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High-intensity approximations applied to multiphoton ionization

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We show that the most commonly used high-intensity approximations as applied to ionization by strong electromagnetic fields are related. We also discuss the applicability of the steepest-descent method in these approximations, and the relation between them and first-order perturbation theory.

I. INTRODUCTION

The advent of high-power lasers strongly stimulated theoretical studies on multiphoton processes and, in particular, multiphoton ionization. These treatments can be roughly divided in two classes: high-order perturbation theory, which is well known to be poorly convergent for strong electromagnetic fields (EMF), and a class of treatments, which we denominate generally by high-intensity approximations, where one tries to incorporate approximately the EMF to all orders.

Fifteen years ago Keldysh[1] pioneered the work in the area of high-intensity approximations, proposing a scheme based on perturbation theory with a modified basis, which would include already part of the effect of the EMF. For direct ionization, the final states are the exact solutions of the Schrödinger equation of an electron in the presence of a spatially homogeneous EMF (dipole approximation). Two other approaches were suggested one year later, the first by Perelomov *et al.*[2] which was claimed to be more accurate than Keldysh's approximation; the other by Nikishov and Ritus[3] who studied the solutions of the Klein-Gordon equation for a spin-zero particle in the presence of a strong EMF. Recently Berson[4] and Manakov and Rapoport[5] have proposed approximations that are similar to the one in Ref. 2. Henneberger[6] and Faisal[7] applied Kramer's unitary transformation[8] and, starting from the transformed Hamiltonian, sought to build an approximate solution suitable for the case of intense fields. An alternative treatment was proposed by Gersten and Mittleman,[9] who expressed the transition matrix in terms of an approximate Green's function, constructed from the exact solutions of the Schrödinger equation of an electron acted by an EMF in the dipole approximation.

Recently, Brandi and Davidovich[10] have shown that the approximations proposed in Refs. 1, 7, and 9 correspond to the first term of the same kind of expansion in the intra-atomic or intramolecular potential, and differ at most by unitary transformations.

In the present paper, we use the Green's function formalism to show the equivalence of the treatments by Perelomov *et al.*[2] and by Keldysh.[1] Together with the results of Ref. 10, the most frequently used high-intensity approximations are thus related. Using the gauge $\phi = 0$, $\mathrm{div}\vec{A} = 0$, instead of the choice of Refs. 1 and 2 ($\phi = -\vec{E}\cdot\vec{r}$, $\vec{A} = 0$), we retrieve the result for the ionization problem obtained by Faisal,[7] which we denominate space-translation approximation (STA), except for the presence of surface terms, which, however, do not contribute to the ionization rate.

Several criticisms have been presented against the applicability of the STA to the ionization problem in the region of strong EMF (see, for instance, Ref. 11, p. 1381). The retrieval of this approximation through the usual Green's function formalism shows, however, that in the case of ionization the STA should indeed be valid for fields sufficiently intense such that the expansion in the intraatomic or intramolecular potential is rapidly convergent.

Vaidyanathan *et al.*[12] have recently commented on the agreement between Keldysh's approximation and first-order perturbation theory when applied to one-photon transitions in semiconductors. They argued that the good agreement obtained could be fortuitous due to eventual cancellation of errors introduced by the use of the steepest-descent method and of approximated Bloch's function.

In Ref. 10, however, it was formally shown that the high-intensity approximations lead exactly to the first-order perturbation-theory results. In the present work the simplification introduced by the choice of gauge allows the identification of the approximations used in the application of the steepest-descent method as the unique source of the discrepancy between first-order perturbation theory and the proper limit of the calculated high-intensity approximation. We show explicitly that there is good agreement between both theories near the one-photon ionization threshold, which is the region where the saddle-point method is less accurate. This fact increases the confidence in

the application of this method to evaluate multiphoton absorption processes, except for the restrictions discussed in Sec. III and in the Appendix.

In Sec. II, we briefly review the Green's function approach, using it to establish the relation between the treatment of Perelomov *et al.*[2] and the other approaches, discussed in Ref. 10.

In Sec. III, we calculate the ionization rate using the gauge $\phi=0$, $\mathrm{div}\vec{A}=0$, as in the STA (Ref. 7), and show in the appendix that the saddle-point method cannot be applied to fields of arbitrary intensity. We discuss the region of applicability of this method.

