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Bifurcation and stability of families of waves in uniformly driven spatially extended systems

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The destabilization of a normal mode of the uniform stationary state of a spatially extended system may initiate a whole cascade of bifurcations of families of states of increasing complexity. We discuss methods for the linear-stability analysis of the new solutions based on symmetry and continuity requirements which may be applied when no explicit analytic representation is available. We always find a family of simply periodic traveling-wave states bifurcating from the uniform stationary state (primary bifurcation). Stabilization of these simply periodic states is connected with a secondary bifurcation of a family of doubly periodic states, which upon stabilization may in turn be connected with a tertiary bifurcation of triply periodic states, etc. Each of the families may contain solitary states as limiting cases. The general theory is applied to a few representative examples.

INTRODUCTION

Ever since Hopf published the pioneering paper, *Abzweigung einer Periodischen Lösung von einer Stationären Lösung eines Differentialsystems*,[1] bifurcation theory has found continuous interest, and there exists today a wide and vast literature on the subject (see Ref. 2 and references contained therein). However, most of the work refers to dynamical systems with a finite number of state variables. The study of bifurcation and branching in field theories has found attention in mathematics only very recently.[2-6] In physics, this interest has been mainly stimulated by the study of instabilities in spatially extended dynamical systems described by a set of fields.[7-11]

Bifurcations occurring in systems at thermodynamic equilibrium are known as phase transitions and have been extensively studied. On a deterministic level, their description forms the well-established Landau theory, and in the last decade the renormalization group has brought clarification and understanding of the effects of fluctuations.

Of high current interest are instabilities occurring in systems driven by a set of (time-independent) control parameters into states away from thermodynamic equilibrium. Instabilities occurring in hydrodynamic systems, such as the Bênard and Taylor instabilities,[12] as well as current instabilities in strongly temperature-dependent metallic conductors[13] have already been studied for a long time. Interest in these phenomena was renewed by the development of the laser,[14] by the observation of current instabilities in semiconductors,[15] of instabilities in chemical reactions,[16] of optical instabilities,[17] and others. Today, the occurrence of such instabilities has been observed or conjectured in many different areas both in physics and in other sciences. The theoretical description of these phenomena is to a considerable extent independent of the specific system.

Such instabilities in nonequilibrium systems are phase-transitionlike phenomena, showing many similarities to phase transitions in systems at thermodynamic equilibrium: An instability is associated with the undamping of a normal mode and the breaking of symmetry, and in spatially extended systems one expects the occurrence of critical fluctuations with important effects on the bulk behavior. But there exists one important difference: In driven nonequilibrium systems, not only spatial symmetries but also the symmetry under time translations may be spontaneously broken at an instability, leading to time-dependent (nonstationary) states even under stationary driving conditions. (The special case of the bifurcation of periodic states from a stationary state in a system with a finite number of degrees of freedom was the subject of the paper by Hopf mentioned above.)

In the present paper, we study such instabilities on a deterministic level by means of time-evolution equations for the set of fields $\phi(\vec{r}, t)$ describing the macroscopic states of the system, i.e., we disregard the effects of the fluctuations. This theory will thus be the analog of Landau theory for driven systems. As mentioned above, fluctuations will become important close to the bifurcation point, but the macroscopic field equations supplemented by stochastic Langevin forces are expected to be a good starting point also for the investigation of fluctuation behavior, in the same way as the thermodynamic Landau potential is a good starting point for the renormalization group.

We focus attention on a special class of instabilities where a spatially uniform time-independent state ϕ_s loses stability at a critical value α_c of the control parameter α, and bifurcates into a set of stationary waves or of traveling waves (TW) (which are stationary in a moving frame).[3-6,18] In a spatially extended system, the successive destabilization of a branch of normal modes leads to the bifurcation of a whole family of new states ϕ_{TW} with broken space-time translation symmetry.

We label each member of the family at a given value α of the control parameter by its bifurcation point α_b, using the bifurcation wave number $q = q_b = 2\pi/\Lambda_b$ for identification. In the infinite system, there exists, in general, a continuum of bifurcation points, i.e., a bifurcation line[5,18] $\alpha_b \geq \alpha_c$ (Fig. 1). In a finite system, or in a system with periodic boundary conditions in a finite periodicity volume, the set of bifurcation points will be discrete with a spacing going to zero as the volume goes to infinity.

In order to be a candidate for physical realization at asymptotically large times (an "attractor"), the state ϕ_{TW} must be stable. The task is therefore to select the stable members of the family $\phi_{TW}(\alpha, \alpha_b)$. The linear stability of such a state is governed by an eigenvalue problem. Our main goal is to show what general conclusions can be drawn about the stability of the TW states on the basis of symmetry and continuity requirements alone, when no explicit analytic solution is available.[18]

A simple continuity argument[3,18] shows that all TW states adjacent to the bifurcation line $\alpha = \alpha_b$ are unstable for $\alpha_b \neq \alpha_c$. In special cases, the only stable member of the family is the TW state bifurcating at $\alpha = \alpha_c$. In general, other TW states may become stable above a certain critical value $\alpha = \alpha^* > \alpha_b$ of the bifurcation parameter (see Fig. 1). The existence of such a stability limit α^* in the family of simply periodic states will be connected with a secondary bifurcation of a family of doubly periodic states from each of the simply periodic states. In a certain neighborhood of their birfurcation point all these doubly periodic states will again be unstable, and the question arises if they become stabilized above a stability limit α^{**} which would then, in turn, be connected with a tertiary bifurcation of triply periodic states, etc. The destabilization of a normal mode may thus initiate a whole cascade of bifurcations of families of states of increasing complexity, with interesting topological implications.

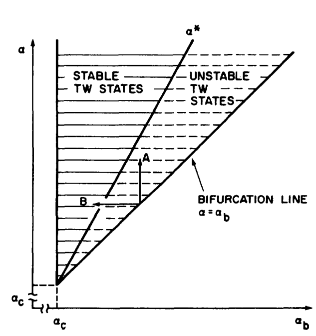

FIG. 1. Normal bifurcation of a family of TW states. Along the bifurcation line $\alpha = \alpha_b \geq \alpha_c$, TW states bifurcate with zero amplitude. To each point $(\alpha_b, \alpha), \alpha \geq \alpha_b \geq \alpha_c$, there corresponds the TW state which has bifurcated at α_b. Any state can be expanded in powers of the amplitude by keeping either α_b fixed (path A) or α fixed (path B), or along a more general path.

It is important to allow for a large enough phase space in the model description of the system. Many attempts have been made to reduce the bifurcation problem of spatially extended systems to the treatment of a single or a few amplitudes. Such an approach is only acceptable, if at least slow spatial variations of the amplitudes are included by taking at least second spatial derivatives in the evolution equations into account. Otherwise, one may miss nonuniform (e.g., solitary) solutions relevant to the problem, and one may find a given solution to be stable which in reality is unstable against nonuniform amplitude modes. Similarly, the proper treatment of secondary or higher bifurcations may require either more amplitudes or higher spatial derivatives.

The rest of the paper is organized as follows: In Sec. I, we study the stability limit of the stationary uniform state for a general class of dynamic systems, and deduce the properties of the bifurcating TW states at their bifurcation points. In Sec. II, the stability of the family of TW states is considered. We present our methods for the stability analysis which may be applied when no explicit analytic representation of the TW states is available. Section III finally contains a detailed discussion of the bifurcation behavior of a few representative systems, which serve to demonstrate our method.

I. STABILITY LIMIT AND BIFURCATION

We consider a general class of systems described by a set of (macroscopic) fields

$$\underline{\phi}(\vec{r}, t) = \{\phi_1(\vec{r}, t), \phi_2(\vec{r}, t), \ldots, \phi_n(\vec{r}, t)\}$$

and a set of parameters $\underline{\alpha} = \{\alpha_1, \alpha_2, \ldots\}$ which can be externally controlled. The time evolution of the fields is assumed to be given by

$$\frac{\partial \underline{\phi}}{\partial t} = \hat{B}[\underline{\phi}, \underline{\alpha}] \tag{1.1}$$

together with appropriate boundary conditions, where $\hat{B}$ is a nonlinear partial-differential operator acting on the state $\underline{\phi}(\vec{r}, t)$, inducing a flow in the state space $\Sigma\{\underline{\phi}\}$. We consider systems which are invariant against translations in space and time, i.e., we exclude an explicit dependence of the operator $\hat{B}$ on $\vec{r}$ and t (uniform and autonomous systems).

The basic assumption of this class of field theories is the locality in time, which is based on the existence of different time scales for the macrovariables and the eliminated microvariables. In order to avoid memory effects, the state space has to be chosen appropriately large, such that all "slow" variables are included. The assumption of locality in space, on the other hand, is not essential, and our results may be generalized to this case by taking $\hat{B}$ as a nonlinear integral operator.

