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High-order perturbation theory of the imaginary part of the resonance eigenvalues of the Stark effect in hydrogen and of the anharmonic oscillator with negative anharmonicity

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The "perturbation theory" for the imaginary part of the resonance energies of the hydrogen atom in the Stark effect and of the two-dimensional anharmonic oscillator with negative anharmonicity, which is a separation constant in the Stark problem, is solved to high order. The solution is based on the Langer-Cherry generalization of the JWKB method, which can be carried out in closed form, order by order. The numerical results should be useful both in interpreting experimental measurements of excited-state lifetimes and in understanding the analytic properties of the Stark and anharmonic-oscillator resonances.

I. INTRODUCTION

The oldest problem of the quantum mechanical Rayleigh-Schrödinger perturbation theory (RSPT) is the Stark effect in hydrogen.[1,2] It is perhaps also the most pathological. The perturbation is singular, the spectrum is absolutely continuous along the entire real axis, and the perturbation series is divergent.[3-8] The bound states of the hydrogen atom become resonances in the Stark effect, moving off the negative real axis into the lower half energy plane.[3-8] The RSPT series for the energy is an asymptotic expansion $E \sim \sum E^{(N)}F^N$, with the energy coefficients increasing factorially rapidly,[3-9] $E^{(N)} \sim N!$ The imaginary part of the complex resonance eigenvalue is related to the ionization rate ($= -2\,\mathrm{Im}E/\hbar$) and is exponentially small ($e^{-k/F}$).

The RSPT energy coefficients $E^{(N)}$ are related to the "perturbation series" for ImE by a dispersion relation.[5,9] Although the $E^{(N)}$ are known to high order,[10] only the first few terms for ImE are known[9,11-13] in an expansion of the form $\mathrm{Im}E \sim F^{-b}e^{-k/F}\sum c_jF^j$. *The main purpose of this paper is to calculate the formal expansion for* ImE *to high order.*

In solving the Stark effect in hydrogen, it is convenient to separate the Schrödinger equation in parabolic coordinates.[1] The separated equations in "squared parabolic coordinates" are identical to the two-dimensional anharmonic oscillator, the separation constant being the oscillator energy. In generating the series for ImE, we first generate the analogous high-order series for the radially symmetric two-dimensional negatively anharmonic oscillator. This result is perhaps equally interesting because of the interest in the analytic properties of anharmonic-oscillator eigenvalues.[14-19]

The calculation of ImE, or more properly the ionization rate, was begun by Oppenheimer,[20] whose calculation unfortunately was erroneous. Lanczos[21-23] next developed the JWKB approach. Many others have used various techniques to calculate ImE numerically.[24-33] The leading asymptotic behavior of ImE has been discovered and rediscovered a few times.[11,12] The possibility for a complete asymptotic series calculated by the Langer-Cherry method[34,35] was first raised by Yamabe, Tachibana, Silverstone,[11] and by Slavjanov.[12] The first term beyond the leading term and the second for a special case were obtained by Damburg and Kolosov[13,33] using a slightly different approach. Silverstone, Adams, Čížek, and Otto[9] obtained three additional terms by numerically fitting the high order $E^{(N)}$ to the asymptotic formula implied by the dispersion relation. Here we obtain the series to arbitrarily high order via a Langer-Cherry-JWKB-like technique; the requisite tedious algebra is done on a computer. We tabulate the coefficients for a few examples, the ground state being taken out to fifty terms.

We do not attempt here to prove rigorously the details of the functional approximations used to produce the expansion for ImE, nor do we give a rigorous discussion of the series itself, although we do investigate numerically the growth of its terms. A rigorous earlier discussion[8] does provide a convergent iterative procedure to calculate Im$E(F)$ for F sufficiently small, and we suspect that, in principle, there is a summability technique applicable to the present series.

Just as the RSPT series is an effective way to calculate perturbed energies, especially when combined with a summation technique,[3,36] so one would expect the perturbation series for ImE to be an effective way to calculate ionization rates. The recent work of Koch[37,38] on the excited states of hydrogen and of Zimmerman, Littman, Kash, and Kleppner[39] on Rydberg states of alkalis makes an effective, simple, economical technique for calculating ionization rates pertinent.

II. ELEMENTARY FORMULAS

The aim of this paper is to derive a small-F expansion for the imaginary part of resonance eigenvalues of the Schrödinger equation for hydrogen in a uniform electrostatic field F,

$$(-\tfrac{1}{2}\nabla^2 - 1/r + Fz - E)\Psi = 0\,. \tag{1}$$

In this section elementary but unavoidable aspects of the derivation are reviewed, with emphasis on two equations important operationally.

Equation (1) conveniently separates in energy-scaled parabolic coordinates,[1,11]

$$\sigma = (-2E)^{1/2}(r+z)\,, \tag{2}$$

$$\rho = (-2E)^{1/2}(r-z)\,, \tag{3}$$

$$\Psi = (\sigma\rho)^{-1/2}\Phi_1(\sigma)\Phi_2(\rho)e^{im\phi}\,, \tag{4}$$

$$\left[-\sigma\left(\frac{d}{d\sigma}\right)^2 + \frac{m^2-1}{4\sigma} + \tfrac{1}{4}\sigma + f\sigma^2 - \beta_1\right]\Phi_1(\sigma) = 0\,, \tag{5}$$

$$\left[-\rho\left(\frac{d}{d\rho}\right)^2 + \frac{m^2-1}{4\rho} + \tfrac{1}{4}\rho - f\rho^2 - \beta_2\right]\Phi_2(\rho) = 0\,, \tag{6}$$

$$f = \tfrac{1}{4}(-2E)^{-3/2}F\,, \tag{7}$$

$$E = -\tfrac{1}{2}(\beta_1+\beta_2)^{-2}\,. \tag{8}$$

Equations (5) and (6) are eigenvalue equations for the separation constants β_1 and β_2 (and are equivalent as differential equations via transformation to squared parabolic coordinates to the radially symmetric two-dimensional anharmonic-oscillator eigenvalue equation). The separation constant β_1 has a purely discrete spectrum. The resonances of E arise from resonances of β_2, which correspond to "eigenfunctions" Φ_2 with an outgoing-wave boundary condition at infinity,[11,40] or, equivalently, to the analytic continuation from Im$f>0$ of the L^2 eigenfunctions of Eq. (6).[3-8]

Since the basic formal computation of Imβ_2 has been discussed in some detail,[11,40] a brief simplified derivation suffices here. If one multiplies Eq. (6) by $\rho^{-1}\Phi_2^*$ and integrates from 0 to ρ, one obtains

$$\mathrm{Im}\beta_2 = -(2i)^{-1}\left(\Phi_2^*\frac{d\Phi_2}{d\rho} - \Phi_2\frac{d\Phi_2^*}{d\rho}\right)\Big/\int_0^\rho \rho^{-1}|\Phi_2|^2 d\rho\,. \tag{9}$$

Equation (9) is rigorously valid for any value of ρ with the exact wave function Φ_2. An approximation to Eq. (9), asymptotically valid as $f\to 0$, can be obtained by using (or anticipating) that Imβ_2 is exponentially small ($\exp[-1/(6f)]$) and by computing Φ_2 with Imβ_2 set to zero. Such an approximate Φ_2 cannot satisfy both boundary conditions, and the value of ρ becomes significant.