In Sec. IV, we compare our results with those predicted by first-order perturbation theory.

II. RELATION AMONG HIGH-INTENSITY APPROXIMATIONS

We consider for simplicity an electron bound by a short-range potential $V(|\vec{x}|)$ in the presence of a spatially homogeneous EMF. The corresponding Hamiltonian is (in atomic units $\hbar=e=m=1$):

$$H=H_0+\bar{H}(t)\,, \tag{2.1}$$

where

$$H_0=-\frac{\nabla^2}{2}+V(|\vec{x}|) \tag{2.2}$$

and, depending on the specific choice of gauge, $\bar{H}(t)$ stands for either

$$H_1(t)=\frac{i}{c}\vec{A}(t)\cdot\vec{\nabla}+\frac{1}{2c^2}\vec{A}^2(t) \tag{2.3}$$

or

$$H_2(t)=-\vec{E}\cdot\vec{x}\,. \tag{2.4}$$

In Refs. 1 and 2 the interaction given by Eq. (2.4) is used, while $H_1(t)$ is assumed in Ref. 7. The term $A^2(t)$ in Eq. (2.3) is eliminated by a contact transformation in Ref. 9.

Let $\phi_i(\vec{x},t)$ and $\phi_f(\vec{x},t)$ be eigenfunctions of H_0 corresponding to the initial and final states of the system, respectively. Let the interaction $\bar{H}(t)$ be turned on and off adiabatically. The transition amplitude from ϕ_i to ϕ_f is given by

$$A_{fi}=\lim_{\substack{t\to\infty\\ t_0\to-\infty}} M_{fi}(t,t_0)\,, \tag{2.5}$$

where

$$M_{fi}(t,t_0)=i\int d^3x\,d^3x'\,\phi_f^*(\vec{x},t)G(\vec{x}t;\vec{x}'t_0)\,\phi_i(\vec{x}',t_0)\,. \tag{2.6}$$

The retarded Green's function G satisfies the equation

$$\left(i\frac{\partial}{\partial t}-H\right)G(x,x')=\delta^4(x-x')\,, \tag{2.7}$$

where we set $x\equiv(\vec{x},t)$. We also introduce the following retarded Green's functions:

$$\left(i\frac{\partial}{\partial t}-H_0\right)G_0(x,x')=\delta^4(x-x') \tag{2.8}$$

and

$$\left(i\frac{\partial}{\partial t}+\frac{\nabla^2}{2}-\bar{H}(t)\right)G_0'(x,x')=\delta^4(x-x')\,. \tag{2.9}$$

The functions G and G_0 are related by the integral equation:

$$G(x,x')=G_0(x,x')+\int d^4x''G(x,x'')\bar{H}(x'')G_0(x'',x') \tag{2.10}$$

(symbolically $G=G_0+G\bar{H}G_0$). An alternative representation for G is

$$G=G_0'+GVG_0'\,. \tag{2.11}$$

The high-intensity approximations discussed in Ref. 10 are obtained considering $G\approx G_0'$ on the rhs of Eq. (2.10) and taking a plane wave for $\phi_f(x,t)$. This involves neglecting in Eq. (2.11) terms depending on V, and should be appropriate for sufficiently strong EMF.

Now we show the equivalence of the above treatment with the one proposed in Ref. 2. Using the approximation described above, $G\approx G_0+G_0'\bar{H}G_0$, we can write

$$M_{fi}(t,t_0)=\delta_{fi}+i\int_{t_0}^{t}dt'\int d^3x\,d^3x'\,\phi_f^*(\vec{x},t)\times G_0'(x;x')\bar{H}(x')\phi_i(\vec{x}',t')\,. \tag{2.12}$$

As finite limits of integration are used in Eq. (2.12), we may set the Heaviside step function in G_0' equal to one. Then we may replace $G_0'\bar{H}$ by $(-i\partial/\partial t'+\frac{1}{2}\nabla'^2)G_0'$ in the integrand of Eq. (2.12), as it is immediately seen in the case of $\bar{H}=H_2$. When $\bar{H}=H_1(t)$, it is necessary first to perform a partial integration with respect to $\vec{x}'$.

