PHYSICAL REVIEW A VOLUME 24, NUMBER 3 SEPTEMBER 1981

Brief Reports

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Coherent state and the damped harmonic oscillator

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(Received 26 January 1981)

Although the method of dual coordinates has been employed in the literature to quantize the damped harmonic oscillator, one cannot reproduce the proper classical limit by this method. This is shown by constructing operators which have the correct damped behavior and obtaining coherent states as their eigenstates. These states turn out to be nonnormalizable; this is shown to be due to the unphysical nature of the Hamiltonian employed in the method of dual coordinates.

I. INTRODUCTION

The quantization of damped harmonic oscillators is of considerable importance these days owing to its prospective application, for instance, in the study of deep-inelastic heavy ion scattering where an essential part of the incident kinetic energy is dissipated as internal energy of the reaction products. Of course, microscopically such processes as dissipation can only be understood as the irreversible leakage of energy from a system to its surrounding considered as an infinite reservoir with which it ineracts.[1-6] However, there are phenomenological theories where these effects are taken into account in terms of bulk properties such as the friction or viscosity constants. As a first step to this end, one tries to understand how damping can occur in a simple quantum system like the harmonic oscillator.

At present, there exist essentially three methods to achieve this end, viz., the time-dependent Lagrangian, the nonlinear Schrödinger equation, and the method of dual coordinates. Although the first method[7-15] has long been shown to be unsatisfactory, articles continue to appear on this subject. This method violates the uncertainty principle by the introduction of an $\exp(-\gamma t)$ time dependence in the Lagrangian. This causes the system to go ultimately to a state of zero energy instead of going to the quantum ground state. Furthermore, the time-dependent Lagrangian has been shown to describe another real physical system, namely an oscillator with a variable mass.[16-18] This interpretation comes out naturally if one identifies the energy with the Hamiltonian of the system instead of the sum of kinetic and potential parts which is assumed by authors who favor a damped-system interpretation. The former assumption seems natural since no time-dependent constraints are included there. We agree with Dodonov and Man'ko[18] that the study of such models is important to the extent that they describe, e.g., stars of variable mass. In the nonlinear Schrödinger approach,[19-25] on the other hand, the restriction that the expectation values satisfy the classical equations of motion is too loose to define a unique Hamiltonian. This means that as long as one neglects quantum fluctuations, one can construct a whole set of nonlinear potentials, which in the classical limit, yields linear damping.

In this paper, we show that the third approach of employing dual coordinates, followed, e.g., by Feshbach and Tikochinsky[26] and others,[27] is also not free of inconsistency. For the quantization of a damped system, there remains, hence, the physical profound theory of extracting a Hamiltonian or Liouvillean from the coupling to a loss mechanism.

The plan of the paper is as follows. In Sec. II we discuss briefly the method of dual coordinates and how quantization is achieved in this method. In Sec. III we find the relevant operators with appropriate damped time dependences and construct the coherent states as their eigenstates. In Sec. IV we discuss the cause for the non-normalizability of the coherent states.

II. THE METHOD OF DUAL COORDINATES

The method of dual coordinates[28] consists in adding to the system governed by the equation

$$\ddot{x} + R\dot{x} + \omega^2 x = 0 \tag{1}$$

another system whose equation of motion is

$$\ddot{y} - R\dot{y} + \omega^2 y = 0, \tag{2}$$

so that the total system is conservative and the equations of motion are derivable as the Euler-Lagrange equations from the Lagrangian

$$L = m\dot{x}\dot{y} + \tfrac{1}{2}R(x\dot{y} - \dot{x}y) - m\omega^2 xy. \tag{3}$$

The corresponding canonical momenta and the Hamiltonian are given by

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{y} - \tfrac{1}{2}Ry, \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{x} + \tfrac{1}{2}Rx, \\ H &= m\dot{x}\dot{y} + m\omega^2 xy. \end{aligned} \tag{4}$$