A. The stability limit of the stationary state

In many systems, there exist branches of stationary states which are time independent and spatially uniform, or which may be considered approximately uniform for certain boundary conditions (e.g., in current instabilities, where the self-Hall field always causes a nonuniformity, which may be neglected under certain conditions). The stability of such a stationary uniform state $\underline{\phi}_s$ is determined by the time evolution of small perturbations $\delta\underline{\phi}(\vec{r}, t)$,

$$\frac{\partial \delta\underline{\phi}(\vec{r}, t)}{\partial t} = -\hat{L}(\underline{\phi}_s, \underline{\alpha}) \cdot \delta\underline{\phi}(\vec{r}, t), \tag{1.2}$$

where $L(\underline{\phi}_s, \underline{\alpha}) = -\nabla_{\phi}\hat{B}|_{\underline{\phi}=\underline{\phi}_s}$ is a linear operator (Fréchet derivative) acting on the tangent space of $\Sigma\{\underline{\phi}\}$ in $\underline{\phi}_s$. Because of spatial and temporal translational invariance, the perturbations $\delta\underline{\phi}(\vec{r}, t)$ have the form of plane waves

$$\delta\underline{\phi}(\vec{r}, t) = \delta\underline{\phi}_{\vec{q}\omega} e^{i(\vec{q}\cdot\vec{r}-\omega t)} . \tag{1.3}$$

This leads to the linear eigenvalue problem

$$[\hat{L}_{\vec{q}}(\underline{\phi}_s, \underline{\alpha}) - i\omega\hat{1}] \cdot \delta\underline{\phi}_{\vec{q}\omega} = 0, \tag{1.4}$$

where $\hat{1}$ is the identity, and we are looking for the branches of (generally complex) frequencies $\omega_\nu(\vec{q}, \underline{\alpha})$ $(\nu = 1, 2, \ldots, n)$ belonging to the real wave vector $\vec{q}$. If $\hat{L}$ is a real operator, then there exists for each mode $(\nu\vec{q})$, with frequency $\omega_\nu(\vec{q}, \underline{\alpha})$ a mode $(\nu, -\vec{q})$, with frequency

$$\omega_\nu(-\vec{q}, \underline{\alpha}) = -\omega_\nu^*(\vec{q}, \underline{\alpha}) . \tag{1.5}$$

Other symmetries cause further degeneracies in the spectrum. The state ϕ_s is called linearly stable if all modes are damped, i.e., if

$$\mathrm{Im}\omega_\nu(\vec{q}, \underline{\alpha}) < 0 \tag{1.6}$$

for all ordinary modes, with $\omega_\nu(0, \underline{\alpha}) \neq 0$. For hydrodynamic modes, with $\omega_\nu(0, \underline{\alpha}) = \overline{0}$, the coefficient of the leading term of $\mathrm{Im}\omega_\nu(\vec{q}, \underline{\alpha})$ in powers of q has to be negative. The stability limit with respect to ordinary modes is given by

$$\max(\nu, \vec{q})\, \mathrm{Im}\omega_\nu(\vec{q}, \underline{\alpha}) = 0 . \tag{1.7}$$

To each point $\underline{\alpha}_c$ on the stability boundary there corresponds a critical mode with wave vector[19] $\vec{q} = \vec{q}_c$, lying on a particular branch which we call $\nu = 1$. The critical frequency

$$\omega_c = \mathrm{Re}\omega_1(\vec{q}_c, \underline{\alpha}_c) \tag{1.8}$$

may either be zero (soft-mode instability) or nonzero (hard-mode instability). In the case of hydrodynamic modes, the corresponding criteria are given in terms of the coefficients of the leading terms of $\mathrm{Im}\omega_\nu(\vec{q}, \underline{\alpha})$ and $\mathrm{Re}\omega_\nu(\vec{q}, \underline{\alpha})$, respectively. In the following, we restrict the discussion to ordinary modes.

We consider a path in control space passing through the point $\underline{\alpha}_c$ on the stability limit, and let α stand for the control coordinate along that path with values increasing as one moves from the stable region into the unstable region. We explicitly assume that there exist stationary uniform states both in the stable and the unstable region. In some cases the control space has to be chosen properly, such that this condition is satisifed. In order to follow the way in which more and more modes become undamped as the control parameter is increased beyond α_c, we expand the dispersion of the critical branch $\omega_1(q, \alpha)$ for fixed α around the most weakly damped mode $q(\alpha)$ determined by $[\partial\, \mathrm{Im}\omega_1(q, \alpha)/\partial q]|_{q=q(\alpha)} = 0$ (Fig. 2):

$$\mathrm{Im}\omega_1(q, \alpha) = \Gamma(\alpha) - D(\alpha)[q - q(\alpha)]^2 + O([q - q(\alpha)]^3), \tag{1.9}$$

$$\mathrm{Re}\omega_1(q, \alpha) = \omega_0(\alpha) + v(\alpha)[q - q(\alpha)] + d(\alpha)[q - q(\alpha)]^2 + O([q - q(\alpha)]^3), \tag{1.10}$$

where $\Gamma(\alpha_c) = 0, D(\alpha) > 0$. Expansion in powers of $\alpha - \alpha_c$ yields

$$\mathrm{Im}\omega_1(q, \alpha) = \Gamma_\alpha(\alpha - \alpha_c) - D_c(q - q_c)^2 + O((\alpha - \alpha_c)^2, (\alpha - \alpha_c)(q - q_c), (q - q_c)^3), \tag{1.11}$$

$$\mathrm{Re}\omega_1(q, \alpha) = \omega_c + (\omega_{c,\alpha} - v_c q_{c,\alpha})(\alpha - \alpha_c) + v_c(q - q_c) + d_c(q - q_c)^2 + O((\alpha - \alpha_c)^2, (\alpha - \alpha_c)(q - q_c), (q - q_c)^3) . \tag{1.12}$$

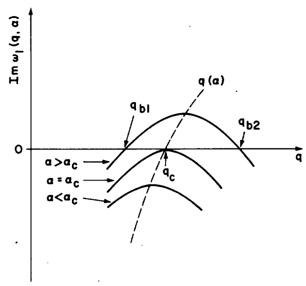

FIG. 2. Behavior of the branch $\omega_1(q,\alpha)$ containing the most weakly damped frequency for values of the control parameter close to α_c. At $\alpha>\alpha_c$, all modes with wave vectors between q_{b1} and q_{b2} have become undamped.

where ω_c, q_c, v_c, D_c, d_c are the values of the corresponding functions at $\alpha=\alpha_c$, and $\omega_{c,\alpha}$, $q_{c,\alpha}$, $v_{c,\alpha}$, Γ_α are their derivatives at $\alpha=\alpha_c$, respectively. We have written these equations for the case of a single component q. The extension to the multidimensional case is straightforward. Setting $\mathrm{Im}\,\omega_1(q,\alpha)=0$, one obtains the "bifurcation line" giving the values $\alpha=\alpha_b(q)$ of the control parameter, where the mode with wave vector q becomes undamped (Fig. 3). One obtains from (1.11)

$$\alpha_b=\alpha_c+(D_c/\Gamma_\alpha)(q-q_c)^2+O((q-q_c)^3). \qquad (1.13)$$

Substitution into (1.12) yields the frequency on the bifurcation line

$$\omega_b(q)=\omega_c+v_c(q-q_c)+\Delta_c(q-q_c)^2+O((q-q_c)^3), \qquad (1.14)$$

where

$$\Delta_c=(\omega_{c,\alpha}-v_c q_{c,\alpha})(D_c/\Gamma_\alpha)+d_c\,. \qquad (1.15)$$

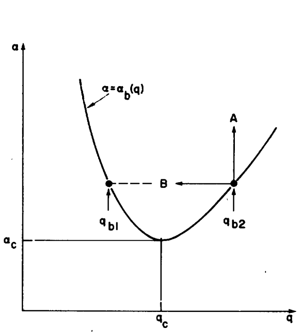

FIG. 3. Bifurcation line $\alpha=\alpha_b(q)$ giving the control parameter α_b at which the mode q becomes undamped. The nonuniform states can be obtained by performing an expansion in $\alpha-\alpha_b$ at constant q (path A) or in q_b at constant α (path B).

