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Analytic structure of the Lorenz system

M. Tabor and J. Weiss

Center for Studies of Nonlinear Dynamics, La Jolla Institute, P. O. Box 1434, La Jolla, California 92038

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The solutions of the Lorenz system are investigated by examination of their complex-time singularities. It is found that the location and type of singularity that occurs for complex time is critical in determining the behavior of the real-time solution. By direct expansion of the solution at a singularity its structure is determined. In general, the solutions are multiple valued in the neighborhood of a singularity; a property that is intimately related to the nonintegrability of the system. A numerical investigation is made of the analytic structure of solutions exhibiting turbulent bursts and undergoing period-doubling bifurcations.

I. INTRODUCTION

The system of equations introduced by Lorenz,[1]

$$\frac{dX}{dt}=\sigma(Y-X),$$

$$\frac{dY}{dt}=-XY+RX-Y, \tag{1.1}$$

$$\frac{dZ}{dt}=XY-BZ,$$

has provided a popular model for investigating the "transition to turbulence" of nonlinear (dissipative) dynamical systems.[2] For various values of the parameters (σ, R, B), the Lorenz system can be integrable or nonintegrable,[3] have a stable limit cycle or "strange attractor",[4] and exhibit "turbulent bursts"[5] and other complicated sequences of bifurcations.[6]

The integrability of a dynamical system is intimately related to its analytic structure. In the 1890's Painlevé and co-workers[7] determined those second-order, nonlinear ordinary differential equations whose only "movable singularities", i.e., singularities whose positions are initial-condition dependent, are poles (thus excluding those systems with algebraic branch points or essential singularities). Of the 50 possible equations in this class, 44 were found to be soluble (integrable) in terms of known functions; the remaining six led to new classes of functions; these are known as the "Painlevé transcendents." Ablowitz, Ramani, and Segur[8] have conjectured that these transcendents may play an important role in determining the integrability of partial differential equations and hence the applicability of the inverse-scattering-transform methods.

The singularity structure of a dynamical system is not only relevant to determining its integrability but can also provide a means of obtaining deeper insights into the "chaotic" or "turbulent" behavior of its nonintegrable regimes. The recent work of Morf, Orszag, and Frisch[9] indicates that the growth of mean-square vorticity ("enstrophy") in a three-dimensional, inviscid, incompressible fluid is caused by a real, finite-time singularity in the solution to the equations of motion (Euler's equations). They conjecture, as did Leray[10] for the Navier-Stokes equation, that the singular behavior of the solutions to Euler's equations plays an important role in the onset of turbulence. Further work of Frisch and Morf[11] contains detailed studies of "intermittency" exhibited by nonlinear systems and shows that the high-frequency behavior is determined by complex time singularities.

In this paper we present an analytical and numerical study of the analytic structure of the Lorenz system. One of the most important features revealed by this study is that, in general, the singularities have a complicated *multivalued* structure. This multiple-valuedness seems to be intimately related to the nonintegrability of the system. Segur[3] has found certain special parameter values of the Lorenz system for which integrals of the motion can be identified. In the cases where more than one integral can be found the solutions can either be expressed in terms of the elliptic functions or else the equations reduced to one of the Painlevé transcendents. Thus the "Painlevé condition," i.e., the property that the only movable singularities are poles, would appear to be a useful guide for determining the integrable regimes of this system. However, for other parameter values, integrals can also be found without the Painlevé condition being satisfied. Here, deeper properties of the analytic structure of the solution are important. In general, however, the system is completely nonintegrable and the singularities are organized in a complicated manner on an infinitely multisheeted Riemannian manifold.

II. LOCAL STRUCTURE OF A SINGULARITY AND THE PAINLEVÉ CONDITION

We first of all introduce the scaling[6]

$$X \to \frac{X}{\epsilon}, \quad Y \to \frac{Y}{\sigma\epsilon^2}, \quad Z \to \frac{Z}{\sigma\epsilon^2}, \quad t \to \epsilon t, \quad \epsilon = \frac{1}{\sqrt{\sigma R}},$$

thereby transforming the Lorenz equations to the form

$$\frac{dX}{dt} = Y - \sigma\epsilon X,$$
$$\frac{dY}{dt} = -XZ + X - \epsilon Y, \tag{2.1}$$
$$\frac{dZ}{dt} = XY - \epsilon BZ.$$

In the limit $\epsilon \to 0$ ($R \to \infty$) these equations reduce to a conservative integrable system and the solutions can be expressed in terms of the Jacobi elliptic functions.[6] These functions are doubly periodic (i.e., periodic in both real and imaginary directions) and have singularities, which are simple poles, arranged on an (infinite) periodic lattice in the complex t plane.

We consider the leading order behavior of a singularity at $t = t^*$ by setting

$$X = \frac{a}{(t-t^*)^\alpha}, \quad Y = \frac{b}{(t-t^*)^\beta}, \quad Z = \frac{c}{(t-t^*)^\gamma} \tag{2.2}$$

from which it is easily deduced that

$$\alpha = 1, \quad \beta = 2, \quad \gamma = 2, \tag{2.3}$$

and

$$a = \pm 2i, \quad b = \pm 2i, \quad c = -2. \tag{2.4}$$

To examine the behavior of the solution in the neighborhood of the singularity at t^* we make the ansatz

$$X = \frac{2i}{(t-t^*)} \sum_{j=0}^{\infty} a_j (t-t^*)^j,$$
$$Y = \frac{-2i}{(t-t^*)^2} \sum_{j=0}^{\infty} b_j (t-t^*)^j, \tag{2.5}$$
$$Z = \frac{-2}{(t-t^*)^2} \sum_{j=0}^{\infty} c_j (t-t^*)^j.$$

