Calculation of the muonic ^{3}He hyperfine structure

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A calculation of the ground-state hyperfine splitting in muonic ^{3}He is given. It is based on a perturbative approach that was applied to the analogous calculation in muonic ^{4}He. The result for the hyperfine splitting is $\Delta\nu = 4164.9 \pm 3.0$ MHz. A semiempirical value for this splitting, based on the measured splitting in muonic ^{4}He, is $\Delta\nu = 4166.5 \pm 0.4$ MHz.

I. INTRODUCTION

In a recent paper, we reported a calculation of the ground-state hyperfine splitting of the muonic ^{4}He atom (^{4}Heμe) based on nonrelativistic perturbation theory.[1] That result is consistent with experiment and other calculations.[2–5] In this paper, we apply the same method to evaluate the ground-state hyperfine splittings in the muonic ^{3}He atom (^{3}Heμe). This requires a generalization to include the effect of the magnetic moment of the ^{3}He nucleus. In this case, the nuclear spin and the muon spin are strongly coupled to form either a spin-zero or spin-one $(^3\text{He}\mu)^+$ effective nucleus. For the spin-one state, there is a subsplitting due to the interaction of the $(^3\text{He}\mu)^+$ effective magnetic moment with the electron spin to form states with total angular momentum $\frac{1}{2}$ or $\frac{3}{2}$. Our main interest here is in this smaller splitting, which should be measurable.[6,7] Comparison of theory and experiment for muonic ^{3}He could provide a test of our understanding of the structure of this unique atom.

The Schrödinger equation for muonic helium is (in units in which $\hbar = c = 1$)

$$\left[-\frac{\nabla_\mu^2}{2M_\mu}-\frac{\nabla_e^2}{2M_e}-\frac{2\alpha}{x_\mu}-\frac{2\alpha}{x_e}+\frac{\alpha}{x_{\mu e}}\right]\psi(\vec{x}_\mu,\vec{x}_e) = E\psi(\vec{x}_\mu,\vec{x}_e)\,, \quad (1)$$

where $\vec{x}_\mu$ and $\vec{x}_e$ are the position vectors of the muon and the electron relative to the nucleus, and where $M_\mu = m_\mu m_N/(m_\mu + m_N)$ and $M_e = m_e m_N/(m_e + m_N)$ are the reduced masses of the muon and the electron with respect to the nucleus. The hyperfine perturbation of the ground state is given by the expectation value of

$$\delta H = -\frac{8\pi}{3}\vec{\mu}_N\cdot\vec{\mu}_\mu\delta(\vec{x}_\mu) - \frac{8\pi}{3}\vec{\mu}_\mu\cdot\vec{\mu}_e\delta(\vec{x}_\mu - \vec{x}_e) - \frac{8\pi}{3}\vec{\mu}_e\cdot\vec{\mu}_N\delta(\vec{x}_e)\,, \quad (2)$$

where $\vec{\mu}_e = -g_e e/(2m_e)\vec{s}_e$, $\vec{\mu}_\mu = -g_\mu e/(2m_\mu)\vec{s}_\mu$, and $\vec{\mu}_N = -g_N e/(2m_p)\vec{I}_N$ are the magnetic-moment vectors of the electron, the muon, and the nucleus, respectively, and where m_p is the proton mass. The nonrelativistic ground-state wave function factorizes into a product of coordinate-space and spin-space parts, so the level shift can be written as the spin-space expectation value of the operator

$$\delta H_s = -a\vec{I}_N\cdot\vec{s}_\mu - b\vec{s}_\mu\cdot\vec{s}_e - c\vec{s}_e\cdot\vec{I}_N\,, \quad (3)$$

where

$$a = \frac{2\pi\alpha}{3}\frac{g_N g_\mu}{m_p m_\mu}\langle\delta(\vec{x}_\mu)\rangle\,, \quad (4a)$$

$$b = \frac{2\pi\alpha}{3}\frac{g_\mu g_e}{m_\mu m_e}\langle\delta(\vec{x}_\mu - \vec{x}_e)\rangle\,, \quad (4b)$$

$$c = \frac{2\pi\alpha}{3}\frac{g_e g_N}{m_e m_p}\langle\delta(\vec{x}_e)\rangle\,, \quad (4c)$$

and where $\langle\ \rangle$ denotes the expectation value in coordinate space. In Ref. 1, we calculated the leading contributions to b in powers of M_e/M_μ. The leading contributions to a and c are calculated in the following section.

