PHYSICAL REVIEW A VOLUME 24, NUMBER 3 SEPTEMBER 1981

Statistical theories of Langmuir turbulence. I. Direct-interaction-approximation responses

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(Received 13 February 1981)

A statistical theory of Langmuir turbulence is developed for the equations of Zakharov in the approximation formally equivalent to the direct-interaction approximation (DIA). At this level the theory is realizable, conserves the mean values of the three invariants of the Zakharov equations, contains the modulational instability, and goes over continuously to weak-turbulence theory in the proper limit. This paper concentrates on the properties of the electrostatic and ion-density response functions which arise in the theory; a new continued-fraction representation is given which is numerically more efficient than direct integration of the DIA equations. These response functions contain a new *nonlinear* dispersion relation for the modulational instability of a fluctuating Langmuir spectrum. Comparison is made between the continued-fraction representation, direct integration of the DIA, and a Monte Carlo simulation. The noninvariance of the DIA to random gauge transformation is discussed as well as the saturation of the modulational instability by spectral broadening.

I. INTRODUCTION

The problem of Langmuir turbulence in plasmas is one of the conceptually simplest and most studied problems in nonlinear plasma physics.[1-3] The physical situation is that of a completely ionized, unmagnetized plasma with electron temperature much higher than the ion temperature, where the *linear* collective modes of the system are longitudinal Langmuir modes and ion-sound modes and transverse electromagnetic modes. A useful model which we will use here and which describes many of the interesting nonlinear features of the system is the model of Zakharov[4] which explicitly separates the fast and slow time scales of the system.

The major purpose of the present paper and its sequel is to present a statistical description of this system which *continuously* connects the regime of weak-turbulence theory with the regime in which modulational instability plays a major role. The weak-turbulence description, which is built on the linear collective modes of the system, has been studied since the early 1960's.[2] It is well known[1,2] that this theory predicts a transfer of Langmuir wave intensity to low wave numbers by a cascade of three wave processes of the form $l \rightleftharpoons l' + s$, i.e., the three wave process which converts a Langmuir wave (l), with the wave number $\vec{k}$ into a Langmuir wave (l'), with wave number $\vec{k}'$ and an ion-sound wave (s), with wave number $\vec{k}-\vec{k}'$ and vice versa. The frequency-matching condition for these processes is the well-known one:

$$\omega_{\vec{k}} = \omega_{\vec{k}'} + |\vec{k}-\vec{k}'|c_s\,, \tag{1.1}$$

where $\omega_{\vec{k}} = (\omega_p^2 + 3k^2v_e^2)^{1/2}$ is the Langmuir frequency and $c_s = (T_e/m_i)^{1/2}$ is the ion-sound speed where we express temperature in energy units. If Langmuir energy is injected at some finite $k \gg \frac{1}{3}(m_e/m_i)^{1/2}k_{De} = k^*$, then Langmuir intensity cascades to lower $k' < k$ until (i) the intensity is low enough for some finite $k_{\min} > k^*$ so that the mode $k_{\min}$ is stabilized by linear collisional damping, or (ii) unstable modes for $k < k^*$ are excited for which further decay by the process (1.1) is not allowed. This piling up of Langmuir intensity at low k has been called the "Langmuir condensate." The question then arises as to how the energy in this condensate is dissipated. Attempts have been made to invoke weak-turbulence processes such as wave-wave scattering, i.e., $l_1 + l_2 \rightleftharpoons l_3 + l_4$. However, as we will show here explicitly, the weak-turbulence scheme breaks down for low wave numbers $k \lesssim k_*$.

It is known[3,4,1] that intense coherent or incoherent Langmuir oscillations at long wavelengths are unstable to modulational instability in which regions of intense Langmuir oscillations deplete the local electron and ion density by action of the ponderomotive force. This instability involves the competition of linear dispersive and dissipative terms with the nonlinear "self-focusing" effect of the ponderomotive force; this is a case of effective "Reynold's number" of order unity. The linear stage of the modulational instability does not directly transfer energy to the high-k region where Landau damping can be an effective dissipation process.

The conceptual picture of the nonlinear evolution of the modulational instability derives from a combination of computer experiments and special analytic solutions.[1] In the case of one dimension, it appears[5,6,7] that the modulational instability can develop into a set of spiky structures which may be loosely related to the known soliton initial value solutions of the undriven, undamped Zakharov or nonlinear Schrödinger equations. When the system is driven, e.g., by a uniform pump, there is conflicting evidence[5] that these spiky structures may, in fact, undergo a singular collapse. In two or three dimensions it is well known from the pioneering work of Zakharov[4] and more recent work by Goldman and Nicholson[6] that a localized Langmuir wave packet will tend to collapse if the electrostatic energy in the packet W satisfies

$$\frac{W}{nT_e} > f\left(\frac{\Delta k}{k_{De}}\right)^2 ,$$

where Δk is the width of the spectrum in wave number and f is a constant of order unity. The ultimate evolution of the collapsing wave packet presumably involves strong wave-particle interaction in which the energy of the packet is given up to very energetic electrons. In two or three dimensions it appears that direct collapse can be initiated without the intermediate stage of a modulational instability.

It is conjectured that the nonlinear stage of the modulational instability results in small scale structures of the order of several Debye lengths which dissipate energy by wave-particle interaction. Galeev *et al.*[8] argue that small-scale density fluctuations are created by sound radiation from supersonically collapsing wave packets; these small-scale density fluctuations provide an enhanced absorption (analogous to inverse bremsstrahlung) of longer wavelength Langmuir waves which terminates the collapse.

In the present paper we attempt to formulate a systematic statistical theory of Langmuir turbulence using the "strong" turbulence or renormalization methods of Kraichnan[9,10] which were originally applied to incompressible fluid turbulence. The simplest renormalized theory, the so-called direct-interaction approximation (DIA), has been successful in describing moderate Reynold's number fluid turbulence, so it seems reasonable to apply it to the Langmuir turbulence problem with Reynolds number of order unity.

In applying these methods to the Langmuir turbulence problem, one is immediately faced with problems not encountered in the usual applications to isotropic, homogeneous Navier-Stokes turbulence. Among these problems is the existence of weakly damped, oscillatory modes corresponding in the limit of large k to the unperturbed (Langmuir and ion-sound) modes of the system and the existence of unstable modes, i.e., modulational instability, which are driven by the electrostatic fluctuations. These problems complicate both the numerical solution of the DIA equations and further analytic approximations such as the "pole approximations" discussed below.

The plan of the paper is as follows: In Sec. II we give the Zakharov equations and their Hamiltonian formulation. The infinitesimal response functions and correlation functions of the statistical theory are introduced and the role of the statistical ensemble in determining the symmetry properties of the statistical description is discussed. The DIA equations are given in Fourier space for the Zakharov system in the special symmetrical statistical ensemble which has zero-mean field. The more general case is studied in the Appendix. The DIA is shown to conserve the mean values of the known invariants of the Zakharov equations: plasmon number, momentum, and Hamiltonian.

In Sec. III we examine the DIA for a steady-state, homogeneous turbulent system in which Fourier transformation with respect to time difference (as well as spatial coordinate difference) variables can be carried out. The weak-turbulence limit is examined in which the response functions are dominated by poles at slightly renormalized values of the linear mode frequencies; contact is made with the weak-turbulence theory of Kadomtsev.

In Sec. IV the representation of the response and correlation functions as a sum of simple pole contributions is discussed. This is a generalization of the weak-turbulence limit in which these functions have poles corresponding to the linear modes of the system. In strong turbulence cases the response functions may have new poles which are unrelated to the linear modes (such as the modulational poles discussed in Sec. V) in addition to poles which are renormalized continuations of the linear modes. A practical iteration scheme for determining the parameters of the pole representation is given in Sec. VI for a narrow-bandwidth Langmuir spectrum. In the first iteration of this procedure, one obtains a new possibly unstable pole in the density response function corresponding to the well-known broadband dispersion relation for modulational instability[1] and a new related but less unstable pole in the electrostatic potential response function. Continued iteration of this procedure is shown in Sec. VI to be equivalent to a continued-fraction representation of the response functions. We carry this out explicitly to convergence for the case of a narrow Langmuir (condensate) spectrum centered at $k=0$. The modulational pole in both response functions appears to approach a fixed point in this procedure. The "trajectories" of the poles of the response functions plotted as a function of k show a continuous transition from large values of $k \gg (m_e/m_i)^{1/2}(W/nT_e)^{1/2}k_{De}$, where the responses are dominated by the Langmuir and ion-sound poles of weak-turbulence theory, to small values of $k \ll (m_e/m_i)^{1/2}(W/nT_e)^{1/2}k_{De}$, where the responses are severely warped away from the weak-turbulence structure and are dominated by the modulational instability.

In Sec. VII the results of direct numerical integration of the DIA response function equations (for fixed covariances) are presented and compared with the continued-fraction iterative procedure of Sec. VI.

In Sec. VIII a Monte Carlo simulation study is described which allows us to compare the DIA results for the modulational growth rates obtained in Secs. VI and VII with a numerical experiment. The experiment treats a three spatial mode truncation of the Zakharov equations, $k=0$ and $k=\pm q$, where the $k=0$ equation is driven by a Gaussian white-noise source of sufficient strength to cause the $k=q$ mode to become modulationally unstable. In the initial growth stage of this model system a modulational growth rate is obtained to compare with the DIA results.