B. Primary bifurcation of a family of periodic traveling waves

We now turn to the problem which set of states the system may assume after the stability limit is exceeded, either as the result of an infinitely slow change of the control parameter, or asymptotically for large times after a sudden change. From now on, we restrict the consideration to the case of one space dimension, and focus on the special class of traveling-wave (TW) states $\underline{\phi}_T(\xi)$ depending on space and time only in the combination $\xi=x-ut$. The TW's are stationary solutions of (1.1) transformed into a frame moving with velocity u,

$$u\left(\frac{d\underline{\phi}}{d\xi}\right)+\hat{B}[\underline{\phi},\alpha]=0\,. \qquad (1.16)$$

We first search for periodic solutions $\underline{\phi}_T(\xi)=\underline{\phi}_T(\xi+\Lambda)$ of period Λ. Such a solution may be expanded into a Fourier series

$$\underline{\phi}_T(\xi)=\underline{\phi}_s+\sum_{n=-\infty}^{n=+\infty}\underline{\phi}^{(n)}e^{inq\xi},\quad q=\frac{2\pi}{\Lambda}\,. \qquad (1.17)$$

For real fields, $\underline{\phi}^{(n)}=\underline{\phi}^{*(-n)}$. We want to follow how any such state bifurcates out of the uniform state $\underline{\phi}_s$, and use the amplitude $A=|\underline{\phi}^{(1)}|$ of the first Fourier coefficient as an expansion parameter. (Solitary states are obtained in the limit $\Lambda\to\infty$. In this case one may use a norm as expansion parameter.) The expansion may be performed either at fixed period $\Lambda=\Lambda_b$ (fixed $q=q_b$, path A in Fig. 3), or at fixed control parameter $\alpha=\alpha_b$ (path B in Fig. 3). We expand all amplitudes $\phi^{(n)}$, $|n|\neq 1$, as well as $\alpha-\alpha_b$ (for fixed q) or $q-q_b$ (for fixed α_b), and the wave velocity u in powers of A. To the lowest order in A, which determines the behavior at bifurcation $\underline{\phi}^{(1)}$ satisfies the equation

$$[\hat{L}_{q_b}(\underline{\phi}_s,\alpha)-iuq_b\hat{1}]\cdot\underline{\phi}^{(1)}=0\,. \qquad (1.18)$$

Therefore, the Fourier coefficient $\underline{\phi}^{(1)}$ is an eigenmode of (1.4), and the period $\Lambda_b=2\pi/q_b$ and the pulse velocity u_b at bifurcation are given by

$$0=\mathrm{Im}\,\omega_1(q_b,\alpha_b), \qquad (1.19)$$

$$q_b u_b=\mathrm{Re}\,\omega_1(q_b,\alpha_b)=\omega_b(q_b)\,. \qquad (1.20)$$

Thus, at every $\alpha=\alpha_b>\alpha_c$ there bifurcate, in general, two TW states out of the unstable stationary state, corresponding to the two normal modes which become undamped at α_b (see Fig. 3). For $\alpha_b\to\alpha_c$, the solutions degenerate into a single one. Near α_c, the relation between α_b and q_b is given by (1.13), and ω_b has the form of (1.14). We now discuss some specific cases of bifurcation behavior.

1. *Hard-mode instability at $q_c \neq 0$*

For $\omega_c \neq 0$, $q_c \neq 0$, one finds the result

$$u_b = u_c + (v_c - u_c)\left(\frac{q_b - q_c}{q_c}\right) + (\Delta_c q_c - v_c + u_c)\left(\frac{q_b - q_c}{q_c}\right)^2 + O((q - q_c)^3), \quad (1.21)$$

where $u_c = \omega_c / q_c$, and Δ_c is given in (1.15). One obtains two pulse velocities, depending on $q_b > q_c$ or $q_b < q_c$. The term proportional to $(q_b - q_c)/q_c$ is determined by the difference of the group velocity v and the pulse velocity u at $\alpha = \alpha_c$. Physical examples of this behavior are the laser threshold[14,20] (with $\Delta_c = 0$) and the laser pulse threshold.[20]

2. *Soft-mode instability at $q_c \neq 0$*

For $\omega_c = 0, q_c \neq 0$, we find

$$u_b = v_c\left(\frac{q_b - q_c}{q_c}\right) + (\Delta_c q_c - v_c)\left(\frac{q_b - q_c}{q_c}\right)^2 + O((q_b - q_c)^3). \quad (1.22)$$

If all undamped modes are soft, i.e., if $\omega_b = 0$ (and therefore $v_c = \Delta_c = 0$), one has $u_b = 0$. This case occurs, for instance, in the Bénard instability.[21]

3. *Hard-mode instability at $q_c = 0$*

In this case, it is more convenient to introduce the variable $\tau = t - x/u$ instead of $\xi = x - ut$, because the solution with $q_b = 0$ is associated with $u = \infty$. The pulse velocity is then given by

$$\frac{1}{u_b} = \frac{1}{\omega_c} q_b - \frac{v_c}{\omega_c^2} q_b^2 + \left(\frac{v_c^2}{\omega_c^3} - \frac{\Delta_c}{\omega_c^2}\right) q_b^3 + O(q_b^4). \quad (1.23)$$

Examples are chemical instabilities.[22]

4. *Soft-mode instability at $q_c \neq 0$*

For $\omega_c = 0$, $q_c = 0$, one finds for $q_b \neq 0$

$$u_b = v_c + \Delta_c q_b + O(q_b^2). \quad (1.24)$$

Examples are current instabilities in semiconductors.[23,24] The $q_b = 0$ limit of this family of periodic states represents a solitary state traveling with $u = v_c$. In addition, there may occur other branches of solitary states, the pulse velocities of which cannot be determined from (1.20) since both sides vanish identically for $q_b = 0$.

C. Primary bifurcation of families of multiple periodic waves

In the above discussion, we have considered the bifurcation of periodic TW states growing from a single undamped mode of wave number q, or from a pair $(q, -q)$. In the case of $q_c \neq 0$, two modes with, in general, incommensurate wave numbers q_{b_1}, q_{b_2} become undamped at the same $\alpha_b > \alpha_c$ (see Fig. 3), each traveling with its own phase velocity $u_{b_{1,2}} = \omega(q_{b_{1,2}})/q_{b_{1,2}}$. This may give rise to the primary bifurcation of a doubly periodic state of the form

$$\underline{\phi}_2(x, t; \alpha, \alpha_b) = \underline{\phi}(x - u_1 t, x - u_2 t; \alpha, \alpha_b),$$

where

$$\underline{\phi}(\xi_1 + n_1\Lambda_1, \xi_2 + n_2\Lambda_2; \alpha, \alpha_b) = \underline{\phi}(\xi_1, \xi_2; \alpha, \alpha_b)$$

$(n_1, n_2$ integer), growing from a linear combination of the two undamped modes. Similarly, undamping of additional (parts of) branches at higher values of α_b may lead to the primary bifurcation of multiply periodic states. In the present paper, we focus attention on systems in which only singly periodic states bifurcate from the uniform state.

II. STABILITY OF BIFURCATING WAVES AND SECONDARY BIFURCATION

A. Expansion to higher powers in A for the singly periodic states

Expansion to higher powers of A at constant $\Lambda = \Lambda_b$ yields power series for $\alpha - \alpha_b$, $u - u_b$ and the higher Fourier coefficients $\underline{\phi}^{(n)}$ of the TW state $\underline{\phi}_T(\xi; \alpha, \Lambda_b)$. The type of bifurcation is determined by the leading term of $\alpha - \alpha_b$,

$$\alpha - \alpha_b = \vartheta_1(\alpha_b) A^{r_1}, \quad (2.1)$$

which defines the "bifurcation exponent" r_1. The bifurcation is called "normal" if the zero-amplitude solution is approached from $\alpha > \alpha_b$, i.e., if $\vartheta_1(\alpha_b) > 0$, and "inverted" in the opposite case. Note that $\vartheta_1(\alpha_b)$ may change sign at a particular value of α_b (Fig. 4). At this point, the type of bi-

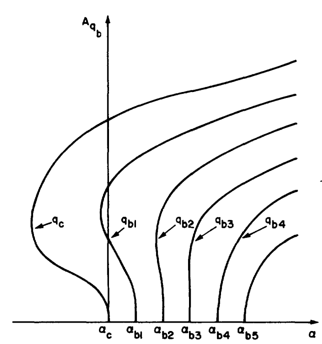

FIG. 4. Amplitude A_{q_b} of the traveling waves as a result of an expansion along path A (see Figs. 1 and 3). Shown is an example in which the bifurcation of periodic TW states changes from inverted to normal as α_b increases.

furcation is determined by the next higher term $\vartheta_2(\alpha_b)A^{r_2}$, with a bifurcation exponent r_2.

