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Fluctuation properties of multiplied-Poisson light: Measurement of the photon-counting distribution for radioluminescence radiation from glass

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We investigate the photon statistics of light whose intensity fluctuations are due to shot noise, such as occurs in certain kinds of luminescence involving multiplied-Poisson processes. The degrees of freedom and degeneracy parameters are obtained and contrasted with those for thermal light. The excess count variance for shot-noise light exhibits particle-like fluctuations and is maximized for large counting times. Radioluminescence photons produced in glass obey the Neyman type-A counting distribution predicted by the theory.

The photon statistics generated by light have evoked continuing interest in the physics community since 1956. In that year, Hanbury-Brown and Twiss carried out an important set of experiments in which they observed correlation in the fluctuations of two photoelectric currents induced by thermal light[1] and by starlight.[2] Purcell[3] recognized that this correlation would also be manifested in the number of photon counts registered in a fixed time interval T, using a single detector. By means of a simple argument, he demonstrated that the variance of this counting distribution, $\text{var}(n)$, would exceed the mean count $\langle n \rangle$ because of the tendency of the photons to "clump," though he was careful to point out that this excess variance would be small for natural light. The full photon-counting distribution $p(n)$ for thermal light was calculated by Mandel[4,5] in 1959. He showed that it could be approximately represented as a negative binomial distribution[6] and emphasized that the number of degrees of freedom (modes) of the light within the resolution time of the counting detector, M, and the degeneracy parameter of the light δ were both crucial determinants of the statistical behavior of n. It has by now been well established that the registration of photons for such sources of light is describable in terms of the doubly stochastic Poisson point process (DSPP).[7–9]

The development of a quantum-electrodynamic theory of coherence and photon counting by Glauber[10,11] in 1963 provided a framework for describing both classical and intrinsically quantum-mechanical states of the radiation field. Quantum-mechanical light for which the count variance lies below the count mean can in principle be created,[12,13] but again a significant deviation from Poisson counting statistics is difficult to produce.[14]

In the following we consider the photon statistics for sources of light whose stochastic intensity is of a shot-noise type, such as occurs in certain kinds of luminescence that involves a multiplicative cascade of two Poisson processes. Expressions are obtained for the degrees of freedom and the degeneracy parameter for shot-noise light; they are shown to be very different in character from those for thermal light. We provide an expression for the count variance and show that it is maximized for large counting times. Finally, we demonstrate that the Neyman type-A distribution,[15] rather than the Poisson, governs the registration of such photons. Only sensitive and carefully designed experiments will reveal deviations from the Neyman type-A for shot-noise light, much as special conditions are required to produce something other than Poisson counting statistics for thermal light.

For the general DSPP, the count variance is related to the count mean by[9,12]

$$\text{var}(n) = \langle n \rangle + \text{var}(W) \ , \tag{1}$$

where W is the integrated intensity of the light $\lambda(t)$,

$$W = \int_0^T \lambda(t)\,dt \ . \tag{2}$$

For $\lambda(t)$ stationary, $\langle n \rangle = \langle W \rangle = \langle \lambda \rangle T$. If the light is thermal (choatic) in nature, the intensity correlation function is[3,4,9]

$$\langle \lambda(t)\lambda(t+\tau) \rangle = \langle \lambda(t) \rangle^2 [1 + |\gamma(\tau)|^2] \ , \tag{3}$$

where $\gamma(\tau)$ is the normalized correlation function of the field. The variance of W can then be written as[4,9,12]

$$\text{var}(W) = \langle W \rangle^2 / M = \langle n \rangle^2 / M \ , \tag{4}$$

where the inverse number of modes (degrees of freedom of the light fluctuations within the resolution time of the detector) is

$$M^{-1}=\frac{2}{T^2}\int_0^T (T-\tau)|\gamma(\tau)|^2 d\tau \ . \qquad (5)$$

Thus

$$\mathrm{var}(n)=\langle n\rangle+\langle n\rangle^2/M \ . \qquad (6)$$

The first term on the right-hand side of Eq. (6), $\langle n\rangle$, is representative of the Poisson distribution and is associated with the irreducible fluctuations of classical particles. The excess fluctuations, on the other hand, are represented by the second term, $\langle n\rangle^2/M$. This contribution reflects the statistical properties of the thermal light intensity and is associated with wave fluctuations.[5]