In Sec. IX an exact invariance property of the statistical ensemble of Zakharov systems under random gauge transformations is demonstrated. The DIA equations are shown to violate this invariance, a situation quite analogous to the noninvariance of the DIA for the Navier-Stokes equations under random Galilean transformations.[11] Some estimates of the seriousness of the problem and suggestions for its cure are given.

In Sec. X the effect of finite bandwidth in k space of the pumping spectrum is considered. It is argued that spreading of the Langmuir spectrum in k space can be an effective saturation mechanism.

Finally, in Sec. XI we summarize the main results of this paper and make some preliminary comparisons with other statistical theories of Langmuir turbulence; in particular, the theory of Khakimov and Tsytovich[12] and the recent renormalization-group calculations of Pelletier.[13] Detailed comparisons are not possible at this point because of the different parameter regimes studied in the various works and other significant differences in the theories.

II. THE DIRECT-INTERACTION APPROXIMATION FOR THE ZAKHAROV EQUATIONS

Let $\psi(\vec{x},t)$ be the (complex valued) low-frequency envelope of the electrostatic potential. Then the Zakharov equations are[4,1,2]

$$\nabla^2\left[i\left(\frac{\partial}{\partial t}+\nu_e\right)+\nabla^2\right]\psi(\vec{x},t)=\vec{\nabla}\cdot[n(\vec{x},t)\vec{\nabla}\psi(\vec{x},t)], \quad (2.1)$$

$$\left(\frac{\partial^2}{\partial t^2}+2\nu_i\frac{\partial}{\partial t}-\nabla^2\right)n(\vec{x},t)=\nabla^2[\vec{\nabla}\psi(\vec{x},t)\cdot\vec{\nabla}\psi^\dagger(\vec{x},t)], \quad (2.2)$$

where $\psi^\dagger(\vec{x},t)$ is the complex conjugate of $\psi(\vec{x},t)$.

We are using a dimensionless set of units here[1] in which distances are measured in units of $\frac{3}{2}(m_i/m_e)^{1/2}\lambda_{D_e}$, where λ_{D_e} is the electron Debye length, time is measured in units of $\frac{3}{2}(m_i/m_e)\omega_{p_e}^{-1}$, where ω_{p_e} is the electron plasma frequency, electric-field strengths are measured in units of $(m_e/m_i)^{1/2}(16\pi n_0 T_e/3)^{1/3}$, where n_0 is the average ion or electron density and T_e is the electron temperature, and the density fluctuation n is measured in units of $n_0(4m_e/3m_i)$. Phenomenological Langmuir wave damping ν_e and ion wave damping ν_i (in the above units) have been added to the equations; ν_e can be chosen to be nonlocal in real space to mimic Landau damping if desired. We will not discuss the conditions for validity of these equations here since they have been given elsewhere.[1] Our goal is to study the statistical theory of these model nonlinear equations; our hope is that further physical refinements can be added later in a straightforward way. They can be cast into Hamiltonian form[1] when the dissipative coefficients, ν_e and ν_i, vanish by formally regarding ψ and $\psi^\dagger$ as independent fields and by introducing the hydrodynamic flux potential u:

$$\nabla^2\left[i\left(\frac{\partial}{\partial t}+\nu_e\right)+\nabla^2\right]\psi=\vec{\nabla}\cdot(n\vec{\nabla}\psi), \quad (2.3a)$$

$$\nabla^2\left[-i\left(\frac{\partial}{\partial t}+\nu_e\right)+\nabla^2\right]\psi^\dagger=\vec{\nabla}\cdot(n\vec{\nabla}\psi^\dagger), \quad (2.3b)$$

$$\frac{\partial n}{\partial t}=\nabla^2 u, \quad (2.3c)$$

$$\left(\frac{\partial}{\partial t}+2\nu_i\right)u=n+\vec{\nabla}\psi\cdot\vec{\nabla}\psi^\dagger. \quad (2.3d)$$

Define the Hamiltonian H as the volume integral of the density $\mathcal{H}$,

$$\mathcal{H}=\nabla^2\psi\nabla^2\psi^\dagger+\tfrac{1}{2}[(\vec{\nabla}u)^2+n^2]+n\vec{\nabla}\psi\cdot\vec{\nabla}\psi^\dagger. \quad (2.4)$$

Then Eqs. (2.1) and (2.2) are equivalent, when $\nu_e=\nu_i=0$, to the canonical system:

$$\frac{\partial n}{\partial t}=-\frac{\delta H}{\delta u},\quad \frac{\partial u}{\partial t}=\frac{\delta H}{\delta n},$$
$$\nabla^2\frac{\partial\psi}{\partial t}=\frac{i\delta H}{\delta\psi^\dagger},\quad \nabla^2\frac{\partial\psi^\dagger}{\partial t}=-\frac{i\delta H}{\delta\psi}. \quad (2.5)$$

In addition to H, Eqs. (2.5) conserve: the plasmon number N,

$$N=\int\vec{\nabla}\psi\cdot\vec{\nabla}\psi^\dagger d\vec{x}=\int\vec{E}\cdot\vec{E}^* d\vec{x}, \quad (2.6)$$

where $\vec{E}$ is the complex electric-field envelope, $-\vec{\nabla}\psi$, the linear momentum

$$P_l=\int\left[\tfrac{1}{2}i\left(E_m\frac{\partial E_l^*}{\partial x_m}-E_m^*\frac{\partial E_l}{\partial x_m}\right)-n\frac{\partial u}{\partial x_l}\right]d\vec{x}, \quad (2.7)$$

and the angular momentum

$$\vec{M}=\int(i\vec{E}\times\vec{E}^*+\vec{x}\times\vec{p})d\vec{x}, \quad (2.8)$$

where $\vec{p}$ is the linear momentum density, and where a summation over repeated indices is implied.

For spatially homogeneous systems it is convenient to work with the Fourier transform of a field f,

$$f(\vec{x}) = \int e^{i\vec{k}\cdot\vec{x}} f(\vec{k}) d\vec{k} , \tag{2.9}$$

where we use the notation $d\vec{k} = d^d k/(2\pi)^d$ to indicate the wave vector space integrations in d dimensions.

Equations (2.3) are then reexpressed as

$$\left(\frac{\partial}{\partial t} + \nu_e + ik^2\right)\psi(\vec{k},t) = -i\int n(\vec{p},t)\psi(\vec{q},t)\frac{\vec{k}\cdot\vec{q}}{k^2}d\vec{p} , \tag{2.10a}$$

$$\left(\frac{\partial}{\partial t} + \nu_e - ik^2\right)\psi^\dagger(\vec{k},t) = i\int n(\vec{p},t)\psi^\dagger(\vec{q},t)\frac{\vec{k}\cdot\vec{q}}{k^2}d\vec{p} , \tag{2.10b}$$

$$\frac{\partial n(\vec{k},t)}{\partial t} = -k^2 u(\vec{k},t) , \tag{2.10c}$$

$$\left(\frac{\partial}{\partial t} + 2\nu_i\right)u(\vec{k},t) = n(\vec{k},t) - \int \vec{p}\cdot\vec{q}\psi(\vec{p},t)\psi^\dagger(\vec{q},t)d\vec{p} , \tag{2.10d}$$

where $\vec{p}+\vec{q}=\vec{k}$. Note that in the Fourier representation the symbol † does not denote complex conjugation. Instead, the relation between ψ and $\psi^\dagger$ is

$$\psi^*(\vec{k}) = \psi^\dagger(-\vec{k}) , \tag{2.11}$$

where the symbol * will always denote complex conjugation. Since $n(\vec{x})$ and $u(\vec{x})$ are real, their Fourier transforms obey the usual

$$n(\vec{k}) = n^*(-\vec{k}) , \quad u(\vec{k}) = u^*(-\vec{k}) . \tag{2.12}$$

Since the spatially uniform component of n is time independent, and hence, only affects the evolution of ψ by a trivial phase change, it can be set equal to zero without loss of generality.

A statistical description of the Zakharov equation deals primarily with the ensemble-averaged or mean value (denoted by angular brackets) of the fields ψ, $\psi^\dagger$, n, and u, and their various products up to order three. Though it is the equal-time products, such as those in the expressions for the conserved quantities, which are of immediate physical interest, we also allow in the following notation for unequal-time products:

$$C^{fg}_{\vec{x}_1 t_1, \vec{x}_2 t_2} = \langle \delta f(\vec{x}_1 t_1)\, \delta g(\vec{x}_2 t_2)\rangle , \tag{2.13}$$

where $\delta f = f - \langle f\rangle$, and f and g are any two of ψ, $\psi^\dagger$, n, and u. If the expectation values are spatially homogeneous, i.e., $\langle n(xt)\rangle$ and $\langle E(xt)\rangle$ are independent of $\vec{x}$, and C^{fg} only depends upon $\vec{x}_1 - \vec{x}_2$, we shall write

$$C^{fg}_{\vec{x}_1 t_1, \vec{x}_2 t_2} = \int C^{fg}_{\vec{k} t_1, t_2} e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)} d\vec{k} . \tag{2.14}$$

The spectra $W(k)$, $N(k)$, and $U(k)$ are defined so that

$$\langle \vec{E}(t)\cdot\vec{E}^\dagger(t)\rangle = W(t) = \int W(\vec{k},t)d\vec{k} + \langle\vec{E}(t)\rangle\cdot\langle\vec{E}^\dagger(t)\rangle , \tag{2.15}$$

$$\langle n^2(t)\rangle = \int N(\vec{k},t)d\vec{k} + \langle n\rangle^2 , \tag{2.16}$$

$$\langle u^2(t)\rangle = \int U(\vec{k},t)d\vec{k} + \langle u\rangle^2 . \tag{2.17}$$

In addition to the covariance functions C, it is convenient to consider the linear response functions R defined in Eq. (2.21).