After partial integration with respect to $\vec{x}'$ and t', one gets:

$$M_{fi}=\delta_{fi}+M_{fi}^{(1)}+M_{fi}^{(2)}\,, \tag{2.13}$$

where

$$M_{fi}^{(1)}=-\delta_{fi}+i\int d^3x\,d^3x'\,\phi_f^*(\vec{x},t)\times G_0'(\vec{x}t;\vec{x}'t_0)\,\phi_i(\vec{x}',t_0) \tag{2.14}$$

and

$$M_{fi}^{(2)} = \int_{t_0}^{t} dt' \int d^3x\, d^3x'\, \phi_f^*(\vec{x}, t) G_0'(x; x') \times \left(i\frac{\partial}{\partial t'} + \frac{\nabla'^2}{2}\right)\phi_i(\vec{x}', t') . \quad (2.15)$$

The term $M_{fi}^{(1)}$ comes from the surface contributions in the partial integration with respect to the variable t'. The partial integration with respect to the space variables leads to no surface contribution.

Since

$$\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2}\right)\phi_i(\vec{x}, t) = V(|\vec{x}|)\phi_i(\vec{x}, t) , \quad (2.16)$$

we may rewrite Eq. (2.15) as

$$M_{fi}^{(2)} = \int_{t_0}^{t} dt' \int d^3x\, d^3x'\, \phi_f^*(\vec{x}, t) G_0'(x, x') \times V(|\vec{x}'|)\phi_i(\vec{x}', t') \quad (2.17)$$

which corresponds to the approximation used by Perelomov *et al.*[2] to calculate the transition rate.

We show below that the surface term $M_{fi}^{(1)}$ does not contribute to the transition rate, which establishes the equivalence between the scheme of Perelomov *et al.* and the high-intensity approximations discussed in Ref. 10.

For definiteness let us choose the gauge $\phi = 0$, $\mathrm{div}\vec{A} = 0$. We then have

$$G_0'(x, x') = -i\theta(t - t') \int \frac{d^3p}{(2\pi)^3} \exp[i\vec{p}\cdot(\vec{x} - \vec{x}')] \exp\left[-\frac{i}{2}\int_{t'}^{t}\left(\vec{p} - \frac{1}{c}\vec{A}(\tau)\right)^2 d\tau\right] . \quad (2.18)$$

Let

$$\phi_i(\vec{x}, t) = \phi_i(\vec{x}) e^{iI_0 t} , \quad (2.19)$$

$$\phi_f(\vec{x}, t) = \phi_f(\vec{x}) e^{-ik^2 t/2} = \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}} e^{-ik^2 t/2} , \quad (2.20)$$

$$\phi_i(\vec{k}) = \int e^{-i\vec{k}\cdot\vec{x}} \phi_i(\vec{x}) d^3x , \quad (2.21)$$

and

$$\alpha(\vec{k}, t) = \int_0^t \left(\frac{A^2(\tau)}{2c^2} - \frac{\vec{A}(\tau)\cdot\vec{k}}{c}\right) d\tau . \quad (2.22)$$

Then

$$M_{fi}^{(1)} = -\delta_{fi} + \frac{\phi_i(\vec{k})}{(2\pi)^{3/2}} e^{i[\alpha(\vec{k}, t) - \alpha(\vec{k}, t_0)]} e^{i(k^2/2 + I_0)t_0} \quad (2.23)$$

and

$$M_{fi}^{(2)} = \frac{\phi_i(\vec{k})}{(2\pi)^{3/2}} \left(I_0 + \frac{\vec{k}^2}{2}\right) e^{i\alpha(\vec{k}, t)} \Delta(t, t_0, \vec{k}, \vec{A}_0, I_0) , \quad (2.24)$$

where

$$\Delta(t, t_0, \vec{k}, \vec{A}_0, I_0) = \int_{t_0}^{t} dt' \exp\left\{+i\int_0^{t'}\left[I_0 + \frac{1}{2}\left(\vec{p} - \frac{1}{c}\vec{A}(\tau)\right)^2\right] d\tau\right\} . \quad (2.25)$$

It is clear that, when $t \to +\infty$ and $t_0 \to -\infty$, Δ becomes a singular function, while $M_{fi}^{(1)}$ remains bounded. Therefore $M_{fi}^{(1)}$ does not contribute to the transition rate, and the scheme proposed in Ref. 2 is indeed equivalent to the other approaches. The different results obtained by Keldysh[1] and Perelomov *et al.*[2] are due only to different choices of the bound-state wave functions, and not to a more exact formulation of Ref. 2, as claimed by Perelomov *et al.* (cf. footnote on p. 931 of Ref. 2).