Let $\Phi_{2,\infty}(\beta_2,\rho)$ and $\Phi_{2,0}(\beta_2,\rho)$ denote the solutions of Eq. (6) for arbitrary β_2 that satisfy, respectively, an outgoing-wave boundary condition at infinity and a regular boundary condition at 0. If β_2 were the resonance eigenvalue, then $\Phi_{2,\infty}$, $\Phi_{2,0}$, and Φ_2 would all be the same function (apart from normalization), but otherwise, especially for real β_2, they are different. Now let β_2 take on a resonance value: We have both $\Phi_2 \sim \Phi_{2,\infty}(\mathrm{Re}\beta_2,\rho) + O(\mathrm{Im}\beta_2)$, and $\Phi_2 \sim \Phi_{2,0}(\mathrm{Re}\beta_2,\rho) + O(\mathrm{Im}\beta_2)$, but neither uniformly in ρ. $\Phi_{2,\infty}$ provides a good approximation for the numerator[41] of Eq. (9) (which is "physically" the current density) if $\rho \gg 0$, while $\Phi_{2,0}$ provides a good approximation for the denominator[41] (which is physically the probability of the electron being in the atomic region) if ρ is well inside the outer turning point, $\rho \ll 1/(4f)$. For sufficiently small f, the region of overlapping validity is large: $0 \ll \rho \ll 1/(4f)$. (Cf. Ref. 8.)

If in the numerator of Eq. (9), Φ_2 is replaced by $\Phi_{2,\infty}(\mathrm{Re}\beta_2,\rho)$, then the numerator is essentially the Wronskian of two solutions of the same equation—i.e., it is a constant. We adjust the normalization of $\Phi_{2,\infty}$ to make the constant $2i$.

If in the denominator of Eq. (9) $|\Phi_2|^2$ is replaced by $|\Phi_{2,0}(\mathrm{Re}\beta_2,\rho)|^2$, the denominator becomes essentially a normalization integral but with finite upper limit. If $|\Phi_{2,0}|^2$ is replaced by an explicitly exponentially decreasing approximation, such as an RSPT partial sum $|\Phi_{2,\mathrm{RS}}|^2$, then the upper integration limit can be extended to infinity with exponentially small error times the dominant behavior $[\sim e^{-\rho}$, where $0 \ll \rho \ll 1/(4f)]$. We take $\Phi_{2,\mathrm{RS}}$ to be normalized to unity (order by order): $\int_0^\infty \rho^{-1}|\Phi_{2,\mathrm{RS}}|^2 d\rho \sim 1$.

In such a manner we obtain the key equation,

$$\mathrm{Im}\beta_2 \sim -|\Phi_{2,\mathrm{RS}}/\Phi_{2,\infty}|^2\,, \tag{10}$$

($\Phi_{2,\mathrm{RS}}$ normalized to unity, $\Phi_{2,\infty}$ normalized to have Wronskian $2i$ with $\Phi_{2,\infty}^*$). In Eq. (10), the symbol "$\sim$" means equality in the sense of asymptotic expansion,[42] and $|\Phi_{2,\mathrm{RS}}/\Phi_{2,\infty}|^2$ means the square of the quotient of the respective asymptotic expansions. A more detailed discussion of the steps leading to Eq. (10) can be found in Refs. 11 and 40.

Since $\Phi_{2,\mathrm{RS}}$ is known to high order,[10] calculation of the outgoing wave $\Phi_{2,\infty}$ to high-order yields, via Eq. (10), $\text{Im}\beta_2$.

When $\text{Im}\beta_2$ has been so obtained as a series expansion in f, the next step is to evaluate $\text{Im}E$ from Eq. (8). The result, neglecting terms of $O((\text{Im}E)^2)$, is

$$\text{Im}E \sim \left.\frac{\text{Im}\beta_2}{(\beta_1+\text{Re}\beta_2)^3 - f\dfrac{d}{df}(\beta_1+\text{Re}\beta_2)^3}\right|_{f=f_r}, \quad (11)$$

where f_r denotes f evaluated at $\text{Re}E$,

$$f_r = \tfrac{1}{4}(-2\,\text{Re}E)^{-3/2}F \quad (12)$$

$$= \tfrac{1}{4}[\beta_1(f_r) + \text{Re}\beta_2(f_r)]^3 F\,. \quad (13)$$

The "extra term" in the denominator of Eq. (11) comes from the dependence of $\beta_1(f)+\beta_2(f)$ on $\text{Im}E$ through f [Eq. (7)].

β_1 and $\text{Re}\beta_2$ as the RSPT power series in both f_r and F are known to high order[10,17]; the series for $\text{Im}E$, to be generated by Eq. (11), waits only for the series for $\text{Im}\beta_2$, which in turn waits for the appropriate expression for $\Phi_{2,\infty}$ to be put into Eq. (10).

III. ADAPTATIONS OF THE LANGER-CHERRY AND JWKB METHODS

To calculate $\text{Im}\beta_2$ with Eq. (10), one needs the outgoing-wave solution $\Phi_{2,\infty}$ of Eq. (6). In the unbound region $[\rho \gg 1/(4f)]$, one has $\Phi_{2,\infty} \sim \text{Ai}^{(+)}[\tfrac{1}{4}f^{-2/3}(1-4f\rho)]$, where $\text{Ai}^{(+)}$ is the outgoing-wave linear combination of the two standard Airy functions,[11,40] $\text{Ai}^{(+)}(z) = \text{Bi}(z) + i\text{Ai}(z)$. The difficulty is to find $\Phi_{2,\infty}$ inside the barrier region $[0 \ll \rho \ll 1/(4f)]$. We calculate $\Phi_{2,\infty}$ by what is in spirit a modified JWKB method, incorporating ideas from Langer[34] and Cherry.[35] The details of the method are developed in this section, the details of the calculation in the next.