Let us now restrict our attention to the spatially homogeneous case. It follows from Eqs. (2.11)–(2.13) that

$$\begin{aligned} C^{\psi\psi^\dagger}_{-\vec{k}t_1,t_2} &= (C^{\psi^\dagger\psi}_{\vec{k}t_1,t_2})^* , \\ R^{\psi^\dagger\psi^\dagger}_{-\vec{k}t_1,t_2} &= (R^{\psi\psi}_{\vec{k}t_1,t_2})^* , \\ C^{nn}_{-\vec{k}t_1 t_2} &= (C^{nn}_{\vec{k}t_1,t_2})^* , \end{aligned} \tag{2.18}$$

and several similar relations which we shall not display.

Even within the homogeneous ensemble we have a choice of whether or not we have a nonzero mean electrostatic envelope field $\langle\vec{E}\rangle = \vec{E}_0$, a spatially constant vector. Physically, this is dictated by the external fields applied to the system. If there are none, the $\vec{k}=0$ component of the electric field could point in any direction and we can reasonably pick an ensemble with $\langle\vec{E}\rangle = 0$. This is consistent with $C^{\psi\psi} = C^{n\psi} = C^{u\psi} = R^{\psi\psi} = R^{\psi n} = R^{\psi u} = R^{u\psi} = R^{n\psi} = 0$ as shown in the Appendix. We call this the "symmetric ensemble" and this will be the choice for most of the detailed considerations in this paper. Such a system might be excited to instability by a charged particle beam which amplifies thermal fluctuations or by an imposed noise source such as we consider in Sec. VII.

If the symmetric ensemble were to show very large fluctuations at $k=0$ associated, say, with the Langmuir condensate, then a better choice of ensemble might have $\vec{E}_0 \neq 0$, where the direction of $\vec{E}_0$ is determined by small applied symmetry breaking fields. So far we have no compelling evidence that there is spontaneous symmetry breaking due to large $k=0$ fluctuations. The unsymmetric ensemble is of intrinsic experimental interest for systems excited by external electromagnetic fields, i.e., parametric excitation experiments. Some of the aspects of the statistical theory for this case have previously been considered by DuBois and Bezzerides[14]; this work was restricted to the weak-turbulence regime and did not account for modulational instability. The symmetric ensemble which we consider in this paper is a simpler model in which to treat modulational instability effects.

Since Eqs. (2) are nonlinear, the equation of motion for $C^{\psi\psi^\dagger}$ is coupled to that of $\langle n\psi\psi^\dagger\rangle$, and so on. A similar "closure problem" exists for the response functions. The direct-interaction approximation (DIA) is an approximation which can be applied to the closure problem associated with any quadratically nonlinear system of equations,[9,15]

$$\frac{\partial}{\partial t_1}\phi_1 = L_{12}\phi_2 + \tfrac{1}{2}A_{123}\phi_2\phi_3 + s_1 , \tag{2.19}$$

where the subscript 1 on the stochastic variable ϕ_1 refers to the time variable t_1 and any set of discrete or continuous variables such as spatial variables and vector indices. In the case of the Zakharov equations ϕ_1 is the four-dimensional vector

$$\phi_1 = \begin{bmatrix} \psi(\vec{k},t_1) \\ \psi^\dagger(\vec{k}_1 t_1) \\ n(\vec{k}_1,t_1) \\ u(\vec{k}_1,t_1) \end{bmatrix} , \tag{2.20}$$

where 1 includes a discrete index σ_1 which runs from $\sigma=1$ to $\sigma=4$. In Eq. (2.19) a summation convention over repeated indices is implied. The coefficients L_{12} and A_{123} can be easily written down by reference to Eqs. (2.10); we will not give these here. The DIA provides a relation between the matrix response functions

$$R_{11'} = \frac{\delta\langle\phi_1\rangle}{\delta\langle s_1'\rangle} , \tag{2.21}$$

which are nonzero (causal) only for $t_1 > t_1'$, and the correlation functions or covariances

$$C_{11'} = \langle\phi_1\phi_1'\rangle - \langle\phi_1\rangle\langle\phi_1'\rangle . \tag{2.22}$$

In general, R and C obey equations of the form

$$\left(\delta(1-2)\frac{\partial}{\partial t_2} - L_{12} - A_{123}\langle\phi_3\rangle - \Sigma_{12}\right)R_{21'} = \delta_{11'} , \tag{2.23}$$

$$\left(\delta(1-2)\frac{\partial}{\partial t_2} - L_{12} - A_{123}\langle\phi_3\rangle - \Sigma_{12}\right)C_{21'} = S_{12}R_{1'2} , \tag{2.24}$$

where $\Sigma_{11'}$ is the "self-energy" which in the DIA is

$$\Sigma_{11'} = A_{123}A_{2'3'1'}R_{22'}C_{33'} , \tag{2.25}$$

and S_{12} is the nonlinear or self-consistent noise function which in the DIA is

$$S_{11'} = \tfrac{1}{2}A_{123}A_{1'2'3'}C_{22'}C_{33'} . \tag{2.26}$$

The explicit matrix components for the Zakharov equations are given below for the homogeneous, symmetric case of zero-mean field $\langle\vec{E}(\vec{k},t)\rangle = \vec{k}\langle\psi(\vec{k},t)\rangle = 0$. In this case all two-point quantities are diagonal in the $\vec{k}$ indices and we can use the notation

$$R_{12} = R^{\sigma_1\sigma_2}_{\vec{k}_1 t_1,\vec{k}_2 t_2} \equiv R^{\sigma_1\sigma_2}_{\vec{k}_1 t_1,t_2}(2\pi)^d\delta^{(d)}(\vec{k}_1+\vec{k}_2)$$

and

$$C_{12'} = C^{\sigma_1\sigma_2}_{\vec{k}_1 t_1,\vec{k}_2 t_2} = C^{\sigma_1\sigma_2}_{\vec{k}_1 t_1,t_2}(2\pi)^d\delta^{(d)}(\vec{k}_1+\vec{k}_2) .$$

The DIA equations then can be explicitly written as

$$\left(\frac{\partial}{\partial t_1}+\nu_e+ik^2\right)R^{\psi\psi}_{\vec{k}t_1,t_3} - \int_{t_3}^{t_1}\Sigma^{\psi\psi}_{\vec{k}t_1,t_2}R^{\psi\psi}_{\vec{k}t_2,t_3}dt_2 = \delta(t_1-t_3) , \tag{2.27}$$

$$\frac{\partial}{\partial t_1}R^{nu}_{\vec{k}t_1,t_3} + k^2R^{uu}_{\vec{k}t_1,t_3} = 0 , \tag{2.28}$$

$$\left(\frac{\partial}{\delta t_1}+2\nu_i\right)R^{uu}_{\vec{k}t_1,t_3} - R^{nu}_{\vec{k}t_1,t_3} - \int_{t_3}^{t_1}\Sigma^{un}_{\vec{k}t_1,t_2}R^{nu}_{\vec{k}t_2,t_3}dt_2 = \delta(t_1-t_3) . \tag{2.29}$$

The limits on the time integrations reflect the causal nature of the response functions, $t_1 \geq t_3 \geq t_0$, where t_0 is the initial value time for the statistical ensemble. It follows that

$$R^{\psi\psi}_{\vec{k}t_1,t_1} = R^{uu}_{\vec{k}t_1,t_1} = 1 , \quad R^{nu}_{\vec{k}t_1,t_1} = 0 ,$$

$$\left(\frac{\partial}{\partial t_1}+\nu_e+ik^2\right)C^{\psi\psi^\dagger}_{\vec{k}t_1,t_3} - \int_{t_0}^{t_1}\Sigma^{\psi\psi}_{\vec{k}t_1,t_2}C^{\psi\psi^\dagger}_{\vec{k}t_2,t_3}dt_2 = \int_{t_0}^{t_3}S^{\psi\psi^\dagger}_{\vec{k}t_1,t_2}(R^{\psi\psi}_{\vec{k}t_3,t_2})^*dt_2 , \tag{2.30}$$

$$\frac{\partial}{\partial t_1}C^{nu}_{\vec{k}t_1,t_3} + k^2C^{uu}_{\vec{k}t_1,t_3} = 0 , \tag{2.31}$$

$$\frac{\partial}{\partial t_1}C^{nn}_{\vec{k}t_1,t_3} + k^2C^{un}_{\vec{k}t_1,t_3} = 0 , \tag{2.32}$$