On substitution of these expansions into Eqs. (2.1) we obtain the following sets of relationships between the coefficients:

$$a_0 = b_0 = c_0 = 1, \tag{2.6}$$

which follows trivially from Eqs. (2.5):

$$a_1 = \frac{(3\sigma - 2B - 1)\epsilon}{6}, \quad b_1 = -\sigma\epsilon,$$
$$c_1 = \frac{(B - 1 - 3\sigma)\epsilon}{3}, \tag{2.7}$$

and for $j = 2, 3, 4\ldots,$

$$\begin{bmatrix} j-1 & 1 & 0 \\ 2 & j-2 & 2 \\ 2 & 2 & j-2 \end{bmatrix} \begin{bmatrix} a_j \\ b_j \\ c_j \end{bmatrix} = \begin{bmatrix} -\sigma\epsilon a_{j-1} \\ -2\sum_{k=1}^{j-1} a_{j-k}c_k - a_{j-2} - \epsilon b_{j-1} \\ -2\sum_{k=1}^{j-1} a_{j-k}b_k - \epsilon B c_{j-1} \end{bmatrix}. \tag{2.8}$$

Owing to the form of the coefficient matrix in the recursion relations (2.8) consistency conditions must be imposed when $j = 2$ and $j = 4$ (for these values it has no unique inverse). If these conditions, which impose restrictions on the parameters (σ, ϵ, B), are satisfied we can solve for the coefficient sets (a_2, b_2, c_2) and (a_4, b_4, c_4) and hence for all (a_j, b_j, c_j). However, the solutions for $j = 0, 2$ will depend on an arbitrary parameter; that is, (a_2, b_2, c_2) and (a_4, b_4, c_4) will be determined up to a vector that belongs to the null space of their respective coefficient matrices. Thus, the general solution at a singularity will depend on three arbitrary parameters, one each for $j = 2$ and $j = 4$ and the actual value of t^*, providing, of course, the conditions on the parameters (σ, ϵ, B) are met. These conditions are, for $j = 2$,

$$\epsilon^2(6\sigma^2 - \sigma B - 2\sigma) = B(B-1)\epsilon^2 \tag{2.9}$$

and, for $j = 4$,

$$\frac{\epsilon^2}{9}(B-1)[57(\sigma-1) - 15(B-1) + 24] = \epsilon^2\sigma(2\sigma - 1) \tag{2.10a}$$

and

$$2(1 + B - \sigma)a_1^2 c_1 \epsilon + \epsilon^2\left(\frac{2\sigma - B - 5}{3}\right)\left(\frac{B(1-\sigma\epsilon^2)}{2} + 2\sigma a_1^2 + B a_1 c_1\right) = 0. \tag{2.10b}$$

There are two conditions when $j=4$ (apart from the "trivial" one $\epsilon=0$), since the inclusion of an arbitrary parameter in the solution for (a_2, b_2, c_2) "splits" the condition for (a_4, b_4, c_4). We could have used the consistency condition at $j=4$ to fix the "arbitrary" parameter and relax condition (2.9). In this way a two-parameter branch of the general solution at a singularity could be found.

In general, the conditions (2.9) and (2.10a) specify σ and B while (2.10b) determines ϵ. For these values of (σ, ϵ, B) the ansatz (2.5) is valid, i.e., the solution satisfies the Painlevé condition. These values have been given by Segur,[3] who notes that the Lorenz system satisfies the Painlevé condition when

(i) $\sigma=0$. The equations are linear and this case is not included in the ansatz (2.5).

(ii) $\sigma=\frac{1}{2}$, $B=1$, $R=0$ ($\epsilon=\infty$). The equations have two exact integrals and the solution reduces to elliptic functions.

(iii) $\sigma=1$, $B=2$, $R=\frac{1}{9}$ ($\epsilon=3$). There is one first integral and the equations reduce to the second Painlevé transcendent.

(iv) $\sigma=\frac{1}{3}$, $B=0$, $R(\epsilon)$ arbitrary. There is one first integral and the equations reduce to the third Painlevé transcendent.

The other cases mentioned by Segur are as follows:

(v) $B=1$, $R=0$, ($\epsilon=\infty$), σ arbitrary. There exists a first integral but the Painlevé condition is not satisfied.

(vi) $B=2\sigma$, $R(\epsilon)$ arbitrary. There exists a first integral but again the Painlevé condition is not satisfied.

For case (vi) we note that when $B=2\sigma$, condition (2.9) is satisfied but conditions (2.10) are not. We interpret this as implying that the satisfaction of one consistency condition indicates the existence of one first integral. In view of this we suggest that since (2.9) is also satisfied for $B=1-3\sigma$, an additional case exists:

(vii) $B=1-3\sigma$, $R(\epsilon)$ arbitrary. The consistency condition (2.9) is satisfied and a first integral may exist (which we have not yet been able to identify) but the Painlevé condition is not satisfied.

We also note that condition (v) is not related to the consistency conditions and is presumably a property of a more general form of solution which we shall discuss in the next section.

III. LOCAL STRUCTURE OF A SINGULARITY: GENERAL PROPERTIES

The expansion about a movable singularity is, in general, not of the Painlevé-type and the resulting psi series contains logarithmic terms. The form of this expansion for the Lorenz system is (where, for notational simplicity, we set the pole position $t^*=0$)

$$
\begin{aligned}
X&=\frac{2i}{t}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}a_{kj}t^{j}(t^2\ln t)^k ,\\
Y&=\frac{-2i}{t^2}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}b_{kj}t^{j}(t^2\ln t)^k ,\\
Z&=\frac{-2}{t^2}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}c_{kj}t^{j}(t^2\ln t)^k .
\end{aligned}
\tag{3.1}
$$