B. Stability of traveling waves

Our next task is to select the members of the family $\underline{\phi}_T(\xi; \alpha, \Lambda_b)$ according to their stability properties. We test the nonuniform solution $\underline{\phi}_T$ against small perturbations $\delta\underline{\phi}(\xi, t)$, which obey the linearized evolution equation

$$\frac{\partial\delta\underline{\phi}(\xi,t)}{\partial t} - u\frac{\partial\delta\underline{\phi}(\xi,t)}{\partial\xi} + \hat{L}(\underline{\phi}_T(\xi;\alpha,\Lambda_b))\cdot\delta\underline{\phi}(\xi,t) = 0, \qquad (2.2)$$

where the linear operator $\hat{L} = -\nabla_{\underline{\phi}}\hat{B}(\underline{\phi})|_{\phi=\phi_T(\xi)}$ is the derivative of $\hat{B}$ at the nonuniform solution $\underline{\phi}_T$. Because $\hat{L}$ is still time independent in the moving frame, we are looking for solutions of the form

$$\delta\underline{\phi}(\xi,t) = \delta\underline{\phi}_\lambda(\xi)e^{-\lambda t},$$

leading to the eigenvalue problem

$$\left(\hat{L}(\underline{\phi}_T(\xi;\alpha,\alpha_b)) - u\frac{d}{d\xi}\hat{1}\right)\cdot\delta\underline{\phi}_\lambda(\xi) = \lambda\delta\underline{\phi}_\lambda(\xi). \qquad (2.3)$$

The stability of the TW states is thus determined by the spectrum of $[\hat{L}(\phi_T) - (ud/d\xi)1]$: The state $\underline{\phi}_T(\xi; \alpha, \Lambda_b)$ is stable if $\mathrm{Re}\lambda > 0$ for all modes except the Goldstone mode (see below).

C. Method

We now list a number of principles[18] which in some cases lead to definite conclusions about the stability of $\underline{\phi}_T$, and, in general, give considerable insight into the spectrum and the eigenmodes of $[\hat{L}(\underline{\phi}_T) - (ud/d\xi)\hat{1}]$.

1. Breaking of translational symmetry

The nonuniform states $\underline{\phi}_T(\xi; \alpha, \Lambda_b)$ are states of broken translational symmetry. Therefore, a set of equivalent states $\underline{\phi}_T(\xi; \alpha, \Lambda_b)$ is generated by translations. The infinitesimal translation represents a Goldstone mode (GM)

$$\delta\underline{\phi}_{\lambda=0} = \frac{\partial\underline{\phi}_T(\xi;\alpha,\Lambda_b)}{\partial\xi},$$

with eigenvalue $\lambda = 0$. Thus, one eigenfunction of (2.3) is always known, and is obtained directly from the nonuniform solution $\underline{\phi}_T$, without solving the eigenvalue equation (2.3)

2. Periodicity of the TW state

The operator $\hat{L}(\underline{\phi}_T(\xi;\alpha,\Lambda_b),\alpha)$ has at least the symmetry of the TW $\underline{\phi}_T$. If $\underline{\phi}_T$ is periodic with period Λ, i.e., invariant under the discrete group $\{T_\Lambda\}$ of translations $n\Lambda$, $\hat{L}(\underline{\phi}_T)$ is at least invariant under $\{T_\Lambda\}$, and under certain conditions even under the higher group $\{T_{\Lambda/m}\}$ of translations $n\Lambda/m$. Therefore, the Bloch theorem applies: The eigenfunctions have the form

$$\delta\underline{\phi}_\lambda(\xi) = \underline{u}_\lambda(\xi)\exp(iq\xi),$$

with

$$\underline{u}_\lambda(\xi + \Lambda/m) = \underline{u}_\lambda(\xi),$$

and the eigenvalues of Eq. (2.3) are multivalued functions $\lambda_n(q, \alpha)$ of the reduced wave vector over the Brillouin zone (BZ) $-m\pi/\Lambda \leq q \leq m\pi/\Lambda$.

3. Zero-amplitude result

The bifurcation analysis rests on the assumption that the amplitude A of $\underline{\phi}_T$ can be used as expansion parameter which implies $A \to 0$, i.e., $\underline{\phi}_T \to \underline{\phi}_s$, as $\alpha \to \alpha_b$. Therefore, at bifurcation, the spectrum $\lambda(q, \alpha_b) = \lambda_b(q)$ is determined by the spectrum $\omega_b(q)$ of the uniform state at $\alpha = \alpha_b$ via

$$\lambda_b(q) = -[\omega_b(Q) - u_b Q],$$

where $Q = q + K$, and K is a reciprocal lattice vector. Thus, the TW state bifurcating at $\alpha_b > \alpha_c$ is at bifurcation unstable against all modes which have become undamped between α_c and α_b. These modes may constitute one or several unstable branches in the BZ.

4. Noncrossing rule

For a general q in the BZ, no crossing of eigenvalues can occur with increasing A. At the symmetry points $q = 0$ and $q = \pm m\pi/\Lambda$ the noncrossing rule holds only for eigenvalues belonging to eigenfunctions with the same transformation behavior.

5. Perturbation theory

The curvature of the $\lambda(q)$ curve near a symmetry point may be obtained from $\vec{k}\cdot\hat{p}$ perturbation theory. This is a standard procedure in band-structure calculations[25] needing no further comment, and we will give the corresponding results without explicit demonstration.

D. Secondary bifurcation of doubly periodic states

Consider a stable TW state $\underline{\phi}_T(\xi; \alpha, \Lambda_b)$. The discussion of bifurcation from this state ("secondary" bifurcation) proceeds along the same lines as that of the primary bifurcation from the uniform state considered in Sec. I. The stability limit $\alpha_c^{(2)}(\alpha_b)$ with respect to ordinary modes is given by

$$\min \mathrm{Re}\lambda_\nu(q, \alpha_c^{(2)}(\alpha_b)) = 0. \qquad (2.4)$$

For hydrodynamic modes and the Goldstone mode,

$\mathrm{Re}\lambda_\nu(q,\alpha)$ has to be replaced by the coefficient of the leading term of $\mathrm{Re}\lambda_\nu(q,\alpha_c^{(2)})$ in powers q or $q-q_{\mathrm{GM}}$, respectively. The undamping of a normal mode at $\alpha=\alpha_b^{(2)}$, $q=q_b^{(2)}$ outside the stability range will give rise to the bifurcation of a doubly periodic state

$$\underline{\phi}_2(x,t;\alpha,\alpha_b^{(1)},\alpha_b^{(2)})=\underline{\phi}_2(x-u_1t,x-u_2t;\alpha,\alpha_b^{(1)},\alpha_b^{(2)}),$$

with

$$\underline{\phi}_2(\xi_1+u_1\Lambda_1,\xi_2+u_2\Lambda_2;\alpha)=\underline{\phi}_2(\xi_1,\xi_2;\alpha),$$

where the second period Λ_2 and the corresponding velocity u_2 are determined at bifurcation by

$$\Lambda_2=2\pi/q_b^{(2)},\quad u_b^{(2)}=u_1+\mathrm{Im}\lambda_\nu(q_b,\alpha_b^{(2)})/q_b^{(2)}. \tag{2.5}$$

Since all periodic TW states $\underline{\phi}_T(\xi;\alpha,\alpha_b)$ with $\Lambda_b\neq 2\pi/q_c$ are unstable against at least one band of modes in the limit $A\to 0$, stabilization of such a state above a critical amplitude A_{crit} requires this branch $\lambda_1(q;\alpha,\Lambda_b)$ to cross the imaginary axis. Thus, there always exists a range of amplitudes below A_{crit} where continuous bifurcation of doubly periodic states occurs. This behavior is illustrated in Fig. 5 for the case that $\hat{L}$ is invariant under $\{T_{\Lambda/2}\}$ so that the BZ has extension $-2\pi/\Lambda\leq q\leq 2\pi/\Lambda$ and the GM lies at the edge of the BZ. For $A\to 0$, the state is unstable against the lowest band $\lambda_1(q)$ [Fig. 5(a)]. As A increases towards A_{crit}, $\mathrm{Re}\lambda_1(q)$ has to intersect the $\lambda=0$ line [Fig. 5(b)], in order to become positive in the whole BZ except at q_{GM} for $A>A_{\mathrm{crit}}$ [Fig. 5(c)].

The same type of arguments may be repeated for the stability of doubly periodic states leading to tertiary bifurcation of triply periodic states, etc. It is a very interesting problem under what conditions a bifurcation of a strongly nonperiodic state ("spatial chaos") may occur.

III. EXAMPLES

We illustrate the general considerations by three specific examples described by a time-evolution equation of the form

$$\frac{\partial\phi}{\partial t}=b(\phi)-v\frac{\partial\phi}{\partial x}+\frac{\partial^2\phi}{\partial x^2}, \tag{3.1}$$

where the "flow vector" $b(\phi)$ is the derivative of a potential $V(\phi)$. The bifurcation behavior is found to depend in an essential way on the symmetry properties of $V(\phi)$. We shall supplement the general considerations by studying the "phase portrait" ("characteristics") of the TW states $\phi_T(\xi)$, $\xi=x-ut$, in the $(\phi,\psi=d\phi/d\xi)$ plane ("phase plane"), given by

$$\frac{d\phi}{d\xi}=\psi, \tag{3.2a}$$

$$\frac{d\psi}{d\xi}=-b(\phi)+(v-u)\psi. \tag{3.2b}$$

A. One-component field without internal symmetry

As a first example, we study a field ϕ with the flow

$$b(\phi)=\alpha\phi-\phi^2-\gamma\phi^3=-\frac{\partial V}{\partial\phi} \tag{3.3}$$

derived from a (2-3-4) potential

$$V(\phi)=-\tfrac{1}{2}\alpha\phi^2+\tfrac{1}{3}\phi^3+\tfrac{1}{4}\gamma\phi^4 \tag{3.4}$$

without any internal symmetry. Such a flow has various applications. We mention here the connection to current instabilities in semiconductors,[24] where $\phi\propto$ excess field (deviation from uniform electric field) in the sample, $\alpha\propto$ total current through the sample, and v is the drift velocity of the carriers. The flow [Eq. (3.3)] is also found in the driven and damped sine-Gordon chain, when an expansion is performed around one of the stable uniform states. Such an expansion is particularly useful for driving fields close to the amplitude of the periodic potential.[26,27]

For simplicity, we study the case $\gamma=0$, although in this case Eq. (3.1) does have an additional symmetry, apart from translational invariance with respect to x and t: It is also left invariant by the transformation $\phi\to\alpha-\phi$.