To show that Eq. (2.12) reduces to the usual first-order perturbation theory it is necessary to assume that $\phi_f(\vec{x}, t)$ is a plane wave. In fact, in first order of $\bar{H}$, G_0' in Eq. (2.12) must be replaced by the free propagator and, of course, after integration with respect to $\vec{x}$, we obtain the usual result.

III. THE IONIZATION RATE

We consider the case of linearly polarized light, so that $\vec{A} = \vec{A}_0 \cos\omega t$. From Eqs. (2.13), (2.23), and (2.24), we get for the transition probability, for

finite times (t,t_0), and $f\neq i$:

$$P_{fi}=|M_{fi}|^2=\frac{|\phi_i(\vec{k})|^2}{(2\pi)^3}\times[1+2(I_0+k^2/2)\times \mathrm{Re}(\Delta^* e^{-i[\alpha(k,t_0)-(k^2/2+I_0)t_0]})+(I_0+k^2/2)^2|\Delta|^2]. \quad (3.1)$$

We may choose, as usual, $t_0=-T/2$, $t=T/2$, and take the limit $T\to\infty$. Using in Eq. (2.25) the relation:

$$e^{ia\sin\omega t}=\sum_{-\infty}^{+\infty}J_n(a)e^{in\omega t}, \quad (3.2)$$

where $J_n(a)$ are cylindrical Bessel functions, it is easily shown that

$$\lim_{T\to\infty}\Delta\left(\frac{T}{2},-\frac{T}{2},\vec{k},\vec{A}_0,I_0\right)=2\pi\sum_{N,n}J_{N+2n}\left(\frac{\vec{k}\cdot\vec{A}_0}{\omega c}\right)J_n\left(\frac{A_0^2}{8\omega c^2}\right)\times\delta\left(\frac{k^2}{2}+\frac{A_0^2}{4}+I_0-N\omega\right). \quad (3.3)$$

Defining, as usual, the transition rate W_{fi} as

$$W_{fi}=\lim_{T\to\infty}\frac{|M_{fi}|^2}{T}, \quad (3.4)$$

it is clear that the only contribution to W_{fi} comes from the third term on the right-hand side of Eq. (3.1), leading to

$$W_{fi}=\frac{|\phi_i(\vec{k})|^2}{(2\pi)^3}\left(I_0+\frac{k^2}{2}\right)^2\sum_{N,n}J_{N+2n}\left(\frac{\vec{k}\cdot\vec{A}_0}{\omega c}\right)J_n\left(\frac{A_0^2}{8\omega c^2}\right)\times\delta\left(I_0+\frac{k^2}{2}+\frac{A_0^2}{4}-N\omega\right). \quad (3.5)$$

Although the transition amplitude obtained by Faisal[7] differs from the present results, the transition rate given by Eq. (3.5) is exactly the same as in the STA. This occurs because $M_{fi}^{(1)}$ can be discarded and $M_{fi}^{(2)}$ differs from Faisal's transition amplitude by a phase factor.

The total ionization rate is obtained from Eq. (3.5) by summing over all final momenta. The main difficulty is to evaluate the summation of the Bessel functions, which can be done approximately by using the method of steepest descent as suggested by Keldysh[1] and Perelomov *et al.*[2]