We first prepare the differential equation (6) by changing the variable to $x = 4f\rho$. The motivation is to fix the outer turning point at $x=1$ as $f\to 0$. (As a function of ρ, the outer turning point moves to ∞ as $f\to 0$.) Equation (6) becomes (with $\text{Re}\beta_2$ for β_2)

$$\left[64f^2\left(\frac{-d^2}{dx^2} + \tfrac{1}{4}(m^2-1)x^{-2}\right) - 16\,\text{Re}\beta_2 f x^{-1} + (1-x)\right]\Phi_{2,\infty} = 0\,. \quad (14)$$

The basic idea of Langer[34] is to transfer determination of $\Phi_{2,\infty}$ to the determination of a new function $\phi(x)$ via,

$$\Phi_{2,\infty} = \pi^{1/2} f^{-1/6}\left(\frac{-d\phi}{dx}\right)^{-1/2}\text{Ai}^{(+)}[\tfrac{1}{4}f^{-2/3}\phi(x)]\,. \quad (15)$$

(The multiplicative constants $\pi^{1/2}f^{-1/6}$ have been chosen to make $\Phi^*_{2,\infty}\,d\Phi_{2,\infty}/d\rho - \Phi_{2,\infty}\,d\Phi^*_{2,\infty}/d\rho = 2i$.) The basic idea of Cherry[35] is to expand $\phi(x)$ in a series in f,

$$\phi(x) = \phi_0(x) + f\phi_1(x) + f^2\phi_2(x) + \cdots\,. \quad (16)$$

By putting Eq. (15) into Eq. (14) one finds the equation for $\phi(x)$:

$$\left(\frac{d\phi}{dx}\right)^2\phi = (1-x) - 16\,\text{Re}\beta_2 f x^{-1} + 64f^2\left[\tfrac{1}{4}(m^2-1)x^{-2} - \left(\frac{-d\phi}{dx}\right)^{1/2}\left(\frac{d^2}{dx^2}\right)\left(\frac{-d\phi}{dx}\right)^{-1/2}\right]. \quad (17)$$

From Eqs. (16) and (17), one has immediately,

$$\phi_0 = 1-x\,, \quad (18)$$

$$\phi_1 = 8\beta_2^{(0)}(1-x)^{-1/2}\ln\frac{1-(1-x)^{1/2}}{1+(1-x)^{1/2}}\,, \quad (19)$$

$$\phi_n = -\tfrac{1}{2}(1-x)^{-1/2}\int_1^x dx(1-x)^{-1/2}\Bigg\{-16\beta^{(n-1)}x^{-1} + \delta_{n,2}16(m^2-1)x^{-2} - \sum_{\substack{i,j,k\\(0\le i\le n-1,\ 0\le j\le n-1,\\ 0\le k\le n-1,\ i+j+k=n)}}\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\phi_k - 64\left[\left(\frac{-d\phi}{dx}\right)^{1/2}\frac{d^2}{dx^2}\left(\frac{-d\phi}{dx}\right)^{-1/2}\right]^{(n-2)}\Bigg\}, \quad (20)$$

where $[\cdot]^{(n-2)}$ means "the term of $[\cdot]$ proportional to f^{n-2}," and where we have used the RSPT expansion[10] for $\text{Re}\beta_2$,

$$\text{Re}\beta_2 \sim \sum_{n=0}\beta_2^{(n)}f^n\,. \quad (21)$$

Note especially for small f, that $\tfrac{1}{4}f^{-2/3}\phi \sim \tfrac{1}{4}f^{-2/3}\times(1-4f\rho)$, so that by Eq. (15) the outgoing-wave boundary condition has been built into $\Phi_{2,\infty}$ at the start. Note also that all the $\phi_n(x)$ are analytic at $x=1$. We have found that *all the ϕ_n can be obtained* *from Eq. (20) in terms of elementary functions.*

It is important to keep in mind that $\Phi_{2,\infty}$ is needed in Eq. (10) for x inside the barrier. Here $0 \ll x \ll 1$, and $\frac{1}{4} f^{-2/3}\phi(x) \sim \frac{1}{4} f^{-2/3} \gg 0$. The asymptotic expansion for $\mathrm{Ai}^{(+)}$ is appropriate,[43] from which we find

$$\Phi_{2,\infty} \sim \sqrt{2}\left(\frac{-dy}{dx}\right)^{-1/2}\left[e^{y/8f}\sum_k c_k\left(\frac{8f}{y}\right)^k + \tfrac{1}{2} i e^{-y/8f}\sum_k c_k\left(\frac{-8f}{y}\right)^k\right], \tag{22}$$

where

$$y = \tfrac{2}{3}\phi^{3/2} = y_0 + f y_1 + f^2 y_2 + \cdots , \tag{23}$$

$$c_k = (6k-1)!!/[(2k-1)!!(216)^k k!]. \tag{24}$$

It would appear that to use the ϕ_n obtained from Eq. (20), one would first compute y from Eq. (23), then $\phi_{2,\infty}$ from Eq. (22). Each step introduces a layer of tedium, however, that can be circumvented by defining a new function S, similar to ϕ and y, that leads to a simpler form for $\Phi_{2,\infty}$, and that is itself simpler to calculate.

We define $S(x)$ by

$$\Phi_{2,\infty} = \sqrt{2}\left(\frac{-dS}{dx}\right)^{-1/2}\left(e^{S/8f} + \tfrac{1}{2} i e^{-S/8f}\right), \tag{25}$$

$$S = S_0 + f S_1 + f^2 S_2 + \cdots , \tag{26}$$

which leads via Eqs. (22) and (23) to

$$S_0 = y_0 = \tfrac{2}{3}\phi_0^{3/2} = \tfrac{2}{3}(1-x)^{3/2}, \tag{27}$$

$$S_1 = y_1 = 8\beta^{(0)} \ln \frac{1-(1-x)^{1/2}}{1+(1-x)^{1/2}}, \tag{28}$$

$$S_2 = y_2 + 64 c_1 y_0^{-1} = y_2 + \tfrac{40}{9} y_0^{-1}, \tag{29}$$

and so forth. Although S is completely determined algebraically from y and dy/dx by Eqs. (22)–(29), direct computation of S from its differential equation is much easier. The equation for S is like that for ϕ, but simpler, and is obtained by putting Eq. (25) into Eq. (14),

$$\left(\frac{dS}{dx}\right)^2 = (1-x) - 16\,\mathrm{Re}\beta_2 f x^{-1} + 64 f^2\left[\tfrac{1}{4}(m^2-1)x^{-2} - \left(\frac{-dS}{dx}\right)^{1/2}\left(\frac{d^2}{dx^2}\right)\left(\frac{-dS}{dx}\right)^{-1/2}\right], \tag{30}$$

$$S_n = -\tfrac{1}{2}\int dx (1-x)^{-1/2}\left\{-16\beta^{(n-1)}x^{-1} + 16\delta_{n,2}(m^2-1)x^{-2} - \sum_{k=1}^{n-1}\frac{dS_k}{dx}\frac{dS_{n-k}}{dx} - 64\left[\left(\frac{-dS}{dx}\right)^{1/2}\frac{d^2}{dx^2}\left(\frac{-dS}{dx}\right)^{-1/2}\right]^{(n-2)}\right\} \quad (n \geq 2). \tag{31}$$

Just as Eq. (20) can be integrated in terms of simple functions, so can Eq. (31). Moreover, S_n turns out to be even simpler than the corresponding ϕ_n.