II. CALCULATION

To evaluate the coordinate-space expectation values in (4a) and (4c), we apply perturbation theory with the division

$$H = H_0 + \delta V \tag{5}$$

in which

$$H_0 = -\frac{\nabla_\mu^2}{2M_\mu} - \frac{\nabla_e^2}{2M_e} - \frac{2\alpha}{x_\mu} - \frac{\alpha}{x_e} \tag{6}$$

and

$$\delta V(\vec{x}_\mu, \vec{x}_e) = \frac{\alpha}{x_{\mu e}} - \frac{\alpha}{x_e} . \tag{7}$$

The zero-order wave function for the ground state is the product of normalized $1s$ hydrogenic wave functions

$$\psi_0(\vec{x}_\mu, \vec{x}_e) = \psi_{\mu 0}(\vec{x}_\mu)\psi_{e0}(\vec{x}_e) = \frac{1}{\pi}(2\alpha^2 M_\mu M_e)^{3/2} e^{-2\alpha M_\mu x_\mu} \times e^{-\alpha M_e x_e} . \tag{8}$$

Thus, the zero-order contribution to the expectation values in (4a) and (4c) are

$$a^{(0)} = \frac{2\pi\alpha}{3}\frac{g_N g_\mu}{m_p m_\mu}\int d\vec{x}_\mu \int d\vec{x}_e \psi_0^\dagger(\vec{x}_\mu, \vec{x}_e) \times \delta(\vec{x}_\mu)\psi_0(\vec{x}_\mu, \vec{x}_e) = \frac{16}{3}\frac{\alpha(\alpha M_\mu)^3}{m_p m_\mu} g_N g_\mu , \tag{9a}$$

$$c^{(0)} = \frac{2\pi\alpha}{3}\frac{g_e g_N}{m_e m_p}\int d\vec{x}_\mu \int d\vec{x}_e \psi_0^\dagger(\vec{x}_\mu, \vec{x}_e) \times \delta(\vec{x}_e)\psi_0(\vec{x}_\mu, \vec{x}_e) = \Delta\nu_F \frac{g_e g_N}{4}\frac{m_\mu}{m_p} , \tag{9b}$$

with $\Delta\nu_F = 8\alpha(\alpha M_e)^3/(3m_e m_\mu)$.

The first-order correction to the wave function is

$$\psi_1(\vec{x}_\mu, \vec{x}_e) = \int d\vec{x}_2 \int d\vec{x}_1 \sum_{n,n' \neq 0,0} \frac{\psi_{\mu n}(\vec{x}_\mu)\psi_{en'}(\vec{x}_e)\psi_{\mu n}^\dagger(\vec{x}_2)\psi_{en'}^\dagger(\vec{x}_1)}{E_{\mu 0} + E_{e0} - E_{\mu n} - E_{en'}} \delta V(\vec{x}_2, \vec{x}_1)\psi_0(\vec{x}_2, \vec{x}_1) , \tag{10}$$

where $E_{\mu 0}$ and E_{e0} are the zero-order hydrogenic $1s$-state muon and electron energies. The first-order correction in a is

$$a^{(1)} = \frac{4\pi\alpha}{3}\frac{g_N g_\mu}{m_p m_\mu}\int d\vec{x}_\mu \int d\vec{x}_e \psi_0^\dagger(\vec{x}_\mu, \vec{x}_e)\delta(\vec{x}_\mu)\psi_1(\vec{x}_\mu, \vec{x}_e) . \tag{11}$$