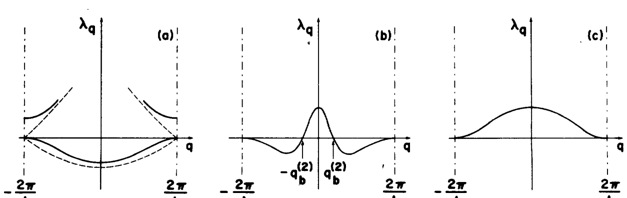

FIG. 5. Change of stability of a TW with period $\Lambda_b\neq 2\pi/q_c$ and bifurcation of a doubly periodic TW. The spectrum of the TW is shown for three different amplitudes: (a) in the limit of vanishing amplitude the spectrum of the TW (heavy line) approaches the spectrum of the unstable uniform state (broken lines), (b) for amplitudes A in a range below A_{crit} part of the lowest band has become stable, (c) for amplitudes $A>A_{\mathrm{crit}}$ the TW is stable. In the range of amplitudes where the lowest band intersects the $\lambda=0$ line at the wave vectors $\pm q_b^{(2)}$ a family of doubly periodic TW's with periods $\Lambda_b^{(1)}=\Lambda$ and $\Lambda_b^{(2)}=2\pi/q_b^{(2)}$ bifurcates from the singly periodic TW's with period Λ.

Equation (3.1) then has two uniform stationary solutions $\phi_s^{(1)}=0$ and $\phi_s^{(2)}=\alpha$. Because of the symmetry mentioned, it is sufficient to study $\phi_s^{(1)}$. Its normal-mode frequencies are

$$\omega(q)=vq+i\alpha-iq^2\,. \tag{3.5}$$

Thus, $\phi_s^{(1)}$ is stable for $\alpha<0$ and becomes unstable at $\alpha_c=0$ against a soft $q_c=0$ relaxation mode, giving rise to the bifurcation of the uniform state $\phi_s^{(2)}=\alpha$ which is stable for $\alpha>0$ ("interchange of stability"). The bifurcation is normal with bifurcation exponent $r=1$. Successive undamping of $q_b\neq 0$ modes along the bifurcation line $q_b=\alpha^{1/2}$ generates by normal bifurcation with bifurcation exponent $r=2$ a family of periodic TW states $\phi_T(\xi;\alpha,\alpha_b)$, with period $\Lambda_b=2\pi/q_b$ traveling with $u=v$. In the limit $\alpha_b=0$ one obtains a solitary wave emerging from and returning to the uniform state $\phi_s^{(2)}=\alpha$. In addition, there bifurcates at $\alpha=0$ a family of solitary states with velocities ranging from $u=-\infty$ to $u=+\infty$, which are all unstable because they connect the state $\phi_s^{(2)}$ with the unstable state $\phi_s^{(1)}=0$.

These statements can be confirmed by studying the phase portrait (3.2). For $u=v$, the characteristics $\psi=\psi(\phi)$ are given by

$$\tfrac{1}{2}\psi^2=V(\phi)+C=-\tfrac{1}{2}\alpha\phi^2+\tfrac{1}{3}\phi^3+C,\quad C\geq 0 \tag{3.6}$$

which are shown in Fig. 6. At $C=0^+$ one has the bifurcating periodic zero-amplitude solution, with period $\Lambda_b=2\pi/\alpha^{1/2}$. For $0<C<\alpha^3/6$, one finds periodic solutions of increasing amplitude, with periods increasing from $\Lambda=\Lambda_b$ to $\Lambda=\infty$. At $C=\alpha^3/6$, one has the uniform state $\phi_s^{(2)}$ and a solitary state emerging from and returning to $\phi_s^{(2)}$. For $C>\alpha^3/6$, all TW solutions are unbounded. For $u\neq v$, no periodic solutions exist. The only bounded solutions are solitary states connecting $\phi_s^{(2)}$ with the unstable state $\phi_s^{(1)}$.

We focus our attention on the periodic TW states $\phi_T(\xi;\alpha,\alpha_b)$ with $u=v$. Their stability is determined by the spectrum of the operator

$$\hat{L}=-\frac{d^2}{d\xi^2}-\alpha+2\phi_T(\xi;\alpha,\alpha_b)\,. \tag{3.7}$$

Since $\hat{L}$ is Hermitian, it has only real eigenvalues λ. The state $\phi_T(\xi;\alpha,\alpha_b)$ is invariant under the translation group T_Λ, and so is the operator (3.7). Therefore, the BZ is $(-\pi/\Lambda,\pi/\Lambda)$, and the GM occurs at $q=0$ (Fig. 7). The spectrum $\lambda=-\alpha_b+q^2$ of the zero-amplitude solution $\alpha=\alpha_b+0$ contains two unstable bands (broken lines in Fig. 7). With increasing amplitude of ϕ_T at fixed α_b, i.e., fixed period $\Lambda=\Lambda_b$ [path A in Fig. 1(a)], the BZ remains unchanged, and gaps develop at $q=0$ and $q=\pm\pi/\Lambda_b$ (full lines in Fig. 7). Since there remains no symmetry in ϕ_T other than T_Λ, all eigenfunctions belong to the unit representation of the small group even at $q=0$, and the lowest band $\lambda_1(q)$ cannot cross the GM fixed at $q=\lambda=0$. Therefore, all periodic TW states with period $\Lambda<\infty$ are unstable against at least part of the band $\lambda_1(q)$.

If we follow the spectrum at fixed value α of the bifurcation parameter (path B in Fig. 3), the period $\Lambda(\alpha_b)$ increases and the BZ shrinks with increasing amplitude. For $\alpha_b\downarrow\alpha_c$ we approach the solitary state which has a discrete spectrum. It contains one negative eigenvalue which again cannot cross the GM at $\lambda=0$, $q=0$. Thus, also the solitary state is unstable, but only against one discrete mode. In the current-instability case, this instability may be removed by coupling the sample to an external circuit with sufficiently low impedance, such that the solitary state can, in fact, be stabilized.[24] In the driven and damped sine-Gordon chain the solitary state corresponds to the critical nucleus for the creation of a kink-antikink pair.[26]

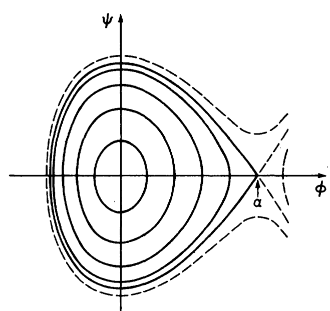

FIG. 6. Phase portrait of the flow [Eq. (3.3)] as determined by Eq. (3.6). The family of periodic TW's (closed curves around the unstable uniform state $\phi_s^{(1)}=0$) is separated by a solitary solution (emerging from and returning to $\phi_s^{(2)}=\alpha$) from unbounded TW's (indicated by broken lines).

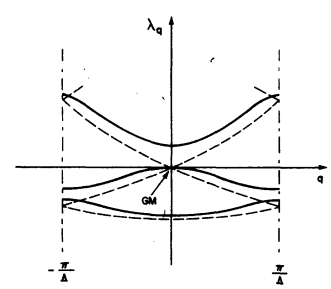

FIG. 7. Spectrum of the unstable state $\phi_s^{(1)}=0$ (broken lines) and of a periodic TW with period Λ for the flow (3.3). The Brillouin zone (BZ) has extension $2\pi/\Lambda$, and the Goldstone mode (GM) lies in the center of the BZ in the second-lowest band.