Quantization is usually achieved with the introduction of the following creation and annihilation operators[26,27]:

$$\begin{aligned} A &= \frac{1}{2\sqrt{m\Omega\hbar}}[(p_x + p_y) - im\Omega(x+y)], \\ B &= \frac{1}{2\sqrt{m\Omega\hbar}}[(p_x - p_y) - im\Omega(x-y)], \\ A^\dagger &= \frac{1}{2\sqrt{m\Omega\hbar}}[(p_x + p_y) + im\Omega(x+y)], \\ B^\dagger &= \frac{1}{2\sqrt{m\Omega\hbar}}[(p_x - p_y) + im\Omega(x-y)], \end{aligned} \tag{5}$$

where

$$\Omega^2 = (\omega^2 - R^2/4m^2) \tag{6}$$

and the commutation relations

$$\begin{aligned} [A, A^\dagger] &= [B, B^\dagger] = 1, \\ [A, B] &= [A^\dagger, B^\dagger] = 0. \end{aligned} \tag{7}$$

The Hamiltonian assumes the form

$$\begin{aligned} H &= \hbar\Omega(A^\dagger A - B^\dagger B) + i\frac{\Gamma}{2}(A^\dagger B^\dagger - AB) \\ &= H_0 + H_1 \end{aligned} \tag{8}$$

$$\Gamma = \hbar R/m. \tag{9}$$

To construct the eigenstates of H one notes that H_0 and H_1 commute. The eigenstates of H_0 are degenerate and the effect of H_1 is to mix in states belonging to the same eigenvalue of H_0. These eigenstates have been constructed and extensively discussed,[26] so we refrain from discussing them any further.

III. THE COHERENT STATE AND ITS CLASSICAL LIMIT

In our search for operators with proper damped time dependence we easily find that the following operators

$$\begin{aligned} A + B^\dagger &= \frac{1}{\sqrt{m\Omega\hbar}}(p_x - im\Omega y), \\ A - B^\dagger &= \frac{1}{\sqrt{m\Omega\hbar}}(p_y - im\Omega x), \end{aligned} \tag{10}$$

and their Hermitean adjoints, will serve our purpose.

The Heisenberg equation of motion for the above operators yield

$$\begin{aligned} \frac{d}{dt}(A \pm B^\dagger) &= \frac{i}{\hbar}[H, (A \pm B^\dagger)] \\ &= \left(-i\Omega \pm \frac{\Gamma}{2\hbar}\right)(A \pm B^\dagger) \end{aligned} \tag{11}$$

and the adjoint equations so that these operators have the simple time dependence

$$(A \pm B^\dagger)_t = (A \pm B^\dagger)_0 \exp\left[\left(-i\Omega \pm \frac{\Gamma}{2\hbar}\right)t\right]. \tag{12}$$

One of these is an exponentially growing solution and the other exponentially decaying. Note that by virtue of the second equality in (11) these operators can be looked upon as raising and lowering operators for the eigenstates of H. They raise and lower the eigenvalues by complex quantities. The raising and lowering operators for the eigenstates of H_1 as given by Feshbach and Tikochinsky are simply products of these operators. Since x and y are related to the operators $(A \pm B^\dagger)$ in the following way

$$\begin{aligned} x &= -\left(\frac{\hbar}{m\Omega}\right)^{1/2} \operatorname{Im}(A - B^\dagger), \\ y &= -\left(\frac{\hbar}{m\Omega}\right)^{1/2} \operatorname{Im}(A + B^\dagger), \end{aligned} \tag{13}$$

if we construct simultaneous eigenstates of the commuting operators $(A - B^\dagger)$ and $(A + B^\dagger)$ with eigenvalues $\alpha(=|\alpha|e^{i\theta})$ and $\beta(=|\beta|e^{i\phi})$, respectively, then the expectation values of x and y in such state will satisfy the equations

$$\begin{aligned} \langle x\rangle_t &= \left(\frac{\hbar}{m\Omega}\right)^{1/2} \exp\left(-\frac{\Gamma}{2\hbar}t\right)|\alpha| \sin(\Omega t - \theta), \\ \langle y\rangle_t &= \left(\frac{\hbar}{m\Omega}\right)^{1/2} \exp\left(\frac{\Gamma}{2\hbar}t\right)|\beta| \sin(\Omega t - \phi). \end{aligned} \tag{14}$$

Finally with the usual prescription[29,30] $\hbar \to 0$, $|\alpha| \to \infty$, $|\beta| \to \infty$ in such a way that $(\hbar/m\Omega)^{1/2}|\alpha| \to A$ (finite), and $(\hbar/m\Omega)^{1/2}|\beta| \to B$ (finite) we recover the classical solutions for x and y with initial amplitudes A and B, respectively.