A straightforward but tedious calculation gives the recursion relations

$$
\begin{pmatrix} 2k+j-1 & 1 & 0\\ 2 & 2k+j-2 & 2\\ 2 & 2 & 2k+j-2 \end{pmatrix}
\begin{pmatrix} a_{kj}\\ b_{kj}\\ c_{kj} \end{pmatrix}
=\begin{bmatrix} -(k+1)a_{k+1,j-2}-\sigma\epsilon a_{kj-1}\\[4pt]
-(k+1)b_{k+1,j-2}-a_{kj-2}-\epsilon b_{kj-1}-2\sum_{m=1}^{j}a_{kj-m}c_{0m}-2\sum_{m=0}^{j-1}a_{0j-m}c_{km}-2\sum_{l=1}^{k-1}\sum_{m=0}^{j}a_{k-l,j-m}c_{km}\\[4pt]
-(k+1)c_{k+1,j-2}-\epsilon B c_{kj-2}-2\sum_{m=1}^{j}a_{kj-m}b_{0m}-2\sum_{m=0}^{j-1}a_{0j-m}b_{km}-2\sum_{l=1}^{k-1}\sum_{m=0}^{j}a_{k-l,j-m}b_{lm} \end{bmatrix} .
\tag{3.2}
$$

To solve for a given (a_{kj}, b_{kj}, c_{kj}) one must know, in general, the coefficients $(a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta})$ for all (α, β) in the range

$$0 \leq \beta \leq j, \quad 0 \leq 2\alpha + \beta \leq 2k + j . \tag{3.3}$$

Thus, for any coefficient set (a_{kj}, b_{kj}, c_{kj}), $k, j \neq 0$, the recursion relations are not closed.

The coefficient matrix in (3.2) is singular when

(i) $k = j = 0$,
(ii) $2k + j = 2$, i.e., $k = 1$, $j = 0$, $k = 0$, $j = 2$,
(iii) $2k + j = 4$, i.e., $k = 2$, $j = 0$, $k = 1$, $j = 2$, $k = 0$, $j = 4$.

At every point where the coefficient matrix is singular, consistency conditions are imposed on the solution and vectors belonging to the null space of the solution are introduced. The cases $k = 0$, $j = 2$, and $k = 0$, $j = 4$ have already been described in the previous section.

The recursion relations for all coefficient sets (a_{k0}, b_{k0}, c_{k0}) are closed and take the form

$$\begin{pmatrix} 2k-1 & 1 & 0 \\ 2 & 2k-2 & 2 \\ 2 & 2 & 2k-2 \end{pmatrix} \begin{pmatrix} a_{k0} \\ b_{k0} \\ c_{k0} \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \sum_{l=1}^{k-1} a_{k-l0} c_{l0} \\ -2 \sum_{l=1}^{k-1} a_{k-l0} b_{l0} \end{pmatrix} . \tag{3.4}$$

The consistency conditions required for $k = 0, 1, 2$ are trivial, i.e., they are identically satisfied without restrictions on (σ, ϵ, B) or previously introduced eigenvectors. It is easy to show that

$$(a_{00}, b_{00}, c_{00}) = (1, 1, 1), \tag{3.5a}$$

$$(a_{10}, b_{10}, c_{10}) = (\lambda, -\lambda, -\lambda), \tag{3.5b}$$

$$(a_{20}, b_{20}, c_{20}) = (\gamma, -3\gamma, 2\gamma) + (0, 0, \lambda^2), \tag{3.5c}$$

where λ and γ are parameters whose values are fixed in the following manner.

The value of λ is determined by the consistency conditions introduced at $k = 0$, $j = 2$. Explicitly, we have (cf. Sec. II)

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} a_{02} \\ b_{02} \\ c_{02} \end{pmatrix} + \begin{pmatrix} \lambda \\ -\lambda \\ -\lambda \end{pmatrix} = \begin{pmatrix} -\sigma\epsilon a_{01} \\ -a_{00} - \epsilon b_{01} - 2a_{01}c_{01} \\ -\epsilon B c_{01} - 2a_{01}b_{01} \end{pmatrix} , \tag{3.6}$$

which yields

$$3\lambda = 2a_{01}(b_{01} - \sigma\epsilon) + \epsilon B c_{01} , \tag{3.7}$$

and using the values of (a_{01}, b_{01}, c_{01}) as given in Eq. (2.7) we obtain

$$\lambda = \frac{\epsilon^2}{9}[B(B-1) - 6\sigma^2 + \sigma B + 2\sigma] . \tag{3.8}$$

If the consistency condition (2.9) is satisfied, then $\lambda = 0$. In this case the logarithmic terms in the psi series enter as powers of $t^4 \ln t$. We will return to this point later.

In a similar manner the parameter γ is determined by the consistency conditions at $k = 1$, $j = 2$ which yield

$$\gamma = \frac{\lambda\epsilon^2}{5}\left[\frac{\sigma}{3}(2\sigma - 1) + \left(\frac{5 - 22\sigma}{3}\right)\left(\frac{B-1}{3}\right) + 6\left(\frac{B-1}{3}\right)^2\right] . \tag{3.9}$$

We note that γ is of order λ and that it does not depend on the eigenvector introduced at $k = 0, j = 2$. The eigenvector at $k = 1$, $j = 2$ is specified by the consistency conditions at $k = 0$, $j = 4$. Finally, we recall that the eigenvectors introduced at $k = 0$, $j = 2$, and $k = 0$, $j = 4$ are, in general, complex constants of integration. The above considerations show that the recursion relations (3.2) are well defined and consequently (3.1) represents the general form of the formal expansion about a singularity.