$$\left(\frac{\partial}{\partial t_1}+2\nu_i\right)C^{uu}_{\vec{k}t_1,t_3} - C^{nu}_{\vec{k}t_1,t_3} - \int_{t_0}^{t_1}\Sigma^{un}_{\vec{k}t_1,t_2}C^{nu}_{\vec{k}t_2,t_3}dt_2 = \int_{t_0}^{t_3}S^{uu}_{\vec{k}t_1,t_2}(R^{uu}_{\vec{k}t_3,t_2})^*dt_2 , \tag{2.33}$$

$$\left(\frac{\partial}{\partial t_1}+2\nu_i\right)C^{un}_{\vec{k}t_1,t_3} - C^{nn}_{\vec{k}t_1,t_3} - \int_{t_0}^{t_1}\Sigma^{un}_{\vec{k}t_1,t_2}C^{nn}_{\vec{k}t_2,t_3}dt_2 = \int_{t_0}^{t_3}S^{uu}_{\vec{k}t_1,t_2}(R^{nu}_{\vec{k}t_3,t_2})^*dt_2 ,$$

$$t_1 \geq t_0 , \quad t_3 \geq t_0 . \tag{2.34}$$

The self-energy or memory functions are given by

$$\Sigma^{\psi\psi}_{\vec{k}t_1,t_2} = -\int \left(\frac{(\vec{q}\cdot\vec{k})^2}{k^2q^2} R^{\psi\psi}_{\vec{q}t_1,t_2} C^{nn}_{\vec{p}t_1,t_2} + \frac{i(\vec{k}\cdot\vec{p})^2}{k^2} R^{nu}_{\vec{q}t_1,t_2} C^{\psi\psi^\dagger}_{\vec{p}t_1,t_2} \right) d\vec{p}\,, \quad (2.35)$$

$$\Sigma^{un}_{\vec{k}t_1,t_2} = i\int \frac{(\vec{p}\cdot\vec{q})^2}{q^2} [(R^{\psi\psi}_{-\vec{q}t_1,t_2})^* C^{\psi\psi^\dagger}_{\vec{p}t_1,t_2} - R^{\psi\psi}_{\vec{q}t_1,t_2} (C^{\psi\psi^\dagger}_{-\vec{p}t_1,t_2})^*] d\vec{p}\,, \quad (2.36)$$

where $\vec{k}=\vec{p}+\vec{q}$. The nonlinear noise functions are given by

$$S^{\psi\psi^\dagger}_{\vec{k}t_1,t_2} = \int \frac{(\vec{k}\cdot\vec{q})^2}{k^4} C^{\psi\psi^\dagger}_{\vec{q}t_1,t_2} C^{nn}_{\vec{p}t_1,t_2} d\vec{p}\,, \quad (2.37)$$

$$S^{uu}_{\vec{k}t_1,t_2} = \int (\vec{p}\cdot\vec{q})^2 C^{\psi\psi^\dagger}_{\vec{p}t_1,t_2} (C^{\psi\psi^\dagger}_{-\vec{q}t_1,t_2})^* d\vec{p}\,. \quad (2.38)$$

If a random source s is added to the equation of motion for $\psi(x)$ (and for consistency, a corresponding source $s^\dagger$ is added to the equation of motion for $\psi^\dagger$) then the "linear noise", $\langle ss^\dagger\rangle$, must be added to $S^{\psi\psi^\dagger}$. Similarly, it is possible to add a linear noise term to S^{uu}. For example, these random sources could represent the coupling to the underlying particle degrees of freedom and in the case of a low dimensional ($d=1$ or $d=2$) model, there could be additional random sources which represent the coupling to modes that live in the omitted dimensions.

It is easy to show that the (first order in time) equations of motion for the various R and C functions associated with n and u can be replaced by the following (second order in time) equations for $\hat{R}^{nn}$ and C^{nn}:

$$\left(\frac{\partial^2}{\partial t^2} + 2\nu_i \frac{\partial}{\partial t} + k^2\right) \hat{R}^{nn}_{\vec{k}t_1,t_3} - \int_{t_3}^{t_1} \Sigma^{nn}_{\vec{k}t_1,t_2} \hat{R}^{nn}_{\vec{k}t_2,t_3} dt_2 = \delta(t_1-t_2)\,, \quad (2.39)$$

$$\left(\frac{\partial^2}{\partial t^2} + 2\nu_i \frac{\partial}{\partial t} + k^2\right) C^{nn}_{\vec{k}t_1,t_3} - \int_{t_0}^{t_1} \Sigma^{nn}_{\vec{k}t_1,t_2} C^{nn}_{\vec{k}t_2,t_3} dt_2 = \int_{t_0}^{t_3} S^{nn}_{\vec{k}t_1,t_2} (\hat{R}^{nn}_{\vec{k}t_3,t_2})^* dt_2\,, \quad (2.40)$$

where

$$\hat{R}^{nn} \equiv -k^{-2} R^{nu}$$

and

$$\Sigma^{nn}_{\vec{k}} = -k^2 \Sigma^{un}_{\vec{k}}\,,$$
$$S^{nn}_{\vec{k}} = +k^4 S^{uu}_{\vec{k}}\,. \quad (2.41)$$

In general, the DIA is a realizable approximation. This means that there exists a dynamical system whose response and fluctuation functions exactly satisfy the DIA. One such system is the random-coupling model introduced by Kraichnan.[16]

It can be shown that for any Hamiltonian system whose energy H contains terms which are cubic in the canonical variables, but no terms of higher order, the DIA conserves $\langle H\rangle$. (This demonstration involves some tedious algebra.) Since in the Zakharov equations, $\langle H\rangle$ contains $\langle n\vec{\nabla}\psi\cdot\vec{\nabla}\psi^\dagger\rangle$, and since the exact equation of motion for $\langle\psi\nabla^2\psi^\dagger\rangle$ contains $\langle n\vec{\nabla}\psi\cdot\vec{\nabla}\Psi^\dagger\rangle$, within the DIA $\langle H\rangle$ is evaluated by replacing the three-field-correlation function by the time derivative of the appropriate two-field-correlation function. Also, it can be shown in general (spatially inhomogeneous and nonsymmetric ensemble) that the DIA conserves $\langle N\rangle$ and $\langle\vec{P}\rangle$.

Since the weak-turbulence approximation (see Sec. III) conserves only the quadratic part of $\langle H\rangle$, and since it is the nonlinearity of the Zakharov equations (which comes from the cubic term in H) which is responsible for solitons ($d=1$) and soliton collapse ($d=2,3$), the conservation of the entire $\langle H\rangle$ by the DIA is a distinctive feature. By itself, this feature has an ambiguous significance because it only implies a consistency between the equation of motion of the two-field-correlation function (referred to above) and $\langle H\rangle$. In principle, the DIA might have qualitative inaccuracies in its predictions while still maintaining this consistency. Numerical studies of the stationary solutions to the DIA for the case of a model in which only the Fourier modes at 0 and $\pm k$ are retained, yield a value for the cubic piece of $\langle H\rangle$ which in sign (it is negative when the modulational instability is significant during the approach to stationarity) and magnitude agree favorably with a numerical simulation of that model.[17] Thus, there is some indication that the conservation of $\langle H\rangle$ by the DIA is significant.

III. THE DIA EQUATIONS IN STEADY STATE

In this section we examine the steady-state form of the DIA equations. We can imagine an ensemble of systems driven unstable by some fluctuating external sources whose mean over the ensemble is zero but whose covariance is nonzero and stationary; i.e.,

$$\langle s_\psi\rangle = \langle s_n\rangle = 0\,,$$
$$\langle s_\psi(t)s_\psi^*(t')\rangle = S_0^{\psi\psi^\dagger}(t-t')\,,$$
$$\langle s_n(t)s_n(t')\rangle = S_0^{nn}(t-t')\,.$$

Or we can imagine the systems driven unstable from initial random noise by the existence of regions in k space of negative damping, i.e., negative $\nu_e(k)$ or $\nu_i(k)$ which might arise, for example, from a two-stream instability of the underlying particle degrees of freedom. We *assume* in this section that the systems saturate and go into a statistically stationary state. In this state all response functions and covariances are functions of time differences only, i.e.,

$$C_{t_1,t_1'}=C(t_1-t_1')\ ,\quad R_{t_1,t_1'}=R(t_1-t_1')\ .$$

Initially, in the unstable, temporally evolving regime the response functions and covariances are exponentially growing functions of $|t-t'|$. Steady state is reached when the renormalization effects represented by the self-energies Σ^{nn} and $\Sigma^{\phi\phi}$ provide enough turbulent or nonlinear damping to stabilize the system and the nonlinear noise represented by S^{nn} and $S^{\phi\phi^\dagger}$ maintain the level of fluctuations. In this limit all response functions and covariances are decaying functions of $|t-t'|$.