To solve for the simultaneous eigenfunctions of $(A \pm B^\dagger)$ given by

$$(A - B^\dagger)|\alpha, \beta\rangle = \alpha|\alpha, \beta\rangle, \quad (A + B^\dagger)|\alpha, \beta\rangle = \beta|\alpha, \beta\rangle \tag{15}$$

we note that with the operators as given in (10), the system of Eqs. (15) are decoupled in the center of mass and relative coordinates defined, respectively, by

$$R = \tfrac{1}{2}(x+y), \quad r = (x-y) \tag{16}$$

with the total wave function written as a product

$$\Psi_{\alpha,\beta}(x,y) = \phi_{\alpha,\beta}(R)\chi_{\alpha,\beta}(r). \tag{17}$$

This is permissible since the total system is Galilean invariant although the individual ones are not.

The solutions are

$$\phi_{\alpha,\beta}(R) \sim \exp\left[-\frac{m\Omega}{\hbar}R^2 + i(\alpha+\beta)\left(\frac{m\Omega}{\hbar}\right)^{1/2}R\right],$$
$$\chi_{\alpha,\beta}(r) \sim \exp\left[\frac{m\Omega}{4\hbar}r^2 + i\,\frac{\alpha-\beta}{2}\left(\frac{m\Omega}{\hbar}\right)^{1/2}r\right]. \tag{18}$$

$\phi_{\alpha\beta}(R)$ is normalizable and has the usual form of the coherent state for a one-dimensional oscillator. $\chi_{\alpha,\beta}(r)$ is not normalizable because of the positive sign of the first term in the exponential. Furthermore, these wave functions cannot be normalized by changing the metric of the Hilbert space and defining the normalization integral as the integral of the product of the wave function and its time reverse as has been done by Feshbach and Tikochinsky for the eigenstates of H_1.

IV. DISCUSSION

The non-normalizability of the coherent state stems from the fact that it is impossible to construct such states as an eigenfunction of annihilation operators alone. This is readily seen by going back to the Hamiltonian (4),

$$H = m\dot{x}\dot{y} + m\omega^2 xy$$
$$= (\tfrac{1}{2}M\dot{R}^2 + \tfrac{1}{2}M\omega^2R^2) - (\tfrac{1}{2}\mu\dot{r}^2 + \tfrac{1}{2}\mu\omega^2r^2), \tag{19}$$

where $M = 2m$ and $\mu = m/2$ are the total and reduced masses, respectively. Thus the total energy of the system instead of being the sum of the center of mass and internal energies turns out to be the difference of the two. The negative internal energy can be formally considered as the energy of a harmonic oscillator with imaginary coordinates; the center-of-mass coordinate being real. This can happen if the x and y oscillators have coordinates which are complex conjugates of each other. The coherent state for the relative coordinate is obtained by replacing x by ix in the usual one-dimensional harmonic oscillator coherent state.[29] Thus the non-normalizability of $\chi_{\alpha\beta}(r)$ is related directly to the form of the Hamiltonian (4) which is just a mathematical Hamiltonian reproducing the equation of motion and has nothing to do with the energy of the physical system.

Thus we have seen that although the Hamiltonian for the damped harmonic oscillator in the method of dual coordinates reproduces the correct equations of motion as given in Eq. (14), we cannot make use of this opportunity since the wave function turns out to be non-normalizable and this has been shown to be directly due to the unphysical nature of the particular Hamiltonian employed.

One of us (G.G.) wishes to thank the Alexander-von-Humboldt Foundation for financial support.

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