Equations (25)–(31), particularly (25) and (31), are the key equations for the calculations described in the next section. They clearly resemble the usual JWKB approximation with f playing the role of $\hbar$, but with some important practical differences. (i) The "potential" has a term linear in f, so that S has both odd and even terms in f. In $\hbar$ – JWKB, only even terms in $\hbar$ occur. (ii) The "outer turning point" at $x=1$ is defined only by part of the "potential" and is a turning point in the classical sense only when $f \to \infty$. In $\hbar$ – JWKB, the entire potential is used to fix the (true) outer turning point. (iii) The integrand for $y_n (n \geq 2)$ has a nonintegrable singularity at $x=1$. Notice in Eq. (31) that no limits have been specified for the integral. This is an important loose end that we now tie up.

The "integration constant" left vague by Eq. (31) is neither determined by Eq. (30), which involves only dS/dx and not S, nor by the form of Eq. (25) *alone*, since an additive constant in S only changes the relative weights of the positive and negative exponential solutions. Rather, it is fixed by the *equality* of Eq. (25) with Eqs. (22) and (15) that insures the *outgoing-wave boundary condition.* Since that equality is difficult to use directly we use it indirectly.

First we note that, by the definition (23) of y, $(1-x)^{-1/2}y$ is order-by-order meromorphic at $x=1$. Second we note that in Eq. (22) the $i\,\mathrm{Im}\phi_{2,\infty}(22)$ is one-half times the analytic continuation of $\mathrm{Re}\Phi_{2,\infty}(22)$ clockwise about $x=1$ from $x = 1 - |1-x|e^{i0}$ to $x = 1 - |1-x|e^{-2\pi i}$. [By $\Phi_{2,\infty}(22)$ we mean the *asymptotic expansion* in Eq. (22). The analytic-continuation relationship holds for $\Phi_{2,\infty}(22)$, but *not* for the function $\Phi_{2,\infty}$ itself, which is analytic at $x=1$. As is well known,[42] analytic properties of asymptotic expansions can differ from those of the function being represented.] Since the asymptotic expansion of Eq. (25) is essentially the same expansion as Eq. (22), it must have the same analytic behavior, which in turn implies that S changes by $e^{-\pi i}$ on the path described above. It is easy to see *directly* from Eq. (31) that S_n has the form

$$S_n = \text{const} + (1-x)^{-1/2} \times (\text{function meromorphic at } x=1)\,. \tag{32}$$

Thus the integration constant in Eq. (32) is necessarily zero; we may specify the integration $\int dx$ in Eq. (31) more definitely as $\frac{1}{2}\int_{\gamma_x} dx$, where γ_x is the path from $x - i0$, counterclockwise about 1, to $x + i0$.

Equations (25) and (31), with $\int dx = \frac{1}{2}\int_{\gamma_x} dx$, may be regarded as a JWKB "connection formula" for an outgoing-wave boundary condition, valid to all orders in f.

IV. CALCULATION OF $\Phi_{2,\infty}$. RELATIONSHIP OF $\Phi_{2,\infty}$ TO $\Phi_{2,\text{RS}}$. CALCULATION OF $\text{Im}\beta_2$ AND $\text{Im}E$

The integrands dS_n/dx of Eq. (31) can be put into a simple form by the method of partial fractions (and by iterative substitution),

$$\frac{dS_n}{dx} = (1-x)^{-1/2}\{p_n(x^{-1}) + q_n[(1-x)^{-1}]\}, \quad n \geq 2 \tag{33}$$

where $p_n(x^{-1})$ is a polynomial of degree n in x^{-1}, and $q_n[(1-x)^{-1}]$ is a polynomial of degree $[3n/2]-1$ in $(1-x)^{-1}$. (Here $[3n/2]$ denotes the largest integer that does not exceed $3n/2$.) Consequently, S_n has the form

$$S_n = (1-x)^{1/2}\{P_n(x^{-1}) + Q_n[(1-x)^{-1}]\} + k_n \ln \frac{1-(1-x)^{1/2}}{1+(1-x)^{1/2}}\,, \tag{34}$$

where P_n and Q_n are polynomials of degrees $n-1$ and $[3n/2]-1$. Given such a simple form, it is straightforward to program a computer to calculate the polynomials p_n, q_n, P_n, and Q_n. The first several S_n so obtained for the ground state are given in Table I.

Observe in Table I that only S_1 has a logarithmic contribution; for all $n \neq 1$, $k_n = 0$. On the other hand, the integral of the term $\beta_2^{(n-1)}(1-x)^{-1/2}x^{-1}$ in Eq. (31) is exactly a logarithmic term. Its absence in S_n ($n \geq 2$) implies that $\beta^{(n-1)}$ has a value that cancels the logarithmic contribution of the rest of the integrand. That is, the RSPT $\beta^{(1)}$, $\beta^{(2)}, \ldots$ can be calculated by requiring that $S_2, S_3, \ldots$ have no logarithmic term. In calculating S, we are also doing RSPT!

TABLE I. Coefficients c_{ni}, d_{ni}, and k_n for the state $n_2 = 0$, $m = 0$:

$$S_n(x) = (1-x)^{1/2} \sum_i c_{ni} x^{-i} + \sum_i d_{ni}(1-x)^{-i} + k_n \ln\{[1-(1-x)^{1/2}]/[1+(1-x)^{1/2}]\}\,.$$

n	i	c_{ni}	d_{ni}	k_n
1				4
2	1	−16	16	
	2		6.666 67	
3	1	−144	144	
	2	−96	48	
	3		80	
4	1	−2 912	2 784	
	2	−1 962.666 67	949.333 33	
	3	−1 109.333 33	864	
	4		800	
	5		491.111 11	
5	1	−83 504	76 912	
	2	−56 864	26 832	
	3	−37 248	21 424	
	4	−18 176	16 592	
	5		13 360	
	6		17 680	
6	1	−2 983 200	2 664 864	
	2	−2 046 464	945 952	
	3	−1 423 667.2	709 715.2	
	4	−879 001.6	525 472	
	5	−377 651.2	417 696	
	6		385 440	
	7		388 960	
	8		220 866.666 67	
7	1	−125 217 952	109 160 864	
	2	−86 396 608	39 304 928	
	3	−62 208 512	28 401 440	
	4	−42 324 992	20 645 920	
	5	−24 489 984	16 314 592	
	6	−9 412 608	14 180 768	
	7		12 576 800	
	8		11 159 306.666 67	
	9		13 252 000	
8	1	−5 985 076 672	5 114 690 240	
	2	−4 147 899 392	1 862 745 024	
	3	−3 056 238 080	1 311 250 368	
	4	−2 196 716 397.714 29	944 072 914.285 71	
	5	−1 442 041 270.857 14	743 035 712	
	6	−782 275 145.142 86	630 531 392	
	7	−272 376 978.285 71	544 525 504	
	8		477 725 120	
	9		445 681 600	
	10		450 568 000	
	11		244 196 480.952 38	