B. One-component field with reflection symmetry

As a second example, we consider a field ϕ with flow

$$b(\phi)=\alpha\phi-\phi^3=-\frac{\partial V}{\partial\phi} \tag{3.8}$$

determined by a (2–4)-potential

$$V(\phi)=-\tfrac{1}{2}\alpha\phi^2+\tfrac{1}{4}\phi^4, \tag{3.9}$$

with reflection symmetry $\phi\to-\phi$. The uniform stationary state $\phi_s^{(1)}=0$ has a spectrum of normal-mode frequencies

$$\omega(q)=vq+i\alpha-iq^2. \tag{3.10}$$

Its softening at $\alpha_c=0$, $q_c=0$ leads to the bifurcation of a pair of stable uniform stationary states $\phi_s^{(2,3)}=\pm\alpha^{1/2}$, and with increasing α along the bifurcation line $\alpha=q_b^2$ to the bifurcation of a family of periodic TW states $\phi_T(\xi;\alpha,\Lambda)$ with period Λ, traveling with $u=v$. The bifurcation is normal and the bifurcation exponent is $r=2$ for both the uniform and the nonuniform states. This family of periodic states has as its limit for $\Lambda\to\infty$ a pair of solitary states connecting $\phi_s^{(2)}$ with $\phi_s^{(3)}$. In addition, there occurs again at $\alpha=0$ a bifurcation of solitary TW states, with $u\neq v$ connecting $\phi_s^{(2)}$ or $\phi_s^{(3)}$ with the unstable state $\phi_s^{(1)}$.

The phase portrait of the TW states with $u=v$ is shown in Fig. 8. The characteristics $\psi=\psi(\phi)$ are determined by

$$\tfrac{1}{2}\psi^2=V(\phi)+C=-\tfrac{1}{2}\alpha\phi^2+\tfrac{1}{4}\phi^4+C,\quad C\geq 0. \tag{3.11}$$

At $C=0^+$ one has the periodic zero-amplitude solution with period $\Lambda_b=2\pi/\alpha^{1/2}$. For $0<C<\alpha^2/4$ one finds periodic solutions with increasing amplitude with periods increasing from $\Lambda=\Lambda_b$ to $\Lambda=\infty$. At $C=\alpha^2/4$ one has the pair of uniform states $\phi_s^{(2,3)}$ and a pair of solitary states connecting $\phi_s^{(2)}$ with $\phi_s^{(3)}$. For $C>\alpha^2/4$, all solutions are unbounded. For $u\neq v$, the only bounded solutions are solitary states connecting $\phi_s^{(2)}$ or $\phi_s^{(3)}$ with the unstable state $\phi_s^{(1)}$.

The stability of the periodic TW states $\phi_T(\xi;\alpha,\Lambda)$ is determined by the spectrum of the operator

$$\hat{L}=-\frac{d^2}{d\xi^2}-\alpha+3\phi_T^2(\xi;\alpha,\Lambda). \tag{3.12}$$

Again, $\hat{L}$ is Hermitian and thus has only real eigenvalues. The periodic states have the symmetry $\phi_T(\xi+\Lambda/2)=-\phi_T(\xi)$. Since $\hat{L}$ is even in ϕ_T, it is invariant under $T_{\Lambda/2}$. The BZ is $(-2\pi/\Lambda, 2\pi/\Lambda)$, the GM lies at the BZ boundary, and the spectrum of the zero-amplitude solution contains one unstable band $\lambda_1(q)$ (Fig. 9). Now, this lowest band is not pinned to negative values by the GM, but could, in principle move up to positive values above a critical amplitude of ϕ_T. We have therefore performed a $\vec{k}\cdot\hat{p}$ perturbation expansion[25] around the GM which shows that the curvature of $\lambda_1(q)$ at $q=q_{\rm GM}=2\pi/\Lambda$ remains negative for large amplitudes. For asymptotically large Λ, we find

$$\lambda_1(q)=-12\alpha\Lambda\exp[-\Lambda(\alpha/2)^{1/2}](q-2\pi/\Lambda)^2. \tag{3.13}$$

Thus, also in this case all periodic states are unstable against at least part of the band $\lambda_1(q)$. The solitary states connecting $\phi_s^{(2)}$ with $\phi_s^{(3)}$, on the other hand, have no negative eigenvalue and are, therefore stable

Actually, in both cases in Secs. III A and III B, the same results could have been obtained by making use of the fact that the operators (3.7) and (3.12) are of the Sturm-Liouville type.[28] We emphasize, however, that our approach is more general and applies also to operators which are not of the Sturm-Liouville type.

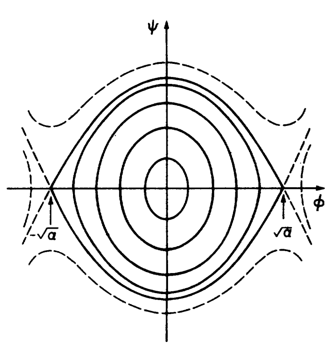

FIG. 8. Phase portrait of the flow [Eq. (3.8)] as determined by Eq. (3.11). The family of periodic TW's (closed curves around the unstable uniform state $\phi_s^{(1)}=0$) is separated by two solitary solutions (connecting the two stable uniform states $\phi_s^{(2,3)}=\pm\sqrt{\alpha}$) from unbounded states (indicated by broken lines).

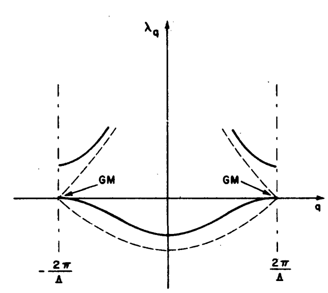

FIG. 9. Spectrum of the unstable uniform state $\phi_s^{(1)}=0$ (broken lines) and of a TW with period Λ for the flow (3.8). The Brillouin zone (BZ) has extension $4\pi/\Lambda$, and the Goldstone mode (GM) lies at the boundary of the BZ and belongs to the lowest band.

C. Two-component field with continuous symmetry

As a third example, we study a complex field ϕ with flow

$$b(\phi)=\alpha\phi-|\phi|^2\phi=-\frac{\partial V}{\partial\phi^*},\quad \alpha \text{ real} \tag{3.14}$$

derived from a potential

$$V(\phi)=-\tfrac{1}{2}\alpha|\phi|^2+\tfrac{1}{4}|\phi|^4, \tag{3.15}$$

which is invariant under the continuous group of rotations and reflections in the complex ϕ plane. In this case, the time-evolution equation (3.1) is known as "time-dependent Ginsburg-Landau equation." It describes with $v=0$ the onset of convection in the Rayleigh-Bénard instability[21,29] and with v = group velocity of electromagnetic waves the onset of coherent laser action at the laser threshold,[14,29] as well as chemical instabilities.[22] It is further related to the dynamics of the superconducting phase in thin wires.[30-32]

The uniform stationary state $\phi_s^{(1)}=0$ has a spectrum of two degenerate normal modes

$$\omega(q)=vq+i\alpha-iq^2, \tag{3.16}$$

which soften at $\alpha=0$, $q=0$. These modes may be taken as an amplitude mode (Re$\delta\phi$ and Im$\delta\phi$ in phase) and a phase mode (Re$\delta\phi$ and Im$\delta\phi$ by $\pi/2$ out of phase). Because of the continuous symmetry, there now occurs a bifurcation (normal $r=2$) of a continuous set of uniform stationary states

$$\phi_s(\alpha,\theta_0)=\sqrt{\alpha}\exp(i\theta_0), \tag{3.17}$$

which are stable with respect to radial variations and neutral with respect to azimuthal variations. Nonuniform TW states $\Phi_T(x-ut)$ are expressed in terms of amplitude $R(\xi)$ and phase $\theta(\xi)$,

$$\theta_T(\xi)=R(\xi)\exp i\theta(\xi),\quad R,\theta \text{ real}. \tag{3.18}$$

One finds from (3.1),

$$R\frac{d^2\theta}{d\xi^2}+(u-v)R\frac{d\theta}{d\xi}+2\frac{dR}{d\xi}\frac{d\theta}{d\xi}=0, \tag{3.19a}$$

$$\frac{d^2R}{d\xi^2}+(u-v)\frac{dR}{d\xi}-R\left(\frac{d\theta}{d\xi}\right)^2+\alpha R-R^3=0. \tag{3.19b}$$

For $u=v$, Eq. (3.19a) yields the first integral

$$J\equiv R^2\frac{d\theta}{d\xi}=\text{const}, \tag{3.20a}$$

and Eq. (3.19b) can also be integrated yielding

$$\frac{1}{2}\left(\frac{dR}{d\xi}\right)^2=-\frac{1}{2}\frac{J^2}{R^2}-\tfrac{1}{2}\alpha R^2+\tfrac{1}{4}R^4+C. \tag{3.20b}$$

Equations (3.20a) and (3.20b) represent the phase portrait in the coordinates $(R, dR/d\xi, J)$ which is shown in Fig. 10. For $u\neq v$, there exists only unbounded solutions except a set of solitary states connecting $\phi_s(\alpha,\theta_0)$ with the unstable state $\phi_s^{(1)}=0$.