In the steady state the self-energies and nonlinear noise sources in (2.35), (2.36), (2.37), and (2.38) become simple convolutions, e.g.,

$$\int_{t_1'}^{t_1} dt_2 \Sigma^{nn}_{\vec{k},t_1,t_2} \hat{R}^{nn}_{\vec{k},t_2,t_1'} = \int_{t_1'}^{t_1} dt_2 \Sigma^{nn}_{\vec{k}}(t_1-t_2)\hat{R}^{nn}_{\vec{k}}(t_2-t_1')\ .$$

Introducing Fourier transforms of each function,[18] e.g.,

$$R(\vec{k},\omega)=\int_{-\infty}^{\infty} dt\, e^{i\omega t} R_{\vec{k}}(t)\ ,$$

the Fourier transform over (t_1-t_1') of this convolution becomes a simple product of the transforms, e.g., $\Sigma^{nn}(\vec{k}\omega)\hat{R}^{nn}(\vec{k}\omega)$. We can then immediately transform Eqs. (2.27)–(2.40) into equations for the Fourier transforms

$$[-i(\omega_1-k_1^2)+\nu_e-\Sigma^{\phi\phi}(\vec{k}_1,\omega_1)]R^{\phi\phi}(\vec{k}_1,\omega_1)=1\ , \tag{3.1}$$

$$\Sigma^{\phi\phi}(\vec{k}_1,\omega_1)=(2\pi)^{-1}\int d\vec{k}_2\int d\omega_2[i(\vec{k}_1\cdot\vec{k}_2)^2k_1^{-2}k_3^2C^{\phi\phi^\dagger}(\vec{k}_2,\omega_2)\hat{R}^{nn}(\vec{k}_3,\omega_3)-(\vec{k}_1\cdot\vec{k}_2)^2k_1^{-2}k_2^{-2}R^{\phi\phi}(\vec{k}_2,\omega_2)C^{nn}(\vec{k}_3,\omega_3)]\ , \tag{3.2}$$

$$[-i(\omega_1-k_1^2)+\nu_e-\Sigma^{\phi\phi}(\vec{k}_1,\omega_1)]C^{\phi\phi^\dagger}(\vec{k}_1,\omega_1)=[R^{\phi\phi}(\vec{k}_1\omega_1)]^*S^{\phi\phi^\dagger}(\vec{k}_1\omega_1)\ , \tag{3.3}$$

$$S^{\phi\phi^\dagger}(\vec{k}_1,\omega_1)=S_0^{\phi\phi^\dagger}(\vec{k}_1,\omega_1)+(2\pi)^{-1}\int d\vec{k}_2\int d\omega_2(\vec{k}_1\cdot\vec{k}_3)^2k_1^{-4}C^{nn}(\vec{k}_2,\omega_2)C^{\phi\phi^\dagger}(\vec{k}_3,\omega)\ , \tag{3.4}$$

$$[-\omega_1^2-2i\omega_1\nu_i+k_1^2-\Sigma^{nn}(\vec{k}_1,\omega_1)]\hat{R}^{nn}(\vec{k}_1,\omega_1)=1\ , \tag{3.5}$$

$$\Sigma^{nn}(\vec{k}_1,\omega_1)=(2\pi)^{-1}\int d\vec{k}_2\int d\omega_2 i(\vec{k}_2\cdot\vec{k}_3)^2k_1^2k_2^{-2}[R^{\phi\phi}(\vec{k}_2\omega_2)C^{\phi^\dagger\phi}(\vec{k}_3,\omega_3)-R^{\phi^\dagger\phi^\dagger}(\vec{k}_2,\omega_2)C^{\phi\phi^\dagger}(\vec{k}_3,\omega_3)]\ , \tag{3.6}$$

$$[-\omega_1^2-2i\omega_1\nu_i+k_1^2-\Sigma^{nn}(\vec{k}_1,\omega_1)]C^{nn}(\vec{k}_1,\omega_1)=[\hat{R}^{nn}(\vec{k}_1,\omega_1)]^*S^{nn}(\vec{k}_1,\omega_1)\ , \tag{3.7}$$

$$S^{nn}(\vec{k}_1\omega_1)=S_0^{nn}(\vec{k}_1,\omega_1)+(2\pi)^{-1}\int d\vec{k}_2\int d\omega_2 k_1^4(\vec{k}_2\cdot\vec{k}_3)^2C^{\phi\phi^\dagger}(\vec{k}_2,\omega_2)C^{\phi^\dagger\phi}(\vec{k}_3,\omega_3)\ , \tag{3.8}$$

where throughout we have used the conventions

$$\vec{k}_1=\vec{k}_2+\vec{k}_3\ ,\qquad \omega_1=\omega_2+\omega_3\ . \tag{3.9}$$

We also have the relations for real $\vec{k}$ and ω:

$$R^{\phi^\dagger\phi^\dagger}(\vec{k},\omega)=R^{\phi\phi}(-\vec{k},-\omega)^*\ ,\qquad C^{\phi^\dagger\phi}(\vec{k},\omega)=C^{\phi\phi^\dagger}(-\vec{k},-\omega)^*\ , \tag{3.10}$$

which follow from (2.18).[19]

The limit of weak-turbulence theory

It is instructive to point out the relationship of the DIA equations to the familiar weak-turbulence formulations of plasma turbulence theory. There are two main steps involved in going from the DIA to this weak-turbulence limit.

(1) Replace the response functions $R^{\phi\phi}$ and $\hat{R}^{nn}$ *as they appear in the self-energies* by their unperturbed values

$$R^{\phi\phi} \to R_0^{\phi\phi} \equiv [-i(\omega - k^2) + \nu_e(k)]^{-1} ,$$
$$\hat{R}^{nn} \to \hat{R}_0^{nn} \equiv [-\omega^2 - 2i\omega\nu_i(k) + k^2]^{-1} . \tag{3.11}$$

With these substitutions, the expressions for $\Sigma^{\phi\phi}$ and Σ^{nn} will be denoted by

$$\Sigma_0^{\phi\phi}(\vec{k}_1\omega_1) = (2\pi)^{-1}\int d\vec{k}_2 \int d\omega_2\left(\frac{i(\vec{k}_1\cdot\vec{k}_2)^2 k_1^{-2}k_3^2 C^{\phi\phi^\dagger}(\vec{k}_2,\omega_2)}{[-\omega_3^2 - 2i\omega_3\nu_i(k_3) + k_3^2]} - \frac{(\vec{k}_1\cdot\vec{k}_2)^2 k_1^{-2}k_2^{-2} C^{nn}(\vec{k}_3,\omega_3)}{[-i(\omega_2 - k_2^2) + \nu_e(k_2)]}\right), \tag{3.12}$$

$$\Sigma_0^{nn}(\vec{k}_1\omega_1) = (2\pi)^{-1}\int d\vec{k}_2 \int d\omega_2 i(\vec{k}_2\cdot\vec{k}_3)^2 k_1^2 k_2^{-2}\left(\frac{C^{\phi^\dagger\phi}(\vec{k}_3,\omega_3)}{[-i(\omega_2 - k_2^2) + \nu_e(k_2)]} - \frac{C^{\phi\phi^\dagger}(\vec{k}_3,\omega_3)}{[-i(\omega_2 + k_2^2) + \nu_e(k_2)]}\right). \tag{3.13}$$

With these substitutions for the self-energies, Eqs. (3.3) and (3.7) for the covariances represent the same level of approximation as Kadomtsev's[2] equations II.6. (Here, this result is generalized to two coupled equations which explicitly takes into account the coupling between Langmuir and ion-sound modes as dictated by the Zakharov equations.) In addition, the symmetry which Kadomtsev introduces in an *ad hoc* way is automatically present here since $R^{\phi\phi}$ or $\hat{R}^{nn}$ rather than $R_0^{\phi\phi}$ and $\hat{R}_0^{nn}$ appear on the right-hand side of these equations.

(2) If we further assume, following Kadomtsev, for example, that $\gamma/\omega \ll 1$ for the unperturbed frequencies and damping rates [i.e., in our case $(\nu_e/k^2) \ll 1$, $\nu_i/k \ll 1$] the weak-turbulence theory retains only the *dissipative parts* of the self-energies which renormalize the damping rates. This allows us to write the equation for $C^{\phi\phi^\dagger}$ as

$$[(\omega - k^2)^2 + \gamma_L(\vec{k})^2]C^{\phi\phi^\dagger}(\vec{k},\omega) = S^{\phi\phi^\dagger}(\vec{k},\omega) , \tag{3.14}$$

where

$$\gamma_L = \nu_e(k) - \mathrm{Re}\Sigma^{\phi\phi}(\vec{k},\omega) , \quad \omega = k^2$$

is a renormalized Langmuir damping rate. Similarly, the equation for C^{nn} can be written as

$$[(\omega^2 - k^2)^2 + 4\omega^2\Gamma_s(\vec{k})^2]C^{nn}(\vec{k}\omega) = S^{nn}(\vec{k},\omega) , \tag{3.15}$$

where

$$\Gamma_s(\vec{k}) = \nu_i(k) + \tfrac{1}{2}\mathrm{Im}\Sigma^{nn}(\vec{k},\omega)\omega^{-1} , \quad \omega = k$$

is a renormalized ion-sound damping. The equations are on the same level of approximation as Kadomtsev (II.7).