Using the integral representation

$$\sum_{n=-\infty}^{+\infty}J_n(a)J_{N+2n}(b)=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\, e^{i(N\theta-b\sin\theta+a\sin2\theta)}, \quad (3.6)$$

we may write the total ionization rate as

$$W=\sum_{N\geq\nu}W_N, \quad (3.7)$$

where

$$W_N=2\pi\int d^3k\,\delta\left(N\omega-\tilde{I}_0-\frac{k^2}{2}\right)|F_N(\vec{k})|^2, \quad (3.8)$$

$$F_N(\vec{k})=\frac{\phi_i(\vec{k})(I_0+k^2/2)}{(2\pi)^{5/2}}\int_{-\pi}^{\pi}d\theta\exp\left\{i\frac{I_0}{\omega}\left[\left(1+\frac{1}{2\gamma^2}+\frac{k^2}{2I_0}\right)\theta-\left(\frac{2}{I_0}\right)^{1/2}\frac{\vec{k}\cdot\hat{x}}{\gamma}\sin\theta+\frac{1}{4\gamma^2}\sin2\theta\right]\right\}, \quad (3.9)$$

and $E_0=cA_0/\omega$, $\tilde{I}_0=I_0[1+1/(2\gamma^2)]$, $\gamma=\omega\sqrt{2I_0}/E_0$, $\nu=\tilde{I}_0/\omega$, and $\vec{A}_0=A_0\hat{x}$. The quantity W_N may be interpreted as the ionization rate associated to the absorption of N photons.

The integral in Eq. (3.9) is evaluated by the saddle-point method, as in Refs. 1 and 2, under the condition $I_0/\omega\gg1$, and as long as E_0 is much smaller than $9E_B$, where E_B is the intraatomic field (see the Appendix). This last condition has not been established in Refs. 1 and 2, and imposes a high-intensity limit on the region of applicability of Keldysh's formula. For $E_0\ll 9E_B$, one has

$$|F_N(\vec{k})|^2=\frac{1}{(2\pi)^4}\frac{\omega}{I_0}\gamma\frac{|\phi_i(\vec{k})|^2}{(1+\gamma^2)^{1/2}}\left(I_0+\frac{k^2}{2}\right)^2\left\{1+(-1)^N\cos\left[\left(\frac{4I_0}{\omega}\right)\frac{(1+\gamma^2)^{1/2}}{\gamma}\frac{k_x}{\sqrt{2I_0}}\right]\right\}\times\exp\left\{-\frac{2I_0}{\omega}\left[\left(1+\frac{1}{2\gamma^2}+\frac{k^2}{2I_0}\right)\gamma\sinh^{-1}-\frac{(1+\gamma^2)}{2\gamma}-\frac{k_x^2}{2I_0}\frac{\gamma}{(1+\gamma^2)^{1/2}}\right]\right\}. \quad (3.10)$$

This result is identical to Eq. (53) of Ref. 2 for the case of a bound wave function of the form $e^{-\alpha r}/r$. For other wave functions the differences are due to the choice of different gauges. In the case of circularly polarized EMF it is also possible to show that the results are equivalent to those of Ref. 2 except for the choice of gauge.

One should note that the dependence of the transition rate on the bound-state wave function, as given by Eq. (3.10), is much simpler than that of Refs. 1 and 2 due to the choice of gauge $\phi=0$, $\vec{\nabla}\cdot\vec{A}=0$.

From Eqs. (3.8) and (3.10), we may proceed as in Ref. 2 to obtain the total ionization rate. This establishes the applicability of the STA to ionization by strong EMF.

IV. PERTURBATION THEORY

As mentioned previously, there exists some interest[12] in the applicability of the high-intensity approximations, in particular the Keldysh method, to few photon processes. In the case $N=1$ it is easily shown from Eq. (3.5), that for $\gamma \gg 1$ the dominant contribution to the summation comes from the Bessel function $J_1(\vec{A}_0 \cdot \vec{k}/\omega c) = J_1(k_x\sqrt{2I_0}/\omega\gamma)$. If one retains only the first term of its series expansion the results of first-order perturbation theory are exactly reproduced.[7] Therefore, we are able to estimate the errors in evaluating the summation in Eq. (3.5) by the method of steepest descent in its most disfavorable situation, that is, near threshold ionization $I_0/\omega \sim 1$.