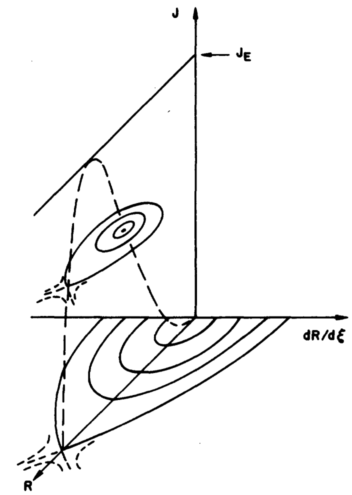

FIG. 10. Phase portrait of the TW's of the Ginzburg-Landau flow [Eq. (3.14)] as determined by Eq. (3.19). In the plane $J=0$ the protrait is identical with Fig. 8. The heavy broken line represents the rolls. Each plane $J=\text{const}<J_E$ contains a stable and an unstable roll. Around the unstable roll we find doubly periodic waves which are separated from unbounded states by a solitary solution emerging from and returning to a stable roll.

In the $(J=0)$ plane, the solutions are periodic amplitude waves $\Phi_T^{(A)}(\xi;\alpha,\Lambda,\theta_0)$ with period Λ as in Eq. (3.11), apart from an arbitrary constant phase factor $\exp(i\theta_0)$. (In this case it is more convenient to return to Cartesian components, $\phi=\phi'+i\phi''$, in order to avoid the phase jumps of $\pm\pi$ at the zeros of R.) They bifurcate due to the undamping of $q\neq 0$ amplitude modes of the uniform stationary state $\phi_s^{(1)}$ along the bifurcation line $\alpha=q_b^2$. This family of solutions has as its limit for $q\to 0$ a set of solitary states (one for each θ_0) connecting $\phi_s(\alpha,\theta_0)$ with $-\phi_s(\alpha,\theta_0)$ (Fig. 10). The bifurcation is normal with bifurcation exponent $r=2$.

The undamping of $q\neq 0$ phase modes (Re$\delta\phi$ and Im$\delta\phi$ by $\pi/2$ out of phase) of the uniform stationary state $\phi_\Lambda^{(1)}$ gives rise to the bifurcation of a family of constant-amplitude TW solutions

$$\begin{aligned}\Phi_T^{(\theta)}(\xi;\alpha,k,\theta_0)&=(\alpha-k^2)^{1/2}\exp[i(k\xi+\theta_0)],\\ R(\xi)&=(\alpha-k^2)^{1/2},\quad \theta(\xi)=k\xi+\theta_0,\end{aligned} \tag{3.21}$$

such that

$$\begin{aligned}J&=k(\alpha-k^2)=\pm R^2(\alpha-R^2)^{1/2},\\ C&=\tfrac{1}{4}(\alpha-k^2)(\alpha+3k^2)=\tfrac{1}{4}R^2(4\alpha-3R^2).\end{aligned} \tag{3.22}$$

These states describe the rolls in the Bénard problem and the coherent waves in the laser. The bifurcation is again normal with bifurcation exponent $r=2$. For fixed $\alpha>\alpha_b=k^2$, the curve $|J(R)|$ has a maximum at $R_E=(2\alpha/3)^{1/2}$ of height $J_E=(4\alpha^3/$

$27)^{1/2}$, corresponding to a roll of wave number $k_E = \sqrt{\alpha/3}$. For every value of J with $|J| < J_E$ there exist two rolls, a large-amplitude roll with wave number $k_1(J, \alpha) < k_E$ which will be shown to be stable, and a small-amplitude roll with wave number $k_2(J, \alpha) > k_E$ which will be shown to be unstable (Fig. 10).

From every unstable roll there bifurcates a family of doubly periodic solutions (see Fig. 10), as will be further discussed below [Eq. (3.35)]. Each of these families has as its limit a solitary solution starting from and returning to the stable roll belonging to the same value of J. These solitary solutions correspond to saddle-point configurations in transitions changing the wave number of the rolls.[31]

The linearized time-evolution equation for the perturbation $\delta\phi(\xi, t)$ of a TW state has the form

$$\frac{\partial}{\partial t}\delta\phi(\xi, t) = \left(\frac{\partial^2}{\partial \xi^2} + \alpha - 2R^2 \;\delta\phi\right) - R^2 e^{2i\theta}\delta\phi^* \equiv -\hat{L}\delta\phi\,, \qquad (3.23)$$

with

$$\hat{L} = -\frac{\partial^2}{\partial \xi^2} - \alpha + 2R^2 + R^2 e^{2i\theta}\hat{K}\,, \qquad (3.24)$$

where $\hat{K}$ is the operator of complex conjugation. Thus, the operator $\hat{L}$ contains linear and antilinear operators. It is most convenient to represent the perturbation $\delta\phi = r + is$ as a two-component spinor

$$\delta\phi = \begin{pmatrix} r \\ s \end{pmatrix}, \qquad (3.25)$$

with real components r, s, such that multiplication by a complex constant $c = a + ib$ is represented by the operator $c = a\hat{1} - ib\sigma_y$, and the operator $\hat{L}$ is represented by the real linear Hermitian operator

$$\hat{L} = \left(-\frac{\partial^2}{\partial \xi^2} - \alpha + 2R^2\right)\hat{1} + R^2\sigma_Z \cos 2\theta + R^2\sigma_x \sin 2\theta\,, \qquad (3.26)$$

where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices. $\hat{L}$ is equivalent to the Hamiltonian of an $S = \frac{1}{2}$ particle in a potential $V = -\alpha + 2R^2$ and a field $\vec{B} = -R^2(\sin 2\theta, 0, \cos 2\theta)$ in spin space. The local σ_z direction may be turned into the direction of $\vec{B}$ by the real unitary transformation

$$\delta\tilde{\phi} = \exp(i\theta\sigma_y)\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} r\cos\theta + s\sin\theta \\ -r\sin\theta + s\cos\theta \end{pmatrix} \equiv \begin{pmatrix} \tilde{r} \\ \tilde{s} \end{pmatrix}. \qquad (3.27)$$

The transformed operator

$$\tilde{\hat{L}} = \exp(i\theta\sigma_y)\hat{L}\exp(-i\theta\sigma_y)$$

depends only on the derivatives of the phase $\theta(\xi)$ which may be expressed by Eq. (3.20a) in terms of J and R,

$$\tilde{\hat{L}} = \left(-\frac{\partial^2}{\partial \xi^2} - \alpha + 2R^2 + \frac{J^2}{R^4}\right)\hat{1} + 2iJ\frac{1}{R}\frac{\partial}{\partial \xi}\frac{1}{R}\sigma_y + R^2\sigma_z\,. \qquad (3.28)$$

We now discuss the spectral properties of this operator in the various states.

The uniform stationary states $\phi_s(\alpha, \theta_0)$, Eq. (3.17), have spectra

$$\lambda_1(q) = 2\alpha + q^2\,, \qquad (3.29a)$$

for the amplitude modes and

$$\lambda_2(q) = q^2\,, \qquad (3.29b)$$

for the phase modes. $\lambda_2(0) = 0$ is the GM which restores the broken rotational symmetry.

For the constant-phase amplitude waves $\Phi_T^{(A)}(\xi;\alpha, \Lambda, \theta_0)$ in the $(J=0)$ plane, the operator $\tilde{\hat{L}}$ is invariant under $T_\Lambda/2$, and the BZ is $(-2\pi/\Lambda, 2\pi/\Lambda)$. The eigenvalue problem decouples into two independent equations for the amplitude modes $(\tilde{r}, 0)$ and the phase modes $(0, \tilde{s})$. The amplitude modes have a potential $-\alpha + 3R^2$, and their spectrum $\lambda^{(A)}(q)$ is identical with that of Eq. (3.12). The phase modes with spectrum $\lambda^{(\theta)}(q)$ have a potential $-\alpha + R^2$, i.e., the periodic potential R^2 splits the twofold degenerate modes of the uniform stationary state $\phi_s^{(1)}$ in such a way that $\lambda^{(A)}(q) \geq \lambda^{(\theta)}(q)$. Since the states $\Phi_T^{(A)}(\xi;\alpha, \Lambda, \theta_0)$ break the translational and the rotational symmetry separately, there occur two GM's. The lowest branch of the amplitude modes contains the GM

$$\tilde{r}_1^{\rm GM} = \frac{\partial \Phi_T^{(A)}}{\partial \xi}\bigg|_{\theta_0 = 0}, \quad \tilde{s}_1^{\rm GM} = 0 \qquad (3.30)$$

restoring the translational symmetry, and the second branch of the phase modes contains the GM

$$\tilde{r}_2^{\rm GM} = 0\,, \quad \tilde{s}_2^{\rm GM} = \frac{1}{i}\frac{\partial \Phi_T^{(A)}}{\partial \theta_0}\bigg|_{\theta_0 = 0} = \Phi_T^{(A)}(\xi)\big|_{\theta_0 = 0} \qquad (3.31)$$

restoring the rotational symmetry. Both GM's occur at the edge of the BZ, $q = 2\pi/\Lambda$ (Fig. 11). Thus, the constant-phase amplitude waves are unstable against both a branch of amplitude modes and a branch of phase modes. The solitary state obtained in the limit $\Lambda \to \infty$ has one negative eigenvalue $\lambda^{(\theta)} < 0$ belonging to a phase mode. The GM (3.31) has one node, and corresponds therefore to the first excited state of the phase modes, whereas (3.30) is nodeless and represents the ground