To bring out more vividly the connection with RSPT, we look at $\Phi_{2,\infty}$ for small f and small ρ. First, S_0 and S_1,

TABLE II. Coefficients $b^{(N)}$ and $a^{(N)}$ for Im$\beta_2(f)$ and Im$E(F)$ for Eqs. (44)–(47).

n_1	n_2	m	N	$b^{(N)}$	$a^{(N)}$
0	0	0	0	$1.000 000 000 000 00 \times 10^{0}$	$1.000 000 000 000 00 \times 10^{0}$
			1	$-1.766 666 666 666 67 \times 10^{1}$	$-8.916 666 666 666 67 \times 10^{0}$
			2	$-7.094 444 444 444 44 \times 10^{1}$	$2.556 597 222 222 22 \times 10^{1}$
			3	$-2.076 771 604 938 27 \times 10^{3}$	$-1.587 464 313 271 60 \times 10^{2}$
			4	$-7.261 242 541 152 26 \times 10^{4}$	$4.690 486 854 584 62 \times 10^{2}$
			5	$-3.007 992 238 854 60 \times 10^{6}$	$-1.027 317 283 344 45 \times 10^{4}$
			6	$-1.421 562 708 112 81 \times 10^{8}$	$5.635 378 188 555 73 \times 10^{3}$
			7	$-7.491 106 257 035 15 \times 10^{9}$	$-1.456 750 142 759 06 \times 10^{6}$
			8	$-4.339 404 851 569 23 \times 10^{11}$	$-4.948 957 447 445 98 \times 10^{6}$
			9	$-2.737 219 336 385 62 \times 10^{13}$	$-3.410 502 625 929 72 \times 10^{8}$
			10	$-1.867 541 477 404 24 \times 10^{15}$	$-2.588 984 466 134 19 \times 10^{9}$
			11	$-1.371 218 133 067 99 \times 10^{17}$	$-1.152 824 958 699 14 \times 10^{11}$
			12	$-1.079 048 402 558 79 \times 10^{19}$	$-1.333 513 920 468 96 \times 10^{12}$
			13	$-9.068 990 246 255 27 \times 10^{20}$	$-5.240 957 408 154 81 \times 10^{13}$
			14	$-8.115 449 919 062 59 \times 10^{22}$	$-8.004 227 004 931 29 \times 10^{14}$
			15	$-7.710 198 082 239 25 \times 10^{24}$	$-3.071 715 246 984 31 \times 10^{16}$
			16	$-7.756 456 789 133 50 \times 10^{26}$	$-5.765 051 302 891 40 \times 10^{17}$
			17	$-8.241 743 808 090 73 \times 10^{28}$	$-2.255 153 289 697 73 \times 10^{19}$
			18	$-9.227 975 479 471 93 \times 10^{30}$	$-4.988 902 332 502 83 \times 10^{20}$
			19	$-1.086 317 951 221 34 \times 10^{33}$	$-2.029 182 061 986 27 \times 10^{22}$
			20	$-1.341 710 427 599 11 \times 10^{35}$	$-5.150 775 341 517 74 \times 10^{23}$
			21	$-1.735 222 787 861 56 \times 10^{37}$	$-2.198 455 031 055 02 \times 10^{25}$
			22	$-2.345 541 072 303 32 \times 10^{39}$	$-6.283 889 382 113 56 \times 10^{26}$
			23	$-3.308 035 224 687 91 \times 10^{41}$	$-2.825 381 133 545 30 \times 10^{28}$
			24	$-4.859 962 365 123 29 \times 10^{43}$	$-8.967 934 457 854 78 \times 10^{29}$
			25	$-7.426 313 765 984 16 \times 10^{45}$	$-4.252 340 256 590 94 \times 10^{31}$
			26	$-1.178 624 821 828 54 \times 10^{48}$	$-1.482 699 008 045 32 \times 10^{33}$
			27	$-1.940 275 228 551 74 \times 10^{50}$	$-7.412 011 738 305 34 \times 10^{34}$
			28	$-3.308 997 142 910 47 \times 10^{52}$	$-2.814 542 922 880 91 \times 10^{36}$
			29	$-5.839 396 695 894 67 \times 10^{54}$	$-1.481 747 125 604 54 \times 10^{38}$
			30	$-1.065 128 686 755 02 \times 10^{57}$	$-6.084 146 692 513 29 \times 10^{39}$
			31	$-2.006 092 513 024 11 \times 10^{59}$	$-3.368 377 250 457 17 \times 10^{41}$
			32	$-3.897 560 703 088 00 \times 10^{61}$	$-1.486 643 351 275 68 \times 10^{43}$
			33	$-7.804 231 009 081 33 \times 10^{63}$	$-8.641 305 361 008 41 \times 10^{44}$
			34	$-1.609 114 577 601 91 \times 10^{66}$	$-4.078 650 190 369 74 \times 10^{46}$
			35	$-3.413 564 823 886 31 \times 10^{68}$	$-2.484 917 565 190 60 \times 10^{48}$
			36	$-7.444 871 598 934 88 \times 10^{70}$	$-1.248 799 541 490 64 \times 10^{50}$
			37	$-1.668 073 211 907 18 \times 10^{73}$	$-7.961 328 532 338 04 \times 10^{51}$
			38	$-3.836 886 554 855 30 \times 10^{75}$	$-4.243 702 342 272 49 \times 10^{53}$
			39	$-9.054 424 830 738 75 \times 10^{77}$	$-2.826 362 569 305 75 \times 10^{55}$
			40	$-2.190 719 978 043 26 \times 10^{80}$	$-1.592 570 348 021 65 \times 10^{57}$
			41	$-5.431 202 081 082 37 \times 10^{82}$	$-1.106 347 449 289 15 \times 10^{59}$
			42	$-1.378 920 516 830 21 \times 10^{85}$	$-6.570 071 757 835 92 \times 10^{60}$
			43	$-3.583 266 013 550 19 \times 10^{87}$	$-4.753 613 658 288 94 \times 10^{62}$
			44	$-9.525 511 928 784 05 \times 10^{89}$	$-2.967 179 694 833 31 \times 10^{64}$
			45	$-2.589 106 052 236 16 \times 10^{92}$	$-2.232 759 962 483 54 \times 10^{66}$
			46	$-7.192 098 487 062 92 \times 10^{94}$	$-1.461 346 137 500 41 \times 10^{68}$
			47	$-2.040 827 951 732 54 \times 10^{97}$	$-1.142 120 632 456 55 \times 10^{70}$
			48	$-5.913 042 608 694 69 \times 10^{99}$	$-7.821 014 911 620 90 \times 10^{71}$
			49	$-1.748 582 003 076 24 \times 10^{102}$	$-6.340 607 068 176 19 \times 10^{73}$
			50	$-5.275 396 874 908 01 \times 10^{104}$	$-4.533 740 159 804 09 \times 10^{75}$
1	0	0	0	$1.000 000 000 000 00 \times 10^{0}$	$1.000 000 000 000 00 \times 10^{0}$
			1	$-1.766 666 666 666 67 \times 10^{1}$	$-1.733 333 333 333 33 \times 10^{2}$
			2	$-7.094 444 444 444 44 \times 10^{1}$	$1.469 022 222 222 22 \times 10^{4}$
			3	$-2.076 771 604 938 27 \times 10^{3}$	$-1.146 492 839 506 17 \times 10^{6}$
			4	$-7.261 242 541 152 26 \times 10^{4}$	$9.239 499 904 526 75 \times 10^{7}$
			5	$-3.007 992 238 854 60 \times 10^{6}$	$-8.932 636 933 934 71 \times 10^{9}$
			6	$-1.421 562 708 112 81 \times 10^{8}$	$8.981 570 199 138 23 \times 10^{11}$
			7	$-7.491 106 257 035 15 \times 10^{9}$	$-1.109 190 986 116 20 \times 10^{14}$