In order to investigate the leading order logarithmic terms in the general expansion (3.1) we introduce the generating functions

$$\Theta(x) = \sum_{k=0}^{\infty} a_{k0} x^k , \tag{3.10a}$$

$$\Phi(x) = \sum_{k=0}^{\infty} b_{k0} x^k , \tag{3.10b}$$

$$\Psi(x) = \sum_{k=0}^{\infty} c_{k0} x^k , \tag{3.10c}$$

where

$$x = t^2 \ln t . \tag{3.11}$$

Using the recursion relations (3.4) we can deduce

$$2x\frac{d\Theta}{dx} - \Theta + \Phi = 0 , \tag{3.12a}$$

$$2x\frac{d\Phi}{dx} - 2\Phi + 2\Theta\Psi = 0 , \tag{3.12b}$$

$$2x\frac{d\Psi}{dx} - 2\Psi + 2\Theta\Phi = 0 , \tag{3.12c}$$

from which we obtain the following (closed) differential equation for Θ:

$$2x^2\frac{d^2\Theta}{dx^2} = x\frac{d\Theta}{dx} - 3\lambda x\Theta + \Theta(\Theta^2 - 1) , \tag{3.13}$$

where

$$\Phi = \Theta - 2x\frac{d\Theta}{dx}$$

and

$$\Psi = \Theta^2 - 3\lambda x\,.$$

We note the special values at $x=0$,

$$\Theta = 1,\quad \frac{d\Theta}{dx} = \lambda,\quad \frac{d^2\Theta}{dx^2} = 2\gamma\,. \tag{3.14}$$

Before attempting to solve Eq. (3.13) it is amusing to determine whether it is actually of the Painlevé-type. We make the ansatz

$$\Theta(x) = \frac{2x_0}{(x-x_0)}\sum_{j=0}^{\infty}\theta_j(x-x_0)^j \tag{3.15}$$

and find that

$$\theta_0 = 1,\quad \theta_1 = \frac{3}{4x_0}\,,$$
$$\theta_2 = -\frac{1}{16x_0^2} + \frac{\lambda}{4x_0}\,,\quad \theta_3 = \frac{1}{32x_0^3} + \frac{21}{32}\frac{\lambda}{x_0^2}\,. \tag{3.16}$$

The coefficient θ_4 is arbitrary and the corresponding consistency condition requires that $\lambda = 0$. Thus, in general, Eq. (3.13) is not of the Painlevé-type. In the special case of $\lambda = 0$, as mentioned earlier, the expansions (3.1) involve powers of $t^4 \ln t$. Some of the properties of the leading-order logarithmic terms in this representation are described in the Appendix.

In order to solve Eq. (3.13) we make the substitution

$$\Theta = x^{1/2} f(x^{1/2})\,, \tag{3.17}$$

which yields

$$f'' = 2f^3 - 6\lambda f\,, \tag{3.18}$$

where primes denote differentiation with respect to the argument of f. Using the special values given in (3.14) we find

$$(f')^2 = f^4 - 6\lambda f^2 + 7\lambda^2 - 10\gamma\,, \tag{3.19}$$

which can be solved in terms of the Jacobi elliptic functions. Let

$$\alpha = 3\lambda + (2\lambda^2 + 10\gamma)^{1/2}\,,$$

$$\beta = 3\lambda - (2\lambda^2 + 10\gamma)^{1/2}\,,$$

and define the (squared) modulus

$$k^2 = 1 - (\beta/\alpha)^2\,, \tag{3.20}$$

where we assume $\alpha,\ \beta > 0$. The solution of (3.19) is

$$f(x) = \frac{\alpha}{\operatorname{sn}(\alpha x, k)}\,,$$

where sn is the Jacobi elliptic function and hence,

$$\Theta(x) = \frac{\alpha x^{1/2}}{\operatorname{sn}(\alpha x^{1/2}, k)}\,. \tag{3.21}$$

Using $x = t^2 \ln t$, we find

$$x(t) = \frac{2i}{t}\,\frac{\alpha t(\ln t)^{1/2}}{\operatorname{sn}(\alpha t(\ln t)^{1/2}, k)} + O(t)\,. \tag{3.22}$$

When α and β are not both positive, or not real, the solution of (3.19) will take different forms that are all expressible in terms of combinations of Jacobi elliptic functions. Rather than write out each case, we note that (α,β) are real when

$$2\lambda^2 + 10\gamma \geq 0\,. \tag{3.23}$$

Furthermore, the modulus k of the elliptic functions used to express the solution is either zero or one when

$$2\lambda^2 + 10\gamma = 0$$

or

$$7\lambda^2 - 10\gamma = 0\,. \tag{3.24}$$

Since

$$\operatorname{sn}(z, 0) = \sin(z)$$

and

$$\operatorname{sn}(z, 1) = \sinh(z)\,,$$

we might expect special types of solutions when $k^2 = 0$ and $k^2 = 1$. By algebraic simplification

$$7\lambda^2 - 10\gamma = \frac{7\lambda\epsilon^2}{9}[(B-1)^2 + (\sigma+1)(B-1) + 3\sigma(1-2\sigma)] \tag{3.25}$$

and for $\lambda\epsilon^2 \neq 0$, $7\lambda^2 - 10\gamma = 0$ if

$$B = 1 + 9\sigma \tag{3.26}$$

or

$$B = \tfrac{2}{5}(3\sigma + 1)\,. \tag{3.27}$$

In addition, we find that

$$2\lambda^2 + 10\gamma = \lambda\epsilon^2[\tfrac{14}{3}(B-1) + 4 - 14\sigma]\left(\frac{B-1}{3}\right) = 0\,, \tag{3.28}$$

which, for $\lambda\epsilon^2 \neq 0$, is satisfied when

$$B = 1 \tag{3.29}$$

or

$$B = 3\sigma + \tfrac{1}{7}\,. \tag{3.30}$$

We recall from Sec. II that there exists an integral of motion when $B = 1$, σ is arbitrary, and $R = 0$ ($\epsilon = \infty$). Of all the known integrals this alone appeared to be unrelated to the structure of the singularities. Our analysis suggests that the existence of this integral may, in fact, be related to the conditions $k^2=0$ and $k^2=1$. However, it is important to emphasize that the above analysis, although interesting in its own right, only indicates that the special values of B given in (3.26)–(3.30) may result in the corresponding solutions having some preferred property *near* a singularity. At this stage the most we can say concerning the existence of integrals of the motion is that the conditions (3.26), (3.27), (3.29), and (3.30) probably provide restrictions on the parameter space in which these integrals may be found. In Fig. 1 we draw the lines given by Eqs. (3.26), (3.27), (3.29), and (3.30) in the B, σ plane along with the lines $B=2\sigma$ (for which an integral of the motion is known) and $B=1-3\sigma$ (for which we believe an integral to exist). There are a number of points at which two or more of these lines intersect and which do not correspond to parameter values for which an integral is definitely known. It is at these points, e.g., $\sigma=\frac{1}{7}$, $B=\frac{4}{7}$, that we would predict an integral is most likely to be found. Our numerical evidence to date is not (and cannot be) conclusive and further analytic work to find restrictions on the corresponding values of ϵ is required.

