and m_k are nonzero integers and, therefore,

$$\chi_k(m_0,0,\ldots,0,m_k)=0 \tag{25}$$

for $k\geq 1$ and $m_0,m_k\neq 0$. In fact, one can show that Eq. (25) holds even for $|\alpha|=2$. This result, with Eq. (17) implies $C_j=0$ for $j>0$ and hence ,

$$D^T=\tfrac{1}{2}\epsilon^2C_0=\frac{(\pi\epsilon)^2}{6} \tag{26}$$

for integral ϵ satisfying $|\epsilon+2|\geq 2$, which proves that the diffusion coefficient exists and is given by the quasilinear value for these values of ϵ.

When $|\alpha|<2$ the eigenvalues $\gamma_\pm$ have modulus one and the sawtooth map is no longer a C system or even ergodic. In spite of this we can determine the correlations. Setting $m_{k-1}=0$ in Eq. (24), we find that $\chi_k(m_0,0,\ldots,0,m_k)$ is nonzero only if $k=3n$ for $|\alpha|=1$, and $k=2n$ for $\alpha=0$, for some integer n. This yields a solution for m_k and finally, from Eqs. (17) and (19), the correlation functions

$$C_k^T=\frac{\pi^2}{3}\begin{cases}1, & \alpha=-1,\ k=3n\\ (-1)^n, & \alpha=0,\ k=2n\\ (-1)^n, & \alpha=1,\ k=3n\end{cases} \tag{27}$$

with $C_k^T=0$ for other values of k. These results may be inserted into the formula (16b) for the diffusion coefficient D. The result is $D=0$ for $\epsilon=-2$ and -1, and $D=\infty$ for $\epsilon=-3$. The vanishing of D is due to the linear stability of the map for $\epsilon=-2$ and -1. The divergence of D in the case $\epsilon=-3$ indicates the presence of accelerator modes.

The accelerator modes for the case $\epsilon=-3$ may be found by searching for fixed points of the reduced mapping which correspond to increases in the momentum of the full mapping. One finds

TABLE I. Properties of the sawtooth map for integral values of ϵ. Stability refers to the linear stability of fixed points. The stream constant is defined in the text.

Value of ϵ	$\epsilon=n$ $\|\epsilon+2\|>2$	$\epsilon=-4$	$\epsilon=-3$	$\epsilon=-2,-1$	$\epsilon=0$
Stability	Hyperbolic	Parabolic	Elliptic	Elliptic	Parabolic
Ergodic	Yes	No	No	No	No
D^T	$\frac{(\pi\epsilon)^2}{6}$	$\frac{(\pi\epsilon)^2}{6}$	∞	0	0
S^T	0	0	$\frac{\pi^2}{2}$	0	0

that the reduced mapping has first-order fixed points at

$$p=0$$

and

$$x=\frac{2\pi m}{\epsilon},\quad 0<|m|<|\epsilon/2|$$

which correspond to momentum increases (in the full mapping) of $2\pi m$ each time step. Hence, -3 is the only integral value of ϵ for which there exist stable, first-order accelerator modes.

When accelerator modes are present, as in the case $\epsilon=-3$, the momentum spread increases linearly with time. To characterize this spread we define the stream coefficient

$$S^R \equiv \lim_{k\to\infty} \frac{1}{2k^2} \langle [p_k(x_0,p_0)-p_0]^2 \rangle_R$$

in analogy with Eq. (15). One can immediately show that

$$S^R=\epsilon^2 \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k-1} \left[1-\frac{j}{k}\right] C_j^R .$$

The results of Eq. (27) imply that the stream constant S^T has the value $\pi^2/2$ for $\epsilon=-3$.

IV. CONCLUSIONS

The results of this paper are presented in Table I. The row labeled "stability" refers to the nature of orbits near fixed points (see Ref. 6, Appendices 27 and 28). In the hyperbolic (elliptic) cases, orbits near the fixed point correspond to a hyperbolic (elliptic) rotation about the fixed point. We note the the diffusion coefficient exists and is nonzero for all of the hyperbolic cases and one of the parabolic cases. The diffusion coefficient vanishes or is infinite in the elliptic cases and one of the parabolic cases.

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