TABLE II. *(Continued)*.

n_1	n_2	m	N	$b^{(N)}$	$a^{(N)}$
			8	$-4.339\,404\,851\,569\,23 \times 10^{11}$	$1.340\,972\,283\,311\,39 \times 10^{16}$
			9	$-2.737\,219\,336\,385\,62 \times 10^{13}$	$-2.023\,560\,932\,103\,26 \times 10^{18}$
			10	$-1.867\,541\,477\,404\,24 \times 10^{15}$	$2.831\,501\,147\,484\,67 \times 10^{20}$
			11	$-1.371\,218\,133\,067\,99 \times 10^{17}$	$-5.043\,115\,734\,357\,03 \times 10^{22}$
			12	$-1.079\,048\,402\,558\,79 \times 10^{19}$	$7.978\,122\,618\,002\,75 \times 10^{24}$
			13	$-9.068\,990\,246\,255\,27 \times 10^{20}$	$-1.637\,706\,361\,759\,52 \times 10^{27}$
			14	$-8.115\,449\,919\,062\,59 \times 10^{22}$	$2.888\,499\,361\,972\,74 \times 10^{29}$
			15	$-7.710\,198\,082\,239\,25 \times 10^{24}$	$-6.719\,456\,394\,907\,50 \times 10^{31}$
0	1	0	0	$1.000\,000\,000\,000\,00 \times 10^{0}$	$1.000\,000\,000\,000\,00 \times 10^{0}$
			1	$-9.766\,666\,666\,666\,67 \times 10^{1}$	$-1.893\,333\,333\,333\,33 \times 10^{2}$
			2	$2.212\,388\,888\,888\,89 \times 10^{3}$	$1.273\,555\,555\,555\,56 \times 10^{4}$
			3	$-1.708\,321\,604\,938\,27 \times 10^{4}$	$-6.598\,083\,950\,617\,28 \times 10^{5}$
			4	$-4.641\,880\,303\,497\,94 \times 10^{5}$	$3.021\,433\,514\,403\,29 \times 10^{7}$
			5	$-4.000\,663\,228\,000\,69 \times 10^{7}$	$-2.992\,693\,902\,362\,69 \times 10^{9}$
			6	$-3.595\,852\,289\,656\,62 \times 10^{9}$	$8.138\,008\,370\,063\,00 \times 10^{10}$
			7	$-3.371\,177\,296\,637\,00 \times 10^{11}$	$-3.564\,224\,520\,862\,39 \times 10^{13}$
			8	$-3.294\,951\,457\,614\,10 \times 10^{13}$	$-1.506\,154\,783\,262\,99 \times 10^{15}$
			9	$-3.352\,506\,364\,290\,28 \times 10^{15}$	$-8.408\,024\,783\,576\,55 \times 10^{17}$
			10	$-3.545\,070\,597\,498\,18 \times 10^{17}$	$-9.997\,671\,864\,571\,75 \times 10^{19}$
			11	$-3.890\,295\,238\,852\,22 \times 10^{19}$	$-3.009\,643\,616\,091\,68 \times 10^{22}$
			12	$-4.425\,273\,520\,293\,74 \times 10^{21}$	$-5.240\,874\,465\,506\,04 \times 10^{24}$
			13	$-5.213\,373\,758\,344\,02 \times 10^{23}$	$-1.405\,025\,266\,781\,30 \times 10^{27}$
			14	$-6.356\,851\,722\,778\,67 \times 10^{25}$	$-2.982\,328\,209\,612\,55 \times 10^{29}$
			15	$-8.018\,812\,621\,172\,92 \times 10^{27}$	$-8.009\,111\,337\,915\,90 \times 10^{31}$
0	0	1	0	$1.000\,000\,000\,000\,00 \times 10^{0}$	$1.000\,000\,000\,000\,00 \times 10^{0}$
			1	$-4.766\,666\,666\,666\,67 \times 10^{1}$	$-1.733\,333\,333\,333\,33 \times 10^{2}$
			2	$2.400\,555\,555\,555\,56 \times 10^{2}$	$1.237\,422\,222\,222\,22 \times 10^{4}$
			3	$-4.121\,438\,271\,604\,94 \times 10^{3}$	$-7.840\,928\,395\,061\,73 \times 10^{5}$
			4	$-2.368\,181\,105\,967\,08 \times 10^{5}$	$4.962\,988\,971\,193\,42 \times 10^{7}$
			5	$-1.447\,953\,560\,613\,85 \times 10^{7}$	$-4.567\,811\,192\,660\,63 \times 10^{9}$
			6	$-9.530\,981\,889\,558\,28 \times 10^{8}$	$3.379\,709\,494\,877\,49 \times 10^{11}$
			7	$-6.732\,920\,041\,634\,54 \times 10^{10}$	$-4.896\,026\,312\,350\,55 \times 10^{13}$
			8	$-5.073\,807\,241\,971\,18 \times 10^{12}$	$3.435\,255\,648\,281\,33 \times 10^{15}$
			9	$-4.057\,941\,687\,293\,67 \times 10^{14}$	$-8.395\,041\,891\,807\,70 \times 10^{17}$
			10	$-3.431\,104\,122\,586\,16 \times 10^{16}$	$4.264\,541\,464\,147\,79 \times 10^{19}$
			11	$-3.058\,202\,384\,831\,12 \times 10^{18}$	$-2.084\,863\,737\,153\,26 \times 10^{22}$
			12	$-2.867\,223\,442\,711\,95 \times 10^{20}$	$4.232\,120\,165\,241\,76 \times 10^{23}$
			13	$-2.822\,861\,997\,840\,79 \times 10^{22}$	$-7.005\,536\,998\,358\,71 \times 10^{26}$
			14	$-2.914\,447\,979\,638\,25 \times 10^{24}$	$-1.100\,994\,385\,763\,16 \times 10^{28}$
			15	$-3.151\,727\,832\,486\,09 \times 10^{26}$	$-3.036\,158\,965\,133\,11 \times 10^{31}$