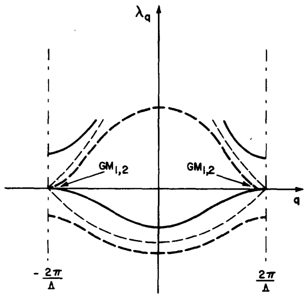

FIG. 11. Spectrum of the unstable state $\phi_s^{(1)} = 0$ of a periodic TW state of the flow [Eq. (3.14)]. For the state $\phi_s^{(1)} = 0$, amplitude and phase modes are degenerate (thin broken line). The spectrum of a periodic TW consists of bands of amplitude modes (heavy lines) and of phase modes (heavy broken lines). The BZ has extension $4\pi/\Lambda$. The amplitude Goldstone mode is at the edge of the BZ and belongs to the lowest amplitude band (see also Fig. 9). The phase Goldstone mode is also at the edge of the BZ but belongs to the second-lowest phase band.

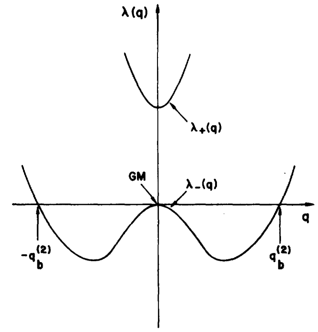

FIG. 12. Spectrum of an unstable (small amplitude) roll according to Eq. (3.32). In addition to the Goldstone mode of the roll located at $q = 0$ there are two eigenfunctions with eigenvalue $\lambda = 0$ and wave vectors $q_b^{(2)} = \pm[2(3k^2 - \alpha)]^{1/2}$, value k is the wave vector of the roll. The undamping of these additional modes gives rise to secondary bifurcation of doubly periodic solutions around the unstable roll (see Fig. 10).

state of the amplitude modes.

For the constant-amplitude states $\Phi_T^{(\theta)}(\xi;\alpha,k)$ Eq. (3.21), the operator $\tilde{L}$ is still translationally invariant. The eigenmodes are coupled amplitude and phase modes with eigenvalues[33]

$$\lambda_\pm(q) = \alpha - k^2 + q^2 \pm [(\alpha - k^2)^2 + 4k^2q^2]^{1/2}. \qquad (3.32)$$

For these states, a translation $\Delta\xi$ can be compensated by a rotation $\Delta\theta = -k\Delta\xi$, i.e., they are still invariant under a continuous group consisting of such combined transformations. Therefore, there exists only one GM which occurs on the λ_- branch at $q=0$. One finds for small q,

$$\lambda_-(q) = [(\alpha - 3k^2)(\alpha + k^2)]q^2 + O(q^4). \qquad (3.33)$$

The branch $\lambda_+(q)$ is positive for all q and $\alpha > \alpha_b \equiv k^2$; the branch $\lambda_-(q)$ is positive for all q and $\alpha > \alpha_E \equiv 3k^2$, but is negative for small q and $\alpha_E > \alpha > \alpha_b$ (Fig. 12). Thus, the large-amplitude rolls $(\alpha > \alpha_E)$ are stable, and the small-amplitude rolls $(\alpha_E > \alpha > \alpha_b)$ are unstable againt long-wavelength perturbations. This stability boundary $\alpha_E = 3k^2$ is known as the "Eckhaus instability."[33]

For the unstable rolls $(\alpha_E > \alpha > \alpha_b)$ with wave number $k = k_2(J,\alpha)$, the eigenvalue $\lambda = 0$ is degenerate: In addition to the GM at $q=0$ there occur two $\lambda = 0$ modes at

$$q = q_b^{(2)} = \pm[2(3k^2 - \alpha)]^{1/2}. \qquad (3.34)$$

These give rise to a secondary bifurcation (normal, $r=2$) of a family of doubly periodic TW states of the form

$$\Phi_T^{(2)}(\xi;\alpha,K,\Lambda,\theta_0) = R(\xi)\exp[i(K\xi + \theta(\xi) + \theta_0)],$$
$$R(\xi + \Lambda) = R(\xi), \quad \theta(\xi+\Lambda) = \theta(\xi), \qquad (3.35)$$

which have already been included in Fig. 10. Here, Λ and K depend on the amplitude A of the periodic part of R: $\Lambda(A;J,\alpha), K(A;J,\alpha)$. For $J \neq 0$ and $A \to 0$ one has $\Lambda \to 2\pi/q_b^{(2)}$, $K \to k_2(J,\alpha)$. For A approaching a maximum value, one obtains a solitary solution with $\Lambda \to \infty, K \to k_1(J,\alpha)$. For $J \to 0$, $K \to 0$, the state $\Phi_T^{(2)}(\xi;\alpha,K,\Lambda,\theta_0)$ becomes singly periodic and coincides with the state $\Phi_T^{(A)}(\xi;\alpha,2\Lambda,\theta_0)$ with a period twice that of the amplitude $R(\xi)$, as follows from continuing $\Phi_T^{(A)}$ in the form (3.18), with $\theta(\xi) = \theta_0$ in the first half and $\theta(\xi) = \theta_0 + \pi$ in the second half of each period. This can also be seen by comparing the bifurcation wave numbers of the two states: From (3.16) we find $q_b^{(A)} = \sqrt{\alpha}$, whereas from (3.34) one obtains with the help of (3.22) for $J \to 0$ the value $q_b^{(2)} = 2\sqrt{\alpha}$.

For these states $\Phi_T^{(2)}(\xi;\alpha,K,\Lambda,\theta_0)$, $\tilde{L}$ is invariant under T_Λ, and the BZ is $(-\pi/\Lambda, \pi/\Lambda)$. Since now the continuous symmetry is completely broken, there occur two GM's

$$\tilde{r}_1^{\rm GM} = \frac{dR}{d\xi}, \quad \tilde{s}_1^{\rm GM} = R\left(Q + \frac{d\theta}{d\xi}\right) \qquad (3.36)$$

restoring the translational symmetry, and

$$\tilde{r}_2^{\rm GM} = 0, \quad \tilde{s}_2^{\rm GM} = R \qquad (3.37)$$

restoring the rotational symmetry, both at $q=0$. Figures 13(a) and 13(b) show two possibilities for the spectrum of these doubly periodic states. The broken lines represent the spectrum at bifurcation, which has threefold degeneracy at $\lambda = q = 0$. Two of these eigenvalues belong to the GM's [(3.36) and (3.37)], the third one λ_3 moves away from $\lambda_3 = 0$ with increasing amplitude. Figures 13(a) and 13(b) correspond to the two cases $\lambda_3 < 0$ and $\lambda_3 > 0$, respectively. The corresponding mode $(\tilde{r}_3, \tilde{s}_3)$ belongs to a different representation as the two GM's,

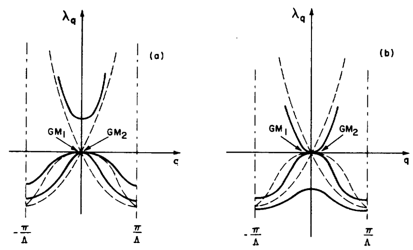

FIG. 13. Spectrum of an unstable roll (broken line) and of a doubly periodic solution (heavy lines). The Brillouin zone (BZ) has extension $\Lambda = 2\pi/q_b^{(2)}$. Two Goldstone modes (GM_1 and GM_2) corresponding to the two periods of the solution are located at $\lambda = 0$, $q = 0$ in the center of the BZ. The third $\lambda = 0$ mode of the uniform unstable state acquires a gap which could open either (a) above $\lambda = 0$ or (b) below $\lambda = 0$.

and λ_3 may therefore cross the value $\lambda_3 = 0$, i.e., one may have a transition from the situation of Fig. 13(a) to that of Fig. 13(b). In any case, the small-amplitude states $\Phi_T^{(2)}$ are unstable. Information on the stability for larger amplitudes could, in principle, be obtained from a $\vec{k} \cdot \hat{p}$ perturbation calculation for the curvature of the GM's at $q = 0$. However, the solitary state has already been shown to be unstable.[31] This suggests that one has for large amplitudes the situation of Fig. 13(b) which would render all doubly periodic states unstable. For $J \to 0$, the situation of Fig. 13(b) is consistent with the results of Fig. 11 for all amplitudes. This may be verified by replacing Λ in Fig. 11 by 2Λ and shifting the origin to the edge of the BZ, as required by the correspondence between the states $\Phi_T^{(2)}$ and $\Phi_T^{(4)}$ described above.

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