To do this we cannot proceed as in Ref. 2, neglecting the term $\cos\{(4I_0/\omega)[(1+\gamma^2)^{1/2}/\gamma](k_x/\sqrt{2I_0})\}$, since in the case $\gamma \gg 1$ this term contributes significantly to the ionization rate. In fact from Eq. (3.10) one easily gets in this limit

$$|F_N(\vec{k})|^2 = \frac{(N\omega)^2}{(2\pi)^4}\frac{\omega}{I_0}\frac{|\phi_i(\vec{k})|^2}{(4\gamma^2)^N} \times\left[1+(-1)^N\left(1-\frac{4I_0}{\omega^2}k_x^2\right)\right] \times \exp\left[\frac{I_0}{\omega}\left(1+\frac{k_x^2}{I_0}\right)\right]. \tag{4.1}$$

The relevant contribution comes from $N=1$. If we assume $\omega \sim I_0$ and, consequently, by energy conservation, $k_x^2 \ll 2I_0$, we must use the expression

$$|F_1(\vec{k})|^2 = \frac{\omega|\phi_i(\vec{k})|^2}{(2\pi)^4\gamma^2}k_x^2\exp(1) \tag{4.2}$$

in Eq. (3.8), and compare this result with the first-order perturbation theory ionization rate

$$W_1' = I_0\int\frac{d^3k}{(2\pi)^2}\frac{|\phi_i(\vec{k})|^2}{2\gamma^2}k_x^2\delta\left(\omega-\frac{k^2}{2}-I_0\right). \tag{4.3}$$

This yields

$$\frac{W_1}{W_1'} = \frac{e}{\pi} \approx 0.865 .$$

It is clear then that the differences found by Vaidyanathan *et al.*[12] between Keldysh's method and first-order perturbation theory are basically due to the approximations involved in the use of the saddle-point method. As we mentioned previously the choice of the gauge $\phi=0$, $\vec{\nabla}\cdot\vec{A}=0$ allows an immediate comparison with perturbation theory. This is not the situation in Keldysh's result, since in this case the integral which is evaluated by the steepest-descent method depends on the wave function.

APPENDIX

We show now that, for very intense fields, the integral in Eq. (3.9) cannot be calculated by the usual saddle-point method, as applied by Keldysh, due to the presence of coalescing saddle points.

Let

$$f(\theta) = \frac{I_0}{\omega}\left[\left(1+\frac{1}{2\gamma^2}+\frac{k^2}{2I_0}\right)\theta - \left(\frac{2}{I_0}\right)^{1/2}\frac{k_x}{\gamma}\sin\theta + \frac{1}{4\gamma^2}\sin 2\theta\right]. \tag{A1}$$

The saddle points will be the zeros of $f'(\theta)$, and are given by

$$\cos\theta_s = \frac{\gamma k_x}{\sqrt{2I_0}} \pm i\gamma\left(1+\frac{k_\perp^2}{2I_0}\right)^{1/2} \tag{A2}$$

The positions of the four corresponding saddle points in the θ plane are sketched in Fig. 1, which also shows the paths followed by them as a function of increasing γ, for $k_x<0$, $k_x=0$, and $k_x>0$. As $\gamma \to 0$, the four saddle points coalesce in pairs around $\theta = \pm\pi/2$.

The integral (Eq. 3.9) can be evaluated by deforming the original contour of integration so that it passes through the two saddle points above the real axes.

In the usual procedure, the saddle-point contribution is evaluated by approximating $f(\theta)$ by $f(\theta_s) + \frac{1}{2}f''(\theta_s)(\theta-\theta_s)^2$, where θ_s is the saddle point. This cannot be done for arbitrarily high field in-

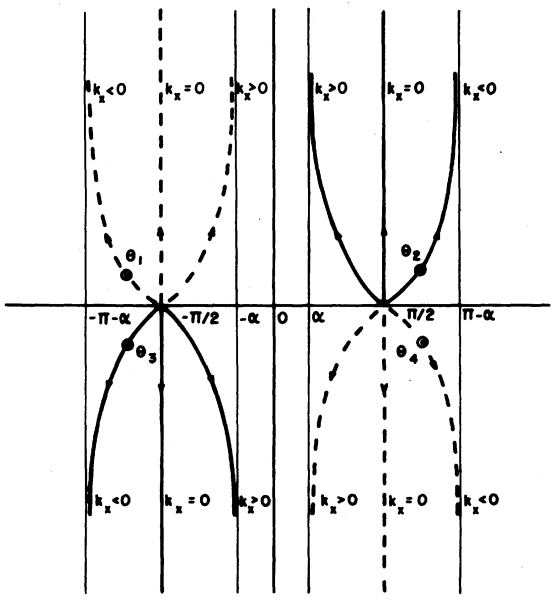

FIG. 1. Paths followed by the four saddle points θ_1, θ_2, θ_3, and θ_4 as γ increases. The dashed and full lines correspond to the plus and minus signs in Eq. (A2), respectively, and