Weak-turbulence theory usually makes the additional assumptions that $\gamma/\omega \ll 1$ and $\Gamma/\omega \ll 1$. Then we can further approximate

$$[(\omega_1 - k_1^2)^2 + \gamma_L(\vec{k})^2]^{-1} \cong +\frac{\pi}{\gamma_L}\delta(\omega_1 - k_1^2)$$

if $k_1^2 \gg \gamma_L$,

$$[(\omega_1^2 - k_1^2)^2 + 4\omega_1^2\Gamma_s(\vec{k})^2]^{-1} \cong +\frac{\pi}{2|\omega_1|\Gamma_s}\delta(\omega_1^2 - k_1^2) \tag{3.16}$$

if $k_1 \gg \Gamma_s$. When these approximations are made in (3.14) and (3.15) we see it is consistent to write

$$C^{\phi\phi^\dagger}(\vec{k},\omega) = \frac{W(\vec{k})}{k^2}2\pi\delta(\omega - k^2) , \tag{3.17}$$

which by (2.18) implies

$$C^{\phi^\dagger\phi}(\vec{k},\omega) = \frac{W(-\vec{k})}{k^2}2\pi\delta(\omega + k^2) \tag{3.17$'$}$$

and

$$C^{nn}(\vec{k},\omega) = |k|N(\vec{k})2\pi\delta(\omega^2 - k^2) , \tag{3.18}$$

where we can identify the coefficients of the delta functions with the electric-field energy density $W(\vec{k})$ and the mean-square density fluctuation $N(\vec{k})$ defined in (2.15) and (2.16) in terms of equal-time covariances.

We make these substitutions in (3.14) and (3.15) and integrate both sides of the equations over ω to obtain the steady-state balance equations

$$2\gamma_L(\vec{k})W(\vec{k}) = k^2 S^{\phi\phi^\dagger}(\vec{k},\omega = k^2) \tag{3.19}$$

and

$$2\Gamma_s(\vec{k})N(\vec{k}) = (4k^2)^{-1}[S^{nn}(\vec{k},\omega = k) + S^{nn}(\vec{k},\omega = -k)] . \tag{3.20}$$

The ω_2 integrations in the nonlinear noise sources can be carried out trivially after the substitutions (3.17) and (3.18) to give

$$k_1^2 S^{\phi\phi^\dagger}(\vec{k}_1,\omega_1 = k_1^2) = \pi\int d\vec{k}_2(\hat{k}_1\cdot\hat{k}_3)^2 N(\vec{k}_2)W(\vec{k}_3)[\delta(k_1^2 - k_3^2 - |k_2|) + \delta(k_1^2 - k_3^2 + |k_2|)] , \tag{3.21}$$

where the carets indicate unit vectors and

$$(2k_1^2)^{-1}S^{nn}(\vec{k}_1,\omega_1 = k_1) = \pi\int d\vec{k}_2 k_1^2(\hat{k}_2\cdot\hat{k}_3)^2 W(\vec{k}_2)W(-\vec{k}_3)\delta(k_1 - k_2^2 + k_3^2) . \tag{3.22}$$

Equations (3.19) and (3.20) with (3.21) and (3.22) are then at the same level of approximation as Kadomtsev

Eq. II.8.

Explicit expressions for the renormalized damping rates in WTT are obtained from (3.12) and (3.13). If we take the limiting expressions for $\nu_e, \nu_i \to 0$, consistent with the assumption (2), we get

$$\gamma_L(\vec{k}_1) = \nu_e(k_1) + \pi \int d\vec{k}_2 (\hat{k}_1 \cdot \hat{k}_2)^2 |k_3| \tfrac{1}{2} \{ W(\vec{k}_2) [\delta(k_1^2 - k_2^2 - |k_3|) - \delta(k_1^2 - k_2^2 + |k_3|)] + |k_3|^{-1} N(\vec{k}_3) [\delta(k_1^2 - k_2^2 + |k_3|) + \delta(k_1^2 - k_2^2 - |k_3|)] \} . \tag{3.23}$$

The term proportional to the Langmuir intensity $W(\vec{k}_2)$ is the usual Langmuir-induced decay ($L \to L \pm S$) term which gives the familiar cascade of Langmuir energy toward lower k; this contribution produces either a turbulent damping or growth in the Langmuir waves depending on the *shape* of $W(\vec{k})$. The term proportional to $N(\vec{k}_3)$ is the corresponding *phonon* induced decay which produces a turbulent *damping* of Langmuir waves. Similarly, we obtain the following expression for the renormalized sound-wave damping:

$$\Gamma_s(\vec{k}_1) = \nu_i(k_1) + \pi |k_1| \int d\vec{k}_2 (\hat{k}_2 \cdot \hat{k}_3)^2 W(\vec{k}_3) \tfrac{1}{2} [\delta(k_2^2 - k_3^2 - |k_1|) - \delta(k_2^2 - k_3^2 + |k_1|)] . \tag{3.24}$$

The turbulent damping or growth is again the process of plasmon-induced decay of $L \to L \pm S$ and its sign depends on the shape of $W(\vec{k})$.

These results of WTT (weak-turbulence theory) are, of course, well known and we record them here for completeness and to aid our further discussion. Zakharov and Kuznetsov[20] and Kats[21] have derived scale invariant power law spectra for the weak-turbulence equations; various solutions were found corresponding to constant fluxes of conserved quantities. We will find that the DIA equations go continuously over to the weak-turbulence equations for sufficiently large values of k.

In the application to the Navier-Stokes equation, Kraichnan[11] has integrated the initial value problem for the DIA until a stationary solution was approached. This result was in qualitative agreement with an approximation to the DIA response and correlation functions, respectively, of the forms

$$R(k,t) = e^{-\zeta(k)t}, \quad C(k,t) = A(k) e^{-\eta(k)t} .$$

The decay rates ζ and η are determined self-consistently from the frequency equal zero transform of the DIA differential-integral equations for R and C while an energy balance equation for $A(k)$ is obtained from setting $t=0$ in the equation for C. Since the low Reynold's number limit for this problem has $R(k,t) = \exp(-\nu k^2 t)$ as its response function, where ν is the kinematic viscosity, $\zeta(k)/k^2$ may be interpreted as a turbulent viscosity.

The exact solution to the DIA, which is nonlinear in R and C, usually has an analytic structure which is more complicated than an exponential in time. In the Fourier transform representation the simple exponential in time behavior yields a simple pole in $R(\vec{k}, \omega)$ at $\omega = -i\zeta(k)$ while the exact solution to the DIA, even for a dynamical system with a finite number of degrees of freedom typically also has a branch discontinuity in $R^{-1}(\vec{k}, \omega)$ in the complex ω plane. These complications are discussed in Sec. IV.

We have seen in (3.14) and (3.15) that the damping rates of the linear normal modes are shifted away from their linear values in WTT and we may speak of "turbulent damping rates" of these modes just as one speaks of "turbulent viscosity" in the Navier-Stokes case.

IV. FREQUENCY-SPACE POLE REPRESENTATIONS OF RESPONSE AND CORRELATION FUNCTIONS

The approximations Σ_0 in (3.12) and (3.13) for the self-energies provide a first approximation beyond weak-turbulence theory to the renormalized response $-R \cong (R_0^{-1} - \Sigma_0)^{-1}$. The various approximations which we will consider for Σ (including the fully self-consistent DIA) allow us to investigate deviations from the unperturbed response functions which in frequency space have poles at complex frequencies corresponding to the unperturbed modes [i.e., $\omega = k^2 - i\nu_e(k)$, and $\omega = \pm k - i\nu_i(k)$ if $\omega_i(k)/k \ll 1$]. These deviations may be small and can therefore be considered as small renormalizations of the unperturbed modes; in this case the results of WTT are only slightly modified. The more interesting case is the appearance of new roots in ω space of $R^{-1}(k, \omega) = 0$ which do not reduce to the unperturbed roots in the limit $\Sigma \to 0$. In this case the turbulent renormalization produces significant new physics; we shall investigate in detail new roots of $(R^{nn})^{-1} = 0$, $(R^{\psi\psi})^{-1} = 0$ which correspond to the modulational instability (MI) of the Langmuir spectrum.

The dispersion relations $R^{-1}(\vec{k}, \omega) = 0$ may have a number of roots (perhaps even an infinite number) which correspond to poles of $R(k, \omega)$ in the complex ω plane. The question then arises: Can $R(k, \omega)$ be represented as a sum of simple pole contributions

$$R(\vec{k}, \omega) = \sum_\nu \frac{iZ_\nu}{\omega - \omega_\nu + i\gamma_\nu}, \tag{4.1}$$

where $\omega = \omega_\nu - i\gamma_\nu$ are zeros of $R^{-1}(\vec{k}, \omega) = 0$? If this is so then

$$R^0(k, \omega_\nu - i\gamma_\nu)^{-1} - \Sigma(\vec{k}, \omega_\nu - i\gamma_\nu) = 0, \tag{4.2}$$

and Z_ν the residue at the pole is

$$-iZ_\nu^{-1} = \partial_\omega [(R^0(k, \omega)^{-1} - \Sigma(\vec{k}, \omega)]|_{\omega=\omega_\nu - i\gamma_\nu}. \tag{4.3}$$

Note that the dependence of the quantities ω_ν, γ_ν, and Z_ν, etc., on k will often be suppressed. This is not a trivial question since if this assumption and a related one for the covariance [see (4.12)] is made, and $\Sigma(\vec{k}, \omega)$ is then computed using these ansatz, it is found in general that $\Sigma(\vec{k}, \omega)$ may have branch discontinuities whose detailed structure depends on the shape of the spectra $W(\vec{k})$ and $N(\vec{k})$. It would therefore appear that the singularities of $R(\vec{k}, \omega)$ are the poles discussed above *plus* possible branch cuts arising from $\Sigma(\vec{k}, \omega)$. This is a more complicated example of a problem well known in linear response and many-body theory. In the theory of the linear dielectric response, for example, $\epsilon(k, \omega)$ has zeros (corresponding to plasmons and phonons) and a branch cut. It is well known that the effect of the branch cut is equivalent to the existence of a (perhaps infinite) set of increasingly damped poles on the adjoining Riemann sheet of the cut complex plane. The response $R(\vec{k}, \omega)$ can then be represented by a series of poles which correspond to deforming the contour of ω integration onto the adjoining Riemann sheet.