Substitution of (10) in (11) yields nonzero terms only for $n'=0$ because of the orthogonality of the electron wave functions. We thus have

$$a^{(1)} = \frac{4\pi\alpha}{3}\frac{g_N g_\mu}{m_p m_\mu}\int d\vec{x}\, \psi_{\mu 0}^\dagger(0) \sum_{n \neq 0} \frac{\psi_{\mu n}(0)\psi_{\mu n}^\dagger(\vec{x})}{E_{\mu 0} - E_{\mu n}} V_e(x)\psi_{\mu 0}(\vec{x}) , \tag{12}$$

where

$$V_e(x) = \int d\vec{x}_e \psi_{e0}^\dagger(\vec{x}_e)\delta V(\vec{x}, \vec{x}_e)\psi_{e0}(\vec{x}_e) = -\frac{\alpha}{x}[\alpha M_e x - 1 + (\alpha M_e x + 1)e^{-2\alpha M_e x}] . \tag{13}$$

As in the previous calculation,[1] only s states contribute to the sum over n in (12), so we may replace the sum by the s-state reduced Green's function for the muon,[8] with one coordinate set equal to zero

$$\sum_{n \neq 0} \frac{\psi_{\mu n s}(0)\psi_{\mu n s}^\dagger(\vec{x})}{E_{\mu 0} - E_{\mu n s}} = -\frac{2\alpha M_\mu^2}{\pi} e^{-2\alpha M_\mu x} \left[\frac{1}{4\alpha M_\mu x} - \ln(4\alpha M_\mu x) + \tfrac{5}{2} - \gamma - 2\alpha M_\mu x \right] . \tag{14}$$

In (14), $\gamma=0.5772\ldots$ is Euler's constant. Evaluation of (12) with the aid of (13) and (14) yields a result of order $(M_e/M_\mu)^3 a^{(0)}$ for $a^{(1)}$, which is negligible to the accuracy considered here. The term $a^{(1)}$ may be regarded as the correction to the muon density at the origin due to the perturbation of the muon wave function by the electron. Only the fraction, of order $(M_e/M_\mu)^3$, of the electron charge distribution inside the muon Bohr radius is effective in modifying this density.

The quantity $c^{(1)}$ is

$$c^{(1)}=\frac{4\pi\alpha}{3}\frac{g_e g_N}{m_e m_p}\int d\vec{x}_\mu \int d\vec{x}_e \psi_0^\dagger(\vec{x}_\mu,\vec{x}_e)\delta(\vec{x}_e)\psi_1(\vec{x}_\mu,\vec{x}_e)\ . \tag{15}$$

Because of the orthogonality of the muon wave functions, only the $n=0$ term in (10) survives upon substitution in (15). Hence,

$$c^{(1)}=\frac{4\pi\alpha}{3}\frac{g_e g_N}{m_e m_p}\int d\vec{x}\, \psi_{e0}^\dagger(0)\sum_{n\neq 0}\frac{\psi_{en}(0)\psi_{en}^\dagger(\vec{x})}{E_{e0}-E_{en}}V_\mu(x)\psi_{e0}(\vec{x})\ , \tag{16}$$

where

$$\begin{aligned} V_\mu(x) &= \int d\vec{x}_\mu \psi_{\mu 0}^\dagger(\vec{x}_\mu)\delta V(\vec{x}_\mu,\vec{x})\psi_{\mu 0}(\vec{x}_\mu) \\ &= -\frac{\alpha}{x}(1+2\alpha M_\mu x)e^{-4\alpha M_\mu x}\ . \end{aligned} \tag{17}$$

Only s states contribute to the sum over n in (16), so we may again employ the s-state reduced Green's function

$$\sum_{n\neq 0}\frac{\psi_{ens}(0)\psi_{ens}^\dagger(\vec{x})}{E_{e0}-E_{ens}}=-\frac{\alpha M_e^2}{\pi}e^{-\alpha M_e x}\left[\frac{1}{2\alpha M_e x}-\ln(2\alpha M_e x)+\tfrac{5}{2}-\gamma-\alpha M_e x\right]\ . \tag{18}$$

Substitution of (17) and (18) in (16) yields

$$c^{(1)}=\Delta\nu_F\frac{g_e g_N}{4}\frac{m_\mu}{m_p}\left[\frac{3}{2}\frac{M_e}{M_\mu}+\left[\frac{M_e}{M_\mu}\right]^2\ln\frac{M_\mu}{M_e}+(\ln 2+\tfrac{1}{4})\left[\frac{M_e}{M_\mu}\right]^2+O\left[\left[\frac{M_e}{M_\mu}\right]^3\ln\frac{M_\mu}{M_e}\right]\right]\ . \tag{19}$$