$$\alpha = \tan^{-1}\left[\frac{2I_0}{|k_x|}\left(1+\frac{k_\perp^2}{2I_0}\right)^{1/2}\right].$$

tensities (when $\gamma \to 0$), since the contribution of the second- and third-order derivatives are comparable in this case.

Indeed, we have, for the saddle point denoted by one in Fig. 1:

$$f''(\theta_1) = \frac{2iI_0}{\gamma\omega}\left(1+\frac{k_\perp^2}{2I_0}\right)^{1/2}\left[1-\frac{k_x^2\gamma^2}{2I_0}+\gamma^2\left(1+\frac{k_\perp^2}{2I_0}\right) - \frac{2i\gamma^2 k_x}{\sqrt{2I_0}}\left(1+\frac{k_\perp^2}{2I_0}\right)^{1/2}\right], \quad \text{(A3)}$$

$$f'''(\theta_1) = \frac{I_0}{\omega}\left[4+\frac{2}{\gamma^2}+\frac{2k_\perp^2}{I_0}-\frac{k_x^2}{I_0}-3ik_x\sqrt{\frac{2}{I_0}}\left(1+\frac{k_\perp^2}{2I_0}\right)^{1/2}\right]. \quad \text{(A4)}$$

When $\gamma \to 0$, we have

$$f''(\theta_1) \to \frac{2iI_0}{\omega\gamma}\left(1+\frac{k_\perp^2}{2I_0}\right)^{1/2}, \quad \text{(A5)}$$

$$f'''(\theta_1) \to \frac{2I_0}{\omega\gamma^2}, \quad \text{(A6)}$$

and the higher-order derivatives are also proportional to $I_0/\omega\gamma$ or $I_0/\omega\gamma^2$. Similar results are valid for θ_2.

Expanding $f(\theta)$ around the saddle point, we have

$$f(\theta) = f(\theta_s) + \tfrac{1}{2}f''(\theta_s)(\theta-\theta_s)^2 + \tfrac{1}{6}f'''(\theta_s)(\theta-\theta_s)^3 + \cdots. \quad \text{(A7)}$$

In the usual saddle-point method, the range of the saddle point is defined by

$$|\theta-\theta_s| \approx \left(\frac{2}{|f''(\theta_s)|}\right)^{1/2}, \quad \text{(A8)}$$

so that when $\gamma \to 0$, the range of θ_1 would be given by

$$|\theta-\theta_1| \approx \left(\frac{\omega\gamma}{I_0}\right)^{1/2}\left(\frac{1}{1+k_\perp^2/2I_0}\right)^{1/4}. \quad \text{(A9)}$$

In this region, the magnitude of the contribution of the next term in the expansion (A7) would be

$$\tfrac{1}{6}|f'''(\theta_s)||\theta-\theta_s|^3 \approx \frac{1}{3}\left(\frac{\omega}{\gamma I_0}\right)^{1/2}\frac{1}{(1+k_\perp^2/2I_0)^{3/4}}. \quad \text{(A10)}$$

So, this term is negligible if and only if

$$\frac{1}{3}\left(\frac{\omega}{\gamma I_0}\right)^{1/2}\frac{1}{\left(1+\frac{k_\perp^2}{2I_0}\right)^{3/4}} \ll 1 \quad \text{(A11)}$$

for all $k_\perp$, which means that $\omega/9\gamma I_0 \ll 1$.

Using that $\gamma = \omega\sqrt{2I_0}/E_0$, and defining the intra-atomic field as $E_B = (2I_0)^{3/2}$, we see that this condition is equivalent to $E \ll 9E_B$, which defines the region of applicability of Keldysh's approach. For the hydrogen atom, $E_B = 5.13 \times 10^9$ V/cm.

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