Of the many forms that var(W) can take, a particularly interesting choice that has not been previously investigated occurs when $\mathrm{var}(W)\propto\langle W\rangle$, for then the excess count flucutations behave in a particle-like way. If $\lambda(t)$ represents shot noise,[16,17] generated by filtering a Poisson process of rate μ with a linear filter whose (non-negative) impulse response function is $h(t)$, the light will have an intensity correlation function

$$\langle\lambda(t)\lambda(t+\tau)\rangle=\langle\lambda^2(t)\rangle+\langle\lambda(t)\rangle\xi(\tau) \ , \qquad (7)$$

with

$$\xi(\tau)=\int_0^\infty h(t)h(t+\tau)dt\Big/\int_0^\infty h(t)dt \ . \qquad (8)$$

Again assuming stationarity, we have

$$\langle W\rangle=\langle n\rangle=\mu\alpha T \ , \qquad (9)$$

where the multiplication parameter α (>0) is

$$\alpha=\int_0^\infty h(t)dt \ . \qquad (10)$$

The variance of the integrated intensity then assumes the form

$$\mathrm{var}(W)=\alpha\langle W\rangle/\mathfrak{M}=\alpha\langle n\rangle/\mathfrak{M} \ , \qquad (11)$$

where

$$\mathfrak{M}^{-1}=\frac{2}{\alpha T}\int_0^T (T-\tau)\xi(\tau)d\tau \ . \qquad (12)$$

In this case, the parameter $\mathfrak{M}$ represents the number of degrees of freedom of the counting time within the correlation time of the light τ_c. The count variance for shot-noise light is therefore

$$\mathrm{var}(n)=(1+\alpha/\mathfrak{M})\langle n\rangle \ , \qquad (13)$$

exhibiting the promised proportionality to $\langle n\rangle$. A relationship in the form of Eq. (13) is a special case of the Burgess variance theorem.[18] The shot-noise driven DSPP has been discussed previously in connection with ecology, and Bartlett has shown that it is a particular Neyman-Scott cluster process.[19]

It will be useful to examine the dependence of $\mathfrak{M}$ on T/τ_c, and the form of the degeneracy parameter for shot-noise light, and to compare the results with those for thermal light. The comparison is carried out for an exponentially decaying correlation in both cases:

$$h(\tau)=(2\alpha/\tau_c)\exp(-2\tau/\tau_c) \ , \qquad (14)$$

$$|\gamma(\tau)|^2=\exp(-2\tau/\tau_c) \ . \qquad (15)$$

Combining Eqs. (8), (12), and (14) leads to

$$\mathfrak{M}=\frac{2(T/\tau_c)}{\exp(-2T/\tau_c)+2T/\tau_c-1} \qquad (16)$$

for the shot-noise light. The result for thermal light is well known[9,12,20]:

$$M=\frac{2(T/\tau_c)^2}{\exp(-2T/\tau_c)+2T/\tau_c-1} \ . \qquad (17)$$

The consequences of what appears to be a minor difference between Eqs. (16) and (17) turn out to be far ranging. This is demonstrated in Fig. 1 where it is apparent that M vs T/τ_c and $\mathfrak{M}$ vs τ_c/T behave very similarly. In particular, $M=1$ for $T \ll \tau_c$ whereas $\mathfrak{M}=1$ for $\tau_c \ll T$. Also $M=T/\tau_c$ for $T \gg \tau_c$ whereas $\mathfrak{M}=\tau_c/T$ for $\tau_c \gg T$. It is important to note that the asymptotic results for M and $\mathfrak{M}$ are maintained for arbitrary $|\gamma(\tau)|^2$ and $h(\tau)$.

The degeneracy parameter δ, as it is usually defined for thermal light,[4,5,21] is the ratio of the coher-

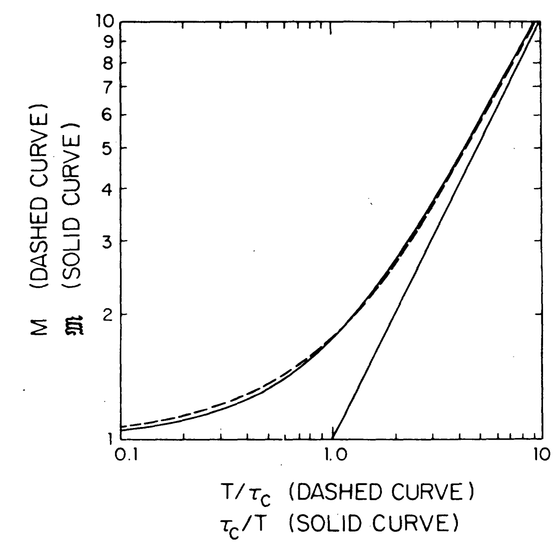

FIG. 1. Dashed curve represents degrees-of-freedom parameter M vs T/τ_c for thermal light. Solid curve represents degrees-of-freedom parameter $\mathfrak{M}$ vs τ_c/T for shot-noise light. The correlation function is assumed to be exponentially decaying in both cases. M and $\mathfrak{M}$ behave in opposite ways, though the curves do intersect at $T=\tau_c$. The solid line represents unity slope.

ence time to the average photon inter-arrival time or, equivalently, it is the average number of detected photons contained in a coherence time ($\delta = \langle\lambda\rangle\tau_c = \langle n\rangle\tau_c/T$). It depends on the average intensity of the light, but not on the counting time T, and provides a measure of the degree to which excess fluctuations are inherent in the detected radiation. The analogous quantity for shot-noise light is simply the multiplication parameter α; indeed, it is immediately apparent from Eq. (13) that the count variance approaches the mean as $\alpha \rightarrow 0$.