The leading term in (19) can also be obtained by applying Zemach's formula to take into account the effect of the finite charge distribution of the effective $(^3\mathrm{He}\mu)^+$ nucleus on the electron-nucleus hyperfine interaction.[9]

III. RESULTS

Diagonalization of δH_s in (3) yields the eigenvalues

$$\lambda_{1,2}=\tfrac{1}{4}(a+b+c)\pm\tfrac{1}{2}(a^2+b^2+c^2-ab-bc-ca)^{1/2}\ , \tag{20a}$$

$$\lambda_3=-\tfrac{1}{4}(a+b+c)\ . \tag{20b}$$

Both λ_1 and λ_2 are doubly degenerate and λ_3 is quadurply degenerate, corresponding to angular momentum $\frac{1}{2}$ and $\frac{3}{2}$, respectively. In the present case, $a>>b$ and $a>>c$, so λ_1 and λ_2 are well approximated by

$$\lambda_1=\tfrac{3}{4}a+\cdots\ , \tag{21a}$$

$$\lambda_2=-\tfrac{1}{4}a+\tfrac{1}{2}(b+c)+\cdots\ , \tag{21b}$$

where the omitted terms are higher order in b/a or c/a. The smaller splitting is given by

$$\Delta\nu=\lambda_2-\lambda_3=\tfrac{3}{4}(b+c) \tag{22}$$

to lowest order in b/a and c/a.

The lowest-order results for a, b,[1] and c are

$$a=\tfrac{16}{3}\alpha(\alpha M_\mu)^3\frac{g_N g_\mu}{m_p m_\mu}=3.3\times10^8\ \text{MHz}\ , \tag{23a}$$

$$b=\Delta\nu_F\frac{g_e g_\mu}{4}\left[1-3\frac{M_e}{M_\mu}+\tfrac{2}{3}S_{1/2}\left[\frac{M_e}{M_\mu}\right]^{3/2}\right]$$

$$=4461.7\ \text{MHz}\ , \tag{23b}$$

$$c=\Delta\nu_F\frac{g_e g_N}{4}\frac{m_\mu}{m_p}\left[1+\frac{3}{2}\frac{M_e}{M_\mu}\right]=1091.5\ \text{MHz}\ , \tag{23c}$$

based on the constants $S_{1/2}=2.8\pm0.2$, $m_\mu/m_e=206.7686$, $m_p/m_e=1836.15$, $m_N/m_p=2.993$, $g_N=4.25525$, $g_e\approx g_\mu\approx 2(1+\alpha/2\pi)$, $1/\alpha=137.0360$, and $R_\infty=3.289\,842\times10^9$ MHz.

We thus have for the ^{3}He hyperfine splitting

$$\Delta\nu=4164.9\pm3\ \text{MHz}\ , \tag{24}$$

where the uncertainty arises from uncalculated terms, including terms of relative order $(M_e/M_\mu)^2\ln(M_\mu/M_e)$.

It is of interest to compare the ^{4}He and ^{3}He hyperfine splittings. To the accuracy considered here, we have $\Delta\nu(^4\text{He})=b(^4\text{He})$, where the difference, $b(^4\text{He})-b(^3\text{He})=1.2$ MHz, is due to the differences in the reduced masses. Hence, employing the experimental value,[2,3] $\Delta\nu(^4\text{He})=4465.0$ MHz, we can obtain a semiempirical estimate for the ^{3}He hyperfine splitting:

$$\begin{aligned}\Delta\nu(^3\text{He})&=\tfrac{3}{4}(b+c)\\&=\tfrac{3}{4}\Delta\nu(^4\text{He})+\tfrac{3}{4}c\\&\quad+\tfrac{3}{4}[b(^3\text{He})-b(^4\text{He})]\\&=4166.5\pm0.4\ \text{MHz}\ .\end{aligned} \tag{25}$$

The error estimate in (25) is based on the assumption that the uncalculated contributions are weakly dependent on the nuclear mass.

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