$$S_0 \sim \tfrac{2}{3} - x + \tfrac{1}{4}x^2 + \cdots , \qquad (35)$$

$$S_1 \sim 8\beta_2^{(0)} \ln \tfrac{1}{4}x + 8\beta_2^{(0)}\left(\tfrac{1}{2}x + \tfrac{3}{16}x^2 + \cdots\right) , \qquad (36)$$

show that

$$\Phi_{2,\infty} \sim \sqrt{2}\,\left[(f\rho)^{\beta_2^{(0)}} e^{1/12f - \rho/2} + \tfrac{1}{2} i (f\rho)^{-\beta_2^{(0)}} e^{-1/12f + \rho/2}\right]. \qquad (37)$$

Compared to the real part, the imaginary part is exponentially small as $f \to 0$,

$$\Phi_{2,\infty} \sim \sqrt{2}\left(\frac{-dS}{dx}\right)^{-1/2} e^{S/8f} \times \left[1 + iO(f^{-2\beta_2}\, e^{-1/6f})\right]. \qquad (38)$$

It would appear (and can be shown more rigorously) from Eqs. (37) and (38) that $\sqrt{2}\,(-dS/dx)^{-1/2} e^{S/8f}$ is the solution of Eq. (6) (with real β_2) *that is regular at the origin.* That is, not only is

$$\Phi_{2,\infty} \sim \sqrt{2}\, f^{\beta_2^{(0)}} e^{1/12f} N(f) \Phi_{2,\mathrm{RS}} \quad (f \to 0, \quad 0 \ll \rho \ll \tfrac{1}{4} f) \qquad (39)$$

where $N(f)$ is a relative normalization factor, but also

$$\left(\frac{-dS}{dx}\right)^{-1/2} e^{S/8f} = \sqrt{2}\, f^{\beta_2^{(0)}} e^{1/12f} N(f) \Phi_{2,\mathrm{RS}} . \qquad (40)$$

The positive-exponential JWKB-like function

TABLE III. Values of $(\frac{2}{3})^N n^{-3N} a^{(N)}$: Comparison of exact values with values extracted numerically from a "Bender-Wu analysis" of high-order RSPT $E^{(N)}$ in Ref. 9.

n_1	n_2	m	N	Exact	Numerical fit
0	0	0	3	−47.035 979 652 492 0	−47.0360 ± 0.000 2
			4	92.651 592 189 325 8	92.65 ± 0.2
			5	−1 352.845 805 227 26	−1 350 ± 20
1	0	0	3	−663.479 652 491 998	−663.45 ± 0.1
			4	4 455.777 345 933 04	4 450 ± 20
			5	−35 898.264 427 142 4	−34 000 ± 3000
0	1	0	2	88.441 358 024 691 4	88.441 36 ± 0.000 05
			3	−381.833 561 957 019	−381.832 ± 0.01
			4	1 457.095 637 733 07	1 455.7 ± 1
			5	−12 026.965 592 699 8	−11 700 ± 200

exactly reproduces the RSPT function in the sense that rearrangement of the terms of the former gives exactly the latter, except for normalization.

From the point of view of the $(-dS/dx)^{-1/2}e^{S/(8f)}$ wave function (where S is associated with the outer turning point at $x=1$), the identity (40) is not entirely anticipated, since the small-x expansions of $(-dS/dx)^{-1/2}$ and $e^{S/(8f)}$ separately contain negative powers of x. The negative powers cancel in the product, a fact we first noticed by direct computation.

The final step in computing $\mathrm{Im}\beta_2$ from Eq. (10) is to compute the square of the ratio

$$\frac{\Phi_{2,\mathrm{RS}}}{\Phi_{2,\infty}} \sim \Phi_{2,\mathrm{RS}}\Big/\left[\sqrt{2}\left(\frac{-dS}{dx}\right)^{-1/2} e^{S/(8f)}\right], \tag{41}$$

$$= \frac{\left(\dfrac{-dS_0}{dx}\right)^{1/2} e^{-(S_0+fS_1)/(8f)}\Phi_{2,\mathrm{RS}}}{\left(\dfrac{-dS_0}{dx}\right)^{1/2}\left(\dfrac{-dS}{dx}\right)^{-1/2}\exp\left(\dfrac{1}{8f}\sum_{n=2}^{\infty} f^n S_n\right)}. \tag{42}$$

For technical reasons, it was particularly convenient to calculate that ratio as the ratio of the ρ^0 terms in the numerator and denominator of Eq. (42). Since

$$\Phi_{2,\mathrm{RS}} \sim \{\rho^{\beta_2^{(0)}} e^{-\rho/2}/[n_2!(n_2+|m|)!]^{1/2}\} \times [1+O(\rho^{-1})+O(f)], \tag{43}$$

$\mathrm{Im}\beta_2$ comes out in the form,

$$\mathrm{Im}\beta_2 \sim -\tfrac{1}{2}[n_2!(n_2+|m|)!]^{-1} f^{-2n_2-|m|-1} e^{-1/(8f)} \times (1+b^{(1)}f+b^{(2)}f^2+\cdots), \tag{44}$$

where the $b^{(N)}$ for a few states are listed in Table II. In Eq. (44), n_2 denotes the usual second parabolic quantum number (n_1 being the first) to which $\beta_2^{(0)}$ is related by $\beta_2^{(0)}=n_2+\frac{1}{2}(|m|+1)$.