IV. NUMERICAL RESULTS

In order to investigate the complex-time behavior of the Lorenz system we use two different techniques: integration by Taylor series expansion and integration by finite difference methods. The former method enables a rapid location of the singularities (along with their leading order behavior) and the latter enables a detailed study of the structure of a given singularity.

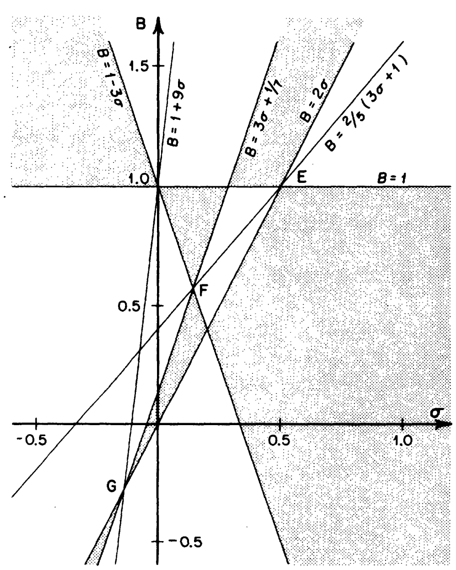

FIG. 1. Lines in the B,σ plane for which the modulus k^2 is either 0 or 1. Shaded areas correspond to regions where k^2 is complex. Intersection points E, F, and G correspond to (B,σ) values of $(\frac{1}{2}, 1)$, $(\frac{1}{7}, \frac{4}{7})$, and $(-\frac{1}{7}, -\frac{2}{7})$, respectively.

Integration by Taylor series expansion is performed here by using an algorithm developed by Chang and Corliss.[13] These authors have developed accurate methods for determining the radius of convergence of a long Taylor series and although the primary use of their algorithm is for rapid integration along the real axis it accurately locates the singularities nearest the real axis and determines their leading order behavior. Applied to the Lorenz system the singularities of $X(t)$, $Y(t)$, and $Z(t)$ are always found to occur coincidentally and in complex conjugate pairs. More detailed information about the singularities, e.g., accurate determination of their order and multivaluedness is provided by using the finite difference methods described below. Frisch and Morf[11] have made a preliminary study of the singularities of the Lorenz system in the strange-attractor regime, $R=28$, also using Taylor series methods. However, since they restricted their integrations to the real axis the multivaluedness of the singularities was not revealed. We add that the Chang-Corliss algorithm can be extended to integration along paths off the real axis and this provides a powerful tool for locating poles deep into the complex plane.

Integration by finite difference methods in the complex t plane is achieved by setting $t=u+iv$ and

$$\begin{aligned} X&=X_R(u,v)+iX_I(u,v)\,,\\ Y&=Y_R(u,v)+iY_I(u,v)\,,\\ Z&=Z_R(u,v)+iZ_I(u,v)\,, \end{aligned} \tag{4.1}$$

where R and I denote real and imaginary parts, respectively, thereby obtaining the set of six (real) first-order equations:

$$\begin{aligned} \frac{dX_R}{du}&=Y_R-\sigma\epsilon X_R\,,\\ \frac{dX_I}{du}&=Y_I-\sigma\epsilon X_I\,,\\ \frac{dY_R}{du}&=-X_RZ_R+X_IZ_I+X_R-\epsilon Y_R\,,\\ \frac{dY_I}{du}&=-X_RZ_I+X_IZ_R+X_I-\epsilon Y_I\,,\\ \frac{dZ_R}{du}&=X_RY_R-X_IY_I-\epsilon BZ_R\,,\\ \frac{dZ_I}{du}&=X_RY_I+X_IY_R-\epsilon BZ_I\,. \end{aligned} \tag{4.2}$$

The corresponding set of equations in terms of derivatives with respect to the imaginary time component v are easily deduced from the Cauchy-Riemann conditions By restricting oneself to contours made up of straight lines parallel to either the real or imaginary axes, standard integrators may be used without modification. Here we use the highly efficient algorithm of Shampine and Gordon[14] which is based on a predictor corrector method. Furthermore, by avoiding the use of complex arithmetic and working with double precision real variables one is able to achieve a high degree of accuracy. The method was thoroughly tested on differential equations of the form

$$\frac{df(z)}{dz}+f^n(z)=0, \quad n>0 \tag{4.3}$$

with the general solution

$$f(z)=\frac{1}{[(n-1)z+f_0^{-(n-1)}]^{1/n-1}}, \tag{4.4}$$

which has a movable singularity whose position is determined by the initial condition f_0. For example, in the case $n=2$, $f(z)$ has a simple pole, whereas for $n=3$ the singularity is a square root branch point. Integration around closed contours not enclosing branch points yielded initial and final values agreeing to within 9 or 10 decimal places. A similarly high degree of accuracy was obtained on repeated integrations around contours enclosing branch points. This sort of accuracy was carried over, in general, to the set of "complexified" Lorenz equations (5.2). We also mention that the scaled equations (2.1) seemed to be much more stable under complex integration than the unscaled equations (1.1).