We will *assume* that $R(\vec{k}, \omega)$ can be represented as a sum of poles as in (4.1). A plausibility argument based on an iteration scheme will be given later in this section. In Sec. VI we examine a degenerate case where the branch cut shrinks to a pole and the results of this study are consistent with the pole ansatz.

We establish the following notation for the pole representations of $R^{\phi\phi}$ and R^{nn}:

$$R^{\phi\phi}(\vec{k}, \omega) = \sum_\nu \frac{iZ_\nu}{\omega - \omega_\nu + i\gamma_\nu}, \tag{4.4}$$

where

$$-i(\omega_\nu - i\gamma_\nu) + \nu_e(k) - \Sigma^{\phi\phi}(\vec{k}, \omega_\nu - i\gamma_\nu) = 0, \tag{4.5}$$

and

$$\hat{R}^{nn}(\vec{k}, \omega) = \sum_\nu \frac{iZ_\nu}{\omega - \Omega_\nu + i\Gamma_\nu}, \tag{4.6}$$

where

$$-(\Omega_\nu - i\Gamma_\nu)^2 - 2i\nu_i(k)(\Omega_\nu - i\Gamma_\nu) + k^2 - \Sigma^{nn}(\vec{k}, \Omega_\nu - i\Gamma_\nu) = 0. \tag{4.7}$$

Note that some of these zeros (or poles) may lie on the analytic continuation of Σ onto the adjoining Riemann sheet as we discuss below. Since $R^{nn}(-\vec{k}, -\omega) = R^{nn}(\vec{k}, \omega)^*$ we must also have a pole at $\omega = -\Omega(-k) - i\Gamma(-k)$, with residue $Z^*(-k)$. For a system symmetric under $\vec{k} \to -\vec{k}$ we can then write for a single pair of poles in (4.6)

$$\hat{R}^{nn}(\vec{k}, \omega) = \frac{2iZ_\nu^{(1)}(\omega + i\Gamma_\nu) - 2Z_\nu^{(2)}\Omega_\nu}{(\omega + i\Gamma_\nu)^2 - \Omega_\nu^2}, \tag{4.8}$$

where $Z_\nu^{(1)}$ and $Z_\nu^{(2)}$ are the real and imaginary parts of Z_ν. Note if $\Gamma_\nu/\Omega_\nu \ll 1$ we can write the denominator approximately as $\omega^2 + 2i\omega\Gamma_\nu - \Omega_\nu^2$ which has the form of a renormalized sound operator. If $\Omega_\nu = 0$, as is sometimes the case for the modulational pole discussed in Sec. V,

$$\hat{R}^{nn}(\vec{k}, \omega) = \frac{2iZ_\nu^{(1)}}{\omega + i\Gamma_\nu}, \tag{4.9}$$

which has a purely imaginary residue at the pole $\omega = -i\Gamma_\nu$.

When the response function can be represented by a sum of simple poles

$$R(\vec{k}, \omega) = \sum_\nu \frac{iZ_\nu(\vec{k})}{\omega - \omega_\nu(\vec{k}) + i\gamma_\nu(\vec{k})}, \tag{4.10}$$

the covariance may also have a simple form provided the noise source $S(\vec{k}, \omega)$ (linear plus nonlinear) varies sufficiently slowly with ω. In (3.3) or (3.7) we have expressions of the form for real ω:

$$C(\vec{k}, \omega) = |R(\vec{k}, \omega)|^2 S(\vec{k}, \omega). \tag{4.11}$$

On using the pole ansatz for $R(\vec{k}, \omega)$ and assuming the resonant frequencies are well separated compared to their width we find

$$C(\vec{k}, \omega) = \sum_\nu \frac{|Z_\nu|^2 S(\vec{k}, \omega)}{(\omega - \omega_\nu)^2 + \gamma_\nu^2},$$

which we can approximate as

$$C(\vec{k}, \omega) \cong \sum_\nu \frac{C_\nu(\vec{k}) 2\gamma_\nu(\vec{k})}{[(\omega - \omega_\nu(\vec{k})]^2 + \gamma_\nu^2(\vec{k})},$$
$$C_\nu(\vec{k}) = \frac{|Z_\nu(\vec{k})|^2 S[\vec{k}, \omega_\nu(\vec{k})]}{2\gamma_\nu(\vec{k})}, \tag{4.12}$$

provided

$$|\omega-\omega_\nu|\,|\partial_\omega S(\vec{k},\omega)|_{\omega=\omega_\nu} \ll |S(\vec{k},\omega_\nu)|\,. \qquad (4.13)$$

Since the Lorentzian factor multiplying $C_\nu(\vec{k})$ is large only for $|\omega-\omega_\nu| \lesssim \gamma_\nu$ we can also write this condition as

$$\gamma_\nu \frac{|\partial_\omega S|_{\omega=\omega_\nu}}{|S|} \ll 1\,. \qquad (4.13')$$

This result implies that in the time domain

$$C(t) \propto \sum_\nu C_\nu \exp(-i\omega_\nu t - \gamma_\nu |t|)\,. \qquad (4.14)$$

On the other hand, one can show directly from (2.23)–(2.26) that $\partial_t C(t)=0$ at $t=0$ if the linear noise is a smooth function of time. This result is reconciled with (4.14) by noting that (4.13) cannot be satisfied for high frequencies which determine the short-time behavior. In the frequency-space region near $\omega=\omega_\nu$, however, (4.12) will be an accurate expression if (4.13) is satisfied.

Note that we can identify $C_\nu(\vec{k})$ with the equal-time covariance since

$$\int \frac{d\omega}{2\pi} C(\vec{k},\omega) = C_{\vec{k}t,t} \cong \sum_\nu C_\nu(\vec{k})\,. \qquad (4.15)$$

In view of (2.15) and (2.16) this then leads to the formulas

$$C^{\psi\psi^\dagger}(\vec{k},\omega) = \sum_\nu \frac{W_\nu(\vec{k})}{k^2} \frac{2\gamma_\nu}{(\omega-\omega_\nu)^2+(\gamma_\nu)^2}\,, \qquad (4.16)$$

where $W(\vec{k}) = \sum_\nu W_\nu(\vec{k})$ and

$$C^{nn}(\vec{k},\omega) = \sum_\nu \left(\frac{\Gamma_\nu}{(\omega-\Omega_\nu)^2+(\Gamma_\nu)^2} + \frac{\Gamma_\nu}{(\omega+\Omega_\nu)^2+(\Gamma_\nu)^2} \right) N_\nu(\vec{k})\,, \qquad (4.17)$$

where $N(\vec{k}) = \sum_\nu N_\nu(\vec{k})$ which are the generalizations of (3.17) and (3.18). The quantity $C^{\psi^\dagger\psi}(\vec{k},\omega)$ is found from $C^{\psi\psi^\dagger}(\vec{k},\omega)$ using (2.18) which can be written as

$$C^{\psi^\dagger\psi}(\vec{k},\omega) = C^{\psi\psi^\dagger}(-\vec{k},-\omega)^*\,. \qquad (4.18)$$

From the initial conditions on the diagonal responses

$$R^{\psi\psi}_{\vec{k},t_1,t_1} = R^{\psi^\dagger\psi^\dagger}_{\vec{k},t_1,t_1} = 1\,, \qquad (4.19)$$

the lowest order sum rules follow:

$$\int \frac{d\omega}{2\pi} R^{\psi\psi}(\vec{k},\omega) = 1 \qquad (4.20)$$

and

$$\int \frac{d\omega}{2\pi} R^{uu}(\vec{k},\omega) = 1\,. \qquad (4.21)$$

From (2.28) and (2.41) we have $R^{uu}(\vec{k},\omega) = i\omega k^{-2} R^{nu}(\vec{k},\omega) = -i\omega \hat{R}^{nn}(\vec{k},\omega)$, and we can write the last sum rule as

$$\int \frac{d\omega}{2\pi}(-i\omega)\hat{R}^{nn}(\vec{k},\omega) = 1\,. \qquad (4.22)$$

These sum rules are a short-time property of the response functions but may provide a useful constraint on the pole ansatz; substitution of (4.4) and (4.6) into (4.20) and (4.22), respectively, yields

$$\sum_\nu z_\nu = 1 \qquad (4.23)$$

and

$$-i\sum_\nu (\Omega_\nu - i\Gamma_\nu) Z_\nu = 1\,. \qquad (4.24)$$

The approximation of Eqs. (4.12) ignores possible poles or cuts arising directly from the nonlinear noise function $S(\vec{k},\omega)$. These additional singularities may be of importance in the full solution of the DIA equations. However, for the detailed considerations of this paper in which we assume the covariances are specified, we do not have to face this problem directly. Therefore, we postpone a detailed discussion of this question for future work.[22]

V. TURBULENCE-INDUCED RESPONSE: MODULATIONAL INSTABILITY

A sufficiently high level of Langmuir fluctuations is known to lead to modulational instability. In the description we are developing in this paper, modulational instability implies that there are new poles in the response functions which are dependent on the fluctuation level and which are not present for zero fluctuations. These responses therefore cannot be described by weak-turbulence theory.