The degrees-of-freedom and degeneracy parameters are important determinants of the statistical behavior of n for shot-noise light, as they are for thermal light, but not in the same way, as we will see presently. The behavior of the count variance for thermal light, given in Eq. (6), is well understood. For $T \ll \tau_c$ $(M=1)$, the wave fluctuations of the light are entirely resolved and Eq. (6) reflects the maximum possible excess variance. This is associated with the Bose-Einstein counting distribution, which reduces to the Poisson as $\langle\lambda\rangle \rightarrow 0$. For $T \gg \tau_c$ $(M = T/\tau_c)$, Eq. (6) can be rewritten as $\mathrm{var}(n) = \langle n\rangle(1+\delta)$. Excess fluctuations therefore disappear when $\delta = \langle n\rangle\tau_c/T \ll 1$. Thus, thermal light produces Poisson counts when $T/\tau_c \gg 1$ and $T/\tau_c \gg \langle n\rangle$. The Poisson limit is very simple to demonstrate experimentally, whereas the Bose-Einstein limit requires quite careful experimentation because of the small value of τ_c.[22,23] An asymptotic formula derived by Glauber[24] is useful when $T/\tau_c \gg 1$ but $T/\tau_c \lesssim \langle n\rangle$.[23]

The analogous results for shot-noise light are obtained from Eq. (13). For $\tau_c \ll T$ $(\mathfrak{M}=1)$, the variance achieves its maximum value, $\mathrm{var}(n) = (1+\alpha)\langle n\rangle$, which is associated with the Neyman type-A distribution.[15,25] Note that this occurs in the limit of *large* counting times and is therefore easily demonstrated experimentally as we will see subsequently. The Neyman type-A reduces to the Poisson when $\alpha \rightarrow 0$ but, unlike the Bose-Einstein, it does not so reduce when $\langle\lambda\rangle \rightarrow 0$. For $\alpha \gg 1$ it approaches the fixed multiplicative Poisson distribution.[25] For $\tau_c \gg T$ $(\mathfrak{M} = \tau_c/T)$, Eq. (13) can be rewritten as $\mathrm{var}(n) = \langle n\rangle(1+\alpha T/\tau_c)$. Excess fluctuations therefore disappear when $\alpha T/\tau_c \ll 1$ so that shot-noise light produces Poisson counts when $\tau_c/T \gg 1$ and $\tau_c/T \gg \alpha$, i.e., in the limit of *small* counting times. This is now the more difficult region to examine experimentally.

The diverse behavior of Eqs. (6) and (13) stems from the different $\langle n\rangle$ dependence of the excess term in the variance. For arbitrary T/τ_c, the statistics of thermal light become Poisson as $\langle\lambda\rangle \rightarrow 0$, whereas the statistics of shot-noise light are independent of $\langle\lambda\rangle$, approaching Poisson when $\alpha \rightarrow 0$. Even barely detectable shot-noise light, therefore, can exhibit substantial photon bunching. This distinction arises because pairs of photons are necessary to generate the wave fluctuations associated with thermal light (this is reflected in the $\langle n\rangle^2$ dependence), whereas single photons suffice to produce the particle fluctuations associated with shot-noise light (this is reflected in the $\langle n\rangle$ dependence). Thus for $T \ll \tau_c$ we conceive of the counting-time interval as a narrow ribbon that in the thermal case resolves the wave fluctuations, but in the shot-noise case cuts apart the natural particle correlations. For $T \gg \tau_c$ the counting time is viewed as a broad ribbon that, for thermal light, averages out the wave fluctuations but for shot-noise light collects all of the natural particle correlations. It is clear that the natural bunched character of shot-noise light is most evident in the long-counting-time limit, which is the opposite from thermal light.