The order of the singularities can be accurately determined by evaluating the integral

$$\frac{1}{2\pi i}\oint_{\mathfrak{C}}\frac{Z'(t)}{Z(t)}dt=\sum_{\text{zeroes}}N_{\text{zero}}-\sum_{\text{poles}}N_{\text{pole}}, \tag{4.5}$$

where prime denotes derivative with respect to (complex) t and N_{pole} and N_{zero} are the orders of the poles and zeroes, respectively, contained within the closed contour $\mathfrak{C}$ (taken in the anticlockwise sense). Similar integrals may be evaluated for $Y(t)$ and $X(t)$: The integrations were performed by Simpson's rule using anything up to 1000 points per side of contour. In the case of the contour enclosing regions devoid of poles and zeroes the integral (4.5) gave a value of zero to at least eight or nine decimal places, even for quite large contours. For all the singularities investigated the integral (4.5) gave values of 2.0, 2.0, and 1.0 to 5 decimal places for $Z(t)$, $Y(t)$, and $X(t)$, respectively, when the singularity was isolated within a fairly small contour (normally a rectangle of side 0.05 in u and 0.005 in v). We remark that our first attempts to locate the singularities were, in fact, performed by repeated applications of Eq. (4.5) evaluated around systematically chosen contours of ever decreasing size. This rather inefficient approach did, however, reveal the all important multivaluedness of the singularities. It was these empirical observations that first led us to consider expansions of the type employed in Sec. III.

For the parameter values of $\sigma=10$ and $B=\frac{8}{3}$ a strange attractor first appears at approximately $R=25$. For higher values of R this attractor "shrinks" down to a periodic orbit. At around $R=166$ the periodic orbit starts to exhibit turbulent bursts and undergoes a complicated sequence of bifurcations which rapidly results in the appearance of a new strange attractor. This behavior has been studied by Manneville and Pomeau.[5] Here we examine in detail the analytic structure of the solution in the vicinity of one such burst which occurs at $R=166.1$ ($\epsilon=0.024\,536\,6$). In Fig. 2 we show the real-time behavior or $Z(t)$ with a fairly low degree of resolution. In Fig. 3 we show the "burst" region in more detail and the corresponding singularity structure, nearest the real axis, in the complex t plane. We observe that the positions of the singularities are correlated with the maxima of the real-time solution and that the amplitude of these maxima is a fairly smooth function of the distance of the corresponding singularity from the real axis. This dependence, which is essentially inverse quadratic as would be expected on the basis of the leading-order behavior (2.3), is plotted in Fig. 4. One extremely useful consequence of this correlation between real-time extrema and singularity position is that the search for the latter is greatly facilitated by inspection of the form-

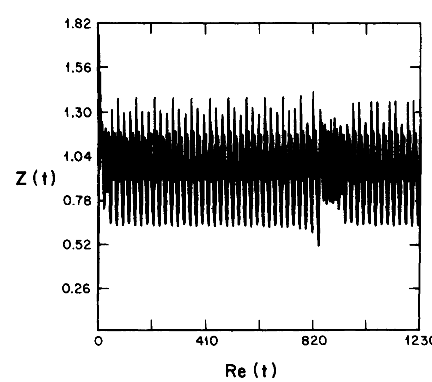

FIG. 2. Real-time behavior of $Z(t)$ for $\epsilon=0.024\,536\,6$ showing turbulent burst beginning at about $t=820$. Irregularity of burst is exaggerated by low degree of resolution. Initial conditions are $(X_0,Y_0,Z_0)=(0.0,\ 0.0602,\ 0.1626)$.

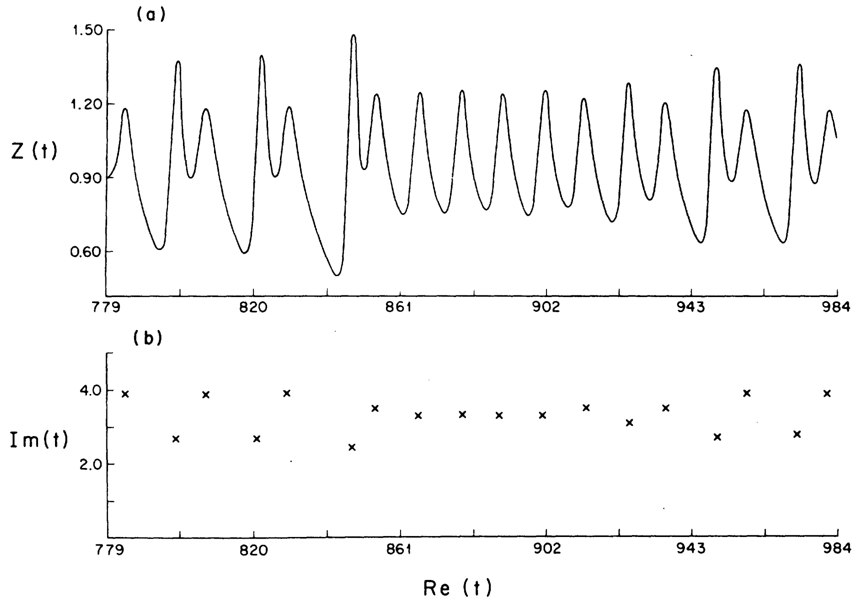

FIG. 3. Turbulent burst in greater detail: (a) real-time solution; (b) corresponding singularity structure in the complex t plane. Only singularities (marked x) nearest real axis are shown.

er. The pre- and post-burst regimes can thus be associated with two lines of singularities: the lower line being associated with the large-scale oscillations and the upper line with the small-scale oscillations. The frequency of the oscillations is clearly determined by the spacing of the singularities. The burst is started by a singularity from the lower line moving down towards the real axis followed by an apparent rearrangement of the double line of poles into a single line which then "unscrambles" itself back into the double-line structure of the periodic post-burst regime. Beyond the poles shown we believe that there are further arrays of poles although we have not yet investigated their disposition in detail. Clearly,

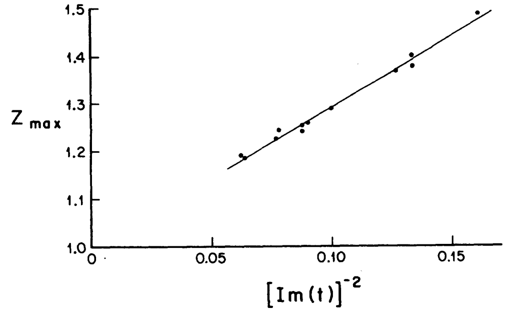

FIG. 4. Amplitude of real-time extrema (Z_{max}) vs inverse square of distance of associated singularity from real-time axis.

though these more distant poles will have little influence on the real-time behavior of the trajectories.