From Eq. (44) and from the RSPT series for β_1 and $\mathrm{Re}\beta_2$ obtained from Ref. 10, it is only a matter of elementary manipulations of power series to obtain $\mathrm{Im}E$ in the form (where $n=n_1+n_2+|m|+1$),

$$\mathrm{Im}E \sim -\tfrac{1}{2}[n^3 n_2!(n_2+|m|)!]^{-1}(\tfrac{1}{4}n^3F)^{-2n_2-|m|-1} \times \exp[-2/(3n^3F)+3(n_1-n_2)] \times (1+a^{(1)}F+a^{(2)}F^2+\cdots). \tag{45}$$

The $a^{(N)}$ for a few states are given in Table II. The values of $a^{(k)}$ extracted numerically[9] from a "Bender-Wu" analysis of the high order $E^{(N)}$ are listed in Table III for comparison.

All the computations described here were initially carried out in double precision on DEC System-10KL computers. The calculation of $\Phi_{2,\mathrm{RS}}$, although very fast (1% of total computation time), turned out to suffer from serious "cancellation error" for $N>20$ arising from the sign alternation of the expansion coefficients of $\Phi_{2,\mathrm{RS}}$ on the unperturbed wave functions. Such cancellation error was circumvented by calculating $\Phi_{2,\mathrm{RS}}$ in quadruple precision on an IBM 3033 computer. The $b^{(N)}$ and $a^{(N)}$ reported to fifteen digits in Table II are probably accurate to ±1 or 2 in the last digit.

V. DISCUSSION AND SUMMARY

In a uniform electrostatic field the hydrogen atom energy eigenvalues turn into resonances with a negative imaginary part. The resonance eigenvalues for the separation constant (radially symmetric two-dimensional negatively anharmonic oscillator) and energy have the asymptotic expansions,

$$\beta_2 \sim \sum_{N=0}^{\infty} \beta_2^{(N)} f^N - i[2n_2!(n_2+|m|)!]^{-1} f^{-2n_2-|m|-1} e^{-1/(8f)} \sum_{N=0}^{\infty} b^{(N)} f^N, \tag{46}$$

$$E \sim \sum_{N=0}^{\infty} E^{(N)}F^N - i\left[2n^3 n_2!\,(n_2+|m|)!\right]^{-1}\left(\tfrac{1}{4}n^3F\right)^{-2n_2-|m|-1} e^{-2/(3n^3F)+3(n_1-n_2)} \sum_{N=0}^{\infty} a^{(N)}F^N. \tag{47}$$

The real part of the expansions is ordinary RSPT. The imaginary part is calculated by a perturbation theory that maintains an outgoing-wave boundary condition at ∞. The uniform version of the "outer" perturbation theory follows Langer and Cherry. Since the "outer wave function" is needed far from the outer turning point, an asymptotic version of the theory, like JWKB with f playing the role of $\hbar$, derived from the Langer-Cherry approach, is more convenient. The "actionlike" function S is expanded in a power series in f, and to each order S can be expressed in closed form in terms of elementary functions.

TABLE IV. Ratio analysis for the Imβ_2 series for the ground state.

N	$b^{(N+1)}/(6b^{(N)})$	$b^{(N+1)}/[(N+1)b^{(N)}]$
0	−2.944 44	−17.666 67
1	0.669 29	2.007 86
2	4.878 87	9.757 74
3	5.827 35	8.741 02
4	6.904 22	8.285 06
5	7.876 59	7.876 59
6	8.782 71	7.528 04
7	9.654 57	7.240 93
8	10.513 04	7.008 69
9	11.371 28	6.822 77
10	12.237 28	6.674 88
11	13.115 45	6.557 72
12	14.007 70	6.465 09
13	14.914 28	6.391 84
14	15.834 40	6.333 76
15	16.766 66	6.287 50
16	17.709 43	6.250 39
17	18.661 05	6.220 35
18	19.620 01	6.195 79
19	20.584 99	6.175 50
20	21.554 86	6.158 53
21	22.528 72	6.144 20
22	23.505 84	6.131 96
23	24.485 64	6.121 41
24	25.467 67	6.112 24
25	26.451 54	6.104 20
26	27.436 99	6.097 11
27	28.423 78	6.090 81
28	29.411 71	6.085 18
29	30.400 65	6.080 13
30	31.390 46	6.075 57
31	32.381 03	6.071 44
32	33.372 29	6.067 69
33	34.364 15	6.064 26
34	35.356 55	6.061 12
35	36.349 45	6.058 24
36	37.342 78	6.055 59
37	38.336 51	6.053 13
38	39.330 61	6.050 86
39	40.325 03	6.048 76
40	41.319 77	6.046 79
41	42.314 77	6.044 97
42	43.310 04	6.043 26
43	44.305 54	6.041 66
44	45.301 26	6.040 17
45	46.297 18	6.038 76
46	47.293 29	6.037 44
47	48.289 57	6.036 20
48	49.286 02	6.035 02
49	50.282 62	6.033 91

The computational details are quite tedious for humans, quite simple for machines. We have written computer programs to generate the $b^{(N)}$ and $a^{(N)}$ for arbitrary states of hydrogen to large N. The $a^{(N)}$ should be useful for comparing theory and experiment, but the most useful summation procedure remains to be determined.

The nature of the divergence of the $\sum a^{(N)}F^N$ and $\sum b^{(N)}f^N$ series is also of interest. The numerical values in Table IV clearly indicate that for the *ground state*, $b^{(N+1)}/[(N+1)b^{(N)}] \to 6$ as $N\to\infty$, which is the same behavior as for the $\beta_2^{(N)}$. The approach, however, is not as rapid as for the $\beta_2^{(N)}$, and we cannot yet distinguish the correct asymptotic behavior, say, from among $6^N N!$, $6^N N!\ln N$, and $6^N N!(a\ln N+b)$, for the ground state.

The $E^{(N)}$ and $\beta_2^{(N)}$ can be expressed as polynomials in the quantum numbers n_1, n_2, and m, with rational coefficients. Similarly, so can the $a^{(N)}$ and $b^{(N)}$, although we have not calculated them as such except for $b^{(1)}$ and $b^{(2)}$, which are

$$b^{(1)} = -34k^2 - 12k - \tfrac{5}{3} + 6M\,, \tag{48}$$

$$b^{(2)} = 578k^4 - 92k^3 - \tfrac{442}{3}k^2 - 90k - \tfrac{155}{18} - 204Mk^2 + 100Mk + 26M + 18M^2\,, \tag{49}$$

where $M=\frac{1}{4}(m^2-1)$, and $k=\beta_2^{(0)}=n_2+\frac{1}{2}|m|+\frac{1}{2}$. The polynomial for $b^{(1)}$ was first obtained by Damburg and Kolosov,[13,33] who also obtained $b^{(2)}$ for the special case $n_2=m=0$.

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