The generation of shot-noise light, in fact, involves a multiplicative cascade of two Poisson processes. The first is associated with the underlying point process of the shot-noise representing the intensity of the light, and the second is associated with the intrinsic nature of light. This can be readily seen by examining the limit $\tau_c \ll T$. The light intensity then takes the form of a random sequence of narrow pulses that arrive at rate μ. The total number of such pulses within the counting time T is a Poisson random variable m with mean $\langle m\rangle = \mu T$. Each pulse produces a random number of photons A, where A is Poisson distributed with mean $\langle A\rangle = \alpha$. (For $0<\alpha<1$, multiplication is reduction.) Therefore, the total number of photons n produced within the counting time T is $n = \sum_{j=1}^{m} A_j$, where the $\{A_j\}$ are independent realizations of the random variable A. This leads directly to the Neyman type-A counting distribution which has the moment-generating function

$$\langle e^{-sn}\rangle = \exp\left[\frac{\langle n\rangle}{\alpha}\{\exp[\alpha(e^{-s}-1)]-1\}\right] \quad . \quad (18)$$

The counting distribution and moments of n for arbitrary τ_c/T can also be obtained, as can the time statistics.[26,27]

We proceed now to demonstrate that the experimental counting distribution for radioluminescence photons produced in glass can be represented in terms of the Neyman type-A. That this is plausible can be understood from the following argument. Consider a stream of Poisson-distributed β^- rays irradiating a glass sample.[28] If each high-energy electron produces a Poisson distributed number of luminescence photons with a maximum time delay that is short in comparison with the counting time ($\tau_c \ll T$), the overall photon count will reflect both sources of independent Poisson randomness, leading to the Neyman type-A photon-counting distribution.

A series of such experiments was carried out.[29] β^-

particles from a ^{90}Sr-^{90}Y equilibrium-mixture source irradiated the Corning 7056 glass faceplate of an EMR type 541N-01 photomultiplier tube from a distance of about 11.5 cm. The maximum β^- energies were 0.54 and 2.23 MeV for the ^{90}Sr and ^{90}Y, respectively, and the β^- flux was $\sim 8.2 \times 10^3$ cm^{-2}s^{-1}. External light was excluded. The photomultiplier anode pulses were passed through a discriminator and standardized. Unavoidable system dead time was $\sim$60 ns. The standardized pulses were counted during consecutive fixed counting intervals ($T = 400$ μs) and the counts were recorded. The experiment was performed repeatedly to obtain good statistical accuracy, and a histogram representing the relative frequency of the counts was constructed. The total duration of a run was about 4 min. In the particular experiment we illustrate, the observed mean count was 85.89 (this number was substantially higher than the mean dark count which could therefore be neglected) and the observed count variance was 429.58. The data are shown as the dots in Fig. 2. The solid curve represents the Neyman type-*A* theoretical counting distribution with the count mean and variance fixed at the experimental values. It is clearly in accord with the data. When one assumes that $\tau_c << T$ ($\mathfrak{M} = 1$), Eq. (13) yields an experimental multiplication parameter $\alpha = 4.0$. A Poisson distribution with mean 85.89 (indicated by arrow) is plotted as the dashed curve in Fig. 2; clearly it bears no relation to the data.

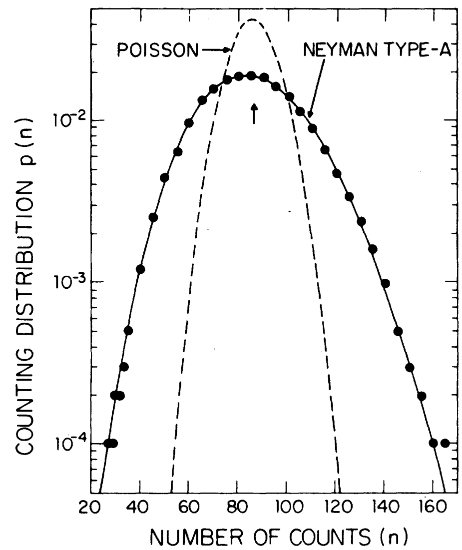

FIG. 2. Photon counting distribution $p(n)$ vs number of photon counts n. Data (dots) represent radioluminescence photon registrations from the glass faceplate of a photomultiplier tube, induced by ^{90}Sr-^{90}Y β^- rays. The counting time $T = 400$ μs. The experimental count mean and variance are 85.89 and 429.58, respectively. The solid curve represents the Neyman type-*A* theoretical counting distribution with the same values of count mean and variance ($\alpha = 4.0$). The Poisson distribution with mean 85.89 (indicated by arrow) is shown as the dashed curve.

The shot-noise light model will also provide a suitable description for certain other types of luminescence radiation where the photons are produced by a multiplicative cascade of two Poisson processes.[26] If the photons generated by the primary process exhibit wave fluctuations, the photon-counting distribution analogous to the Neyman type-*A* is the generalized Polya-Aeppli distribution.[30]

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