Working with the same parameter values of $\sigma = 10$, $B = \frac{8}{3}$ we now examine (see Fig. 5) the changes in analytic structure as ϵ is increased from zero (R decreased from infinity). At $\epsilon = 0$ we see a symmetric periodic orbit and the corresponding regular lattice of poles (of which we show only the first two lines). As ϵ is increased the orbit gradually distorts in the manner shown at $\epsilon = 0.020\,412$ ($R = 240$). The singularities (now no longer simple poles) are now much closer to the real axis and shifted relative to each other. By $\epsilon = 0.021\,320$ ($R = 220$) the orbit has undergone its first period doubling bifurcation. This is reflected in the analytic structure by the lowest line of singularities bifurcating into two distinct levels. By $\epsilon = 0.022\,086$ ($R = 205$) the orbit has undergone further bifurcations with the corresponding appearance of new levels of poles (more clearly visible when viewed over a much longer time period). At $\epsilon = 0.022\,361$ ($R = 200$) the orbit has undergone many more bifurcations and the singularities have become mixed up in a fairly random pattern that reflects the randomness of the motion itself.

V. DISCUSSION

Our results show that much useful information about the Lorenz system can be obtained by examining the structure of its singularities. By making the ansatz that the only singularities are

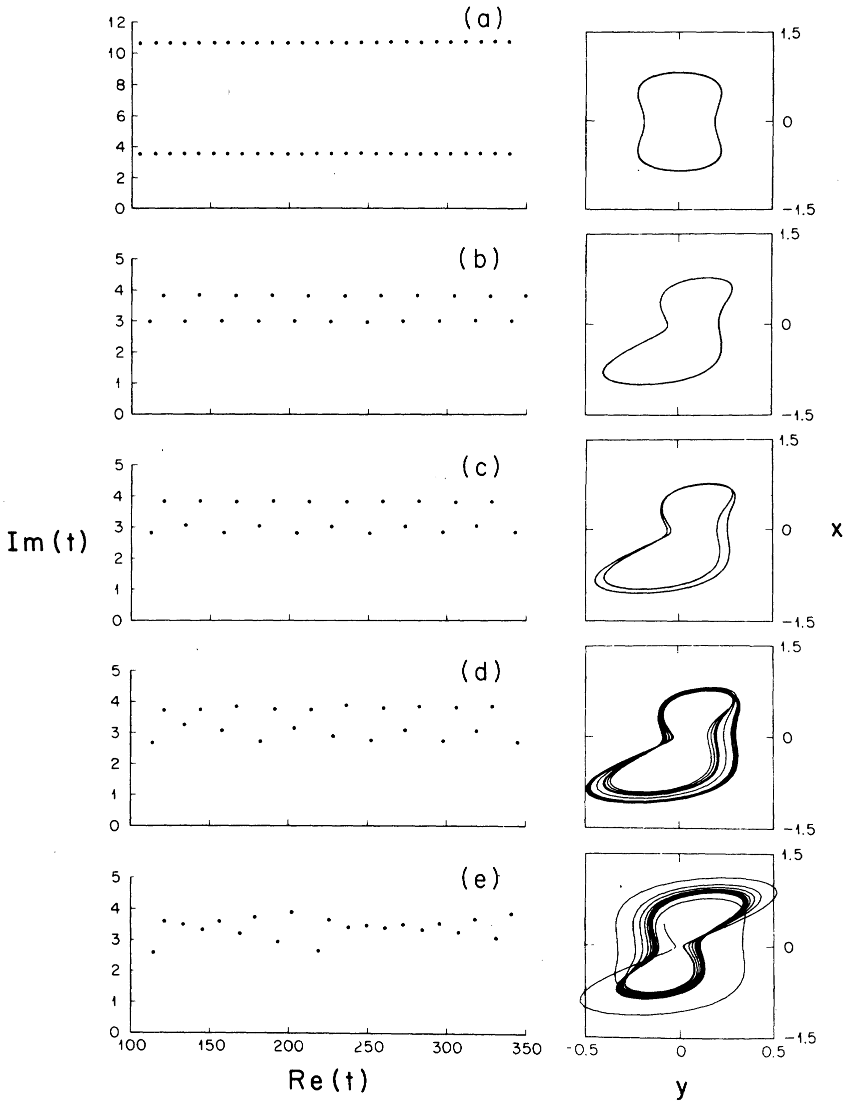

FIG. 5. Real-time solutions (plotted in $X-Y$ plane) and associated singularity structure for (a) $\epsilon = 0$, (b) $\epsilon = 0.020\,412$, (c) $\epsilon = 0.021\,320$, (d) $\epsilon = 0.022\,086$, and (e) $\epsilon = 0.022\,361$.

movable poles (the Painlevê condition) we are able to determine those parameter values for which the system is integrable. This approach would also appear capable of indicating the existence of just one integral of the motion, i.e., less than the full complement required for integrability. In the nonintegrable regimes we show that the poles have logarithmic corrections. Since the leading terms of these corrections can be cast in the form of closed recursion relations we have been able to examine this multivalued property of the solution without recourse to the full solution itself. Although this analysis is not complete it suggests the means by which certain parameter regimes of the system, for which the solutions may have some preferred properties, can be identified. Our various numerical results show that the singularities organize themselves in ways that directly influence the real-time behavior of the solutions.

Finally, we would like to make a few remarks concerning the analytic structure of dynamical systems in general. In the case of the Lorenz system the Painlevê condition seems to provide a successful criterion for identifying (most of) the integrable regimes. Clearly though, there are many other systems which are integrable yet do not satisfy the Painlevê condition. A simple example of this is furnished by the equations of motion for the pole representation of Burger's equation. Here, the movable singularities are branch points of order one-half. Here, our feeling is that for such systems, providing the singularities are of rational order, there will exist some transformation such that the multisheeted Riemannian manifold can be "unfolded" into a single sheet. A Painlevé-type analysis could then be applied. We also add that the equations of motion for systems of point vortices, although very similar in appearance to the poles of Burger's equation have a much more complicated analytic structure.[14]

The changes in analytic structure that occur for nonintegrable motion are, as we have seen in the case of the Lorenz system, very complicated. A further example of this sort of complexity is provided by Hamiltonian systems. Here, it would appear that a property of the nonintegrable regimes may be appearance of a *natural boundary*. We have recently studied such an object for the Henon-Heiles system and our results will be published in a forthcoming note.[15]

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APPENDIX: FORM OF SOLUTION WHEN $\lambda = 0$

When $\lambda = 0$ it is appropriate to expand the solution about a singularity (at $t = 0$) in powers of $t^4 \ln t$:

$$X = \frac{2i}{t} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{kj} t^j (t^4 \ln t)^k , \tag{A1}$$

with similar expression for Y and Z. Using the notation of Sec. III, we find the following recursion relations:

$$\begin{bmatrix} 4k+j-1 & 1 & 1 \\ 2 & 4k+j-2 & 2 \\ 2 & 2 & 4k+j-2 \end{bmatrix} \begin{bmatrix} a_{kj} \\ b_{kj} \\ c_{kj} \end{bmatrix}$$

$$= \begin{bmatrix} -(k+1)a_{k+1,j-4} - \sigma\epsilon a_{kj-1} \\ -(k+1)b_{k+1,j-4} - a_{kj-2} - b_{kj-2} - 2\sum_{m=1}^{j} a_{kj-m} c_{0m} - 2\sum_{m=0}^{j-m} a_{0j-m} c_{km} - 2\sum_{l=1}^{k-1}\sum_{m=0}^{j} a_{k-j,l-m} c_{km} \\ -(k+1)c_{k+1,j-4} - \epsilon B c_{kj-1} - 2\sum_{m=1}^{j} a_{kj-m} b_{0m} - 2\sum_{m=0}^{j-1} a_{0j-m} b_{km} - 2\sum_{l=1}^{k-1}\sum_{m=0}^{j} a_{k-l,j-m} b_{lm} \end{bmatrix} . \tag{A2}$$

The coefficient matrix is singular when

$$\begin{aligned} &\text{(i)}\quad 4k+j-1=1, \quad \text{i.e.,}\quad k=0,\ j=2, \\ &\text{(ii)}\quad 4k+j-2=2, \quad \text{i.e.,}\quad k=1,\ j=0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\ \ k=0,\ j=4. \end{aligned} \tag{A3}$$

By assumption ($\lambda = 0$) the consistency condition will be satisfied when ($k=0$, $j=2$). It is possible to show that the condition at ($k=1$, $j=0$) is trivial and that the eigenvector introduced at ($k=1$, $j=0$) is specified by the consistency condition ($k=0$, $j=4$). In this way the recursion relations are well defined and (A1) will represent the general form of the solution when $\lambda = 0$.

As in Sec. III the leading-order part of the expansion (A1) may be studied by defining

$$\Theta(x) = \sum_{k=0}^{\infty} a_{k0} x^k , \tag{A4}$$

where $x = t^4 \ln t$.

We find that

$$8x^2 \frac{d^2\Theta}{dx^2} + 2x \frac{d\Theta}{dx} = \Theta(\Theta^2 - 1) \tag{A5}$$

and when $x = 0$,

$$\Theta(0) = 1 ,$$

$$\left.\frac{d\Theta}{dx}\right|_{x=0} = \gamma ,$$

where

$$\begin{pmatrix} a_{10} \\ b_{10} \\ c_{10} \end{pmatrix} = \begin{pmatrix} \gamma \\ -3\gamma \\ 2\gamma \end{pmatrix} .$$

By expansion about a movable singularity, x_0 ($x_0 \neq 0$),

$$\Theta(x) = \frac{4x_0}{x - x_0} \sum_{j=0}^{\infty} \theta_j (x - x_0)^j . \tag{A6}$$

We find that Eq. (A5) is of Painlevé-type. For reference we note that

$$\theta_0 = 1,\quad \theta_1 = \frac{5}{8x_0},$$
$$\theta_2 = -\frac{5}{64x_0^2},\quad \theta_3 = \frac{5}{128x_0^3}, \tag{A7}$$

and θ_4 is an arbitrary parameter.

If

$$\Theta(x) = x^{1/2}\psi(x^{1/2}), \tag{A8}$$

ψ will satisfy the equation

$$2y^2 \frac{d^2}{dy^2}\psi = y\frac{d}{dy}\psi + \psi(\psi^2 - 1), \tag{A9}$$

where $y = x^{1/2}$.

Equation (A9) is just Eq. (3.13) when $\lambda = 0$, except here

$$\lim_{y\to 0} \psi \sim 1/y .$$

Furthermore, if we substitute

$$\Theta(x) = x^{1/4} f(x^{1/4}), \tag{A10}$$

it is true that

$$f'' = 2f^3 . \tag{A11}$$

Equation (A11) is integrable in terms of the Jacobi elliptic functions whose squared modulus k^2 is one-half ($k^2 = \frac{1}{2}$). Elliptic functions of this type are called the Lemniscate functions.[16]

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