In this paper we will concentrate on the stability investigations in which we assume the covariances $C^{\psi\psi^\dagger}$ and C^{nn} are given, say, in the form of Eqs. (4.16) or (4.17), and investigate the resulting properties of the response functions. In particular, we are interested in the stability of a Langmuir intensity spectrum peaked near $k=0$, to represent the Langmuir condensate which might arise from cascade of energy to low k. In Sec. VIII we will present a Monte Carlo simulation of the Zakharov equations driven by a Gaussian white-noise source at (or near) $k=0$. In this simulation the stability of the initial, linear Langmuir intensity spectrum will be investigated and compared with the DIA stability analysis.

We find that there may be modes of the system which are unstable for a particular choice of intensities. These modes would grow exponentially in time if they had nonzero initial amplitudes or if linear noise sources were present for these modes. The stability calculations would represent the early time behavior of the response functions for a system with sufficiently weak excitation of the unstable modes. In the limiting case of zero initial amplitudes and noise sources for the unstable modes, the stability calculations would produce a good representation of the DIA response functions. Clearly, this can only happen if the unstable modes are distinct from the modes comprising the assumed initial intensity spectrum (see further discussion in Sec. VII).

In the full self-consistent DIA the growth of these modes to sufficiently high intensities will alter the fluctuation spectrum. The spectrum will evolve as the intensity of the unstable modes increases at the expense of the modes producing the initial instabilities. If the full DIA evolves to a steady state it can be described by the equations in Sec. III. In steady state no unstable modes can occur. The originally unstable modes are stabilized by a redistribution of the fluctuations in k space. This change in spectral shape can affect the growth rate in a quasilinear manner, which will be discussed in a preliminary way in Sec. IX, or in the more nonlinear sense of additional turbulent damping of the modes. The detailed discussion of the full self-consistent DIA will be postponed for a later paper. The understanding of the stability problem is a central part of this problem because it allows us to identify the important modes or poles of the response function such as the modulational pole and the effect of the fluctuation distribution on these poles such as the turbulent damping referred to above.

The ω_2 integrations in Eq. (3.6) for $\hat{R}^{nn}(\vec{k},\omega)$ and in Eq. (3.2) for $R^{\phi\phi}(\vec{k},\omega)$ can be carried out explicitly if we assume that the covariances are given by the expressions (4.16) and (4.17). In the expressions for the self-energies $\Sigma^{nn}(\vec{k}_1,\omega_1)$ or $\Sigma^{\phi\phi}(\vec{k}_1,\omega_1)$ the causal response functions $R^{\phi\phi}(\vec{k}_2,\omega_2)$ or $R^{nn}(\vec{k}_2,\omega_2)$ have their poles (or other singularities) below[18] the contour of integration on ω_2. (This can be taken to be the real ω_2 axis for a stable plasma.) The ω_2 integration can then be performed by Cauchy's theorem by deforming the integration contour into the upper half ω_2 plane; this picks up only the poles of $C^{\phi\phi\dagger}$, $C^{\phi\dagger\phi}$, or C^{nn} as determined from the expressions (4.16) or (4.17). The resulting equations are

$$\Big(-\omega_1^2-2i\omega_1\nu_1+k_1^2-ik_1^2\int d\vec{k}_2\sum_\nu(\hat{k}_2\cdot\hat{k}_3)^2[W^\nu(\vec{k}_2-\vec{k}_1)R^{\phi\phi}(\vec{k}_2,\omega_1+\omega^\nu_{\vec{k}_2-\vec{k}_1}+i\gamma^\nu_{\vec{k}_1-\vec{k}_2}) - W^\nu(\vec{k}_1-\vec{k}_2)R^{\phi\dagger\phi\dagger}(\vec{k}_2,\omega_1-\omega^\nu_{\vec{k}_1-\vec{k}_2}+i\gamma^\nu_{\vec{k}_1-\vec{k}_2})]\Big)\hat{R}^{nn}(\vec{k}_1,\omega_1)=1 \qquad (5.1)$$

and

$$\Big(-i(\omega_1-k_1^2)+\nu_e-i\int d\vec{k}_2\sum_\nu W^\nu(\vec{k}_2)(\hat{k}_1\cdot\hat{k}_2)\,(\vec{k}_1-\vec{k}_2)^2\hat{R}^{nn}(\vec{k}_1-\vec{k}_2,\omega_1-\omega^\nu_{\vec{k}_2}+i\gamma^\nu_{\vec{k}_2}) +\tfrac{1}{2}\int d\vec{k}_2\sum_\nu N^\nu(\vec{k}_1-\vec{k}_2)(\hat{k}_1\cdot\hat{k}_2)^2[R^{\phi\phi}(\vec{k}_2,\omega_1+\Omega^\nu_{\vec{k}_1-\vec{k}_2}+i\Gamma^\nu_{\vec{k}_1-\vec{k}_2})+R^{\phi\phi}(\vec{k}_2,\omega_1-\Omega^\nu_{\vec{k}_1-\vec{k}_2}+i\Gamma^\nu_{\vec{k}_1-\vec{k}_2})]\Big)R^{\phi\phi}(\vec{k}_1,\omega_1)=1. \qquad (5.2)$$

The general DIA stability problem is expressed by the simultaneous solution of (5.1) and (5.2) where the parameters specifying the covariances (i.e., all quantities with superscript ν) are given.

The arguments outlined in Sec. I concerning the formation of the Langmuir condensate near $k=0$ lead us to consider the stability of a Langmuir spectrum concentrated around $k=0$. We assume at first that $N^\nu(\vec{k})=0$ implying that no density fluctuations are excited as would be the case in the early stages of the modulational instability. We will also assume that there is only one pole excited in the Langmuir covariance with intensity $W^\nu(\vec{k})=W(\vec{k})$ and the following expressions for the frequency and damping: $\omega^\nu(k)=k^2$, $\gamma^\nu(k)=\nu_e(k)$. Thus, we are examining the stability of a spectrum of free Langmuir waves depending on the choice of $\nu_e(k)$. The generality of this case will be discussed later.

To make contact with known results we examine the first iteration approximation denoted by $R^{nn}_{(1)}$ in which the zero-order values for the Langmuir response functions

$$R^{\phi\phi}_{(0)}(\vec{k},\omega)=[-i(\omega-k^2)+\nu_e]^{-1} \quad\text{and}\quad R^{\phi\dagger\phi\dagger}_{(0)}(\vec{k},\omega)=[-i(\omega+k^2)+\nu_e]^{-1}$$

are used in the expressions for the self-energies.

In this limit we have (on changing variables of integration)

$$[R^{nn}_{(1)}(\vec{k},\omega)]^{-1}=\Big(-\omega^2-2i\omega\nu_i+k^2+2k^2\int d\vec{q}\,\frac{[(\vec{k}-\vec{q})\cdot\vec{q}]^2}{(\vec{k}-\vec{q})^2q^2}\,\frac{W(\vec{q})(k^2-2\vec{k}\cdot\vec{q})}{[\omega+i\nu_e(q)+i\nu_e(\vec{k}-\vec{q})]^2-(k^2-2\vec{k}\cdot\vec{q})^2}\Big). \qquad (5.3)$$

The dispersion relation for the poles of $R^{nn}_{(1)}$ or $[R^{nn}_{(1)}]^{-1}=0$ is then the broad-band dispersion relation derived on heuristic grounds by Bardwell and Goldman[23] and Tsytovich[24] from the standpoint of linear stability analysis.

In the DIA, Eq. (5.3) is just the first iterative approximation to the response function. In the complete DIA the response functions R^{nn} and $R^{\phi\phi}$ are coupled as in Eqs. (5.1) and (5.2); the response problem for a prescribed covariance is a nonlinear one. We will discuss an iterative solution for the complete DIA response functions in Sec. VI. We may regard this problem as the stability analysis of a plasma subjected to a *fluctuating* pump whose spectrum is given by $W(\vec{k})$. The problem of parametric instabilities driven by fluctuating pumps has been studied by several authors.[23-26] The approximation used in these works is equivalent to the first iteration of the DIA and is called the Bourret approximation in some contexts.[26] This approximation may be valid for *parametric decay* instabilities under certain conditions but we will find it to be inaccurate for the modulational instability driven by a fluctuating pump.

The simplest case is that of the narrow-band pump $[W(\vec{q})\cong(2\pi)^d\delta^{(d)}(\vec{q})W]$ centered at $q=0$. This case avoids problems connected with the branch discontinuities which arise from the integral in (5.3) in the more general case.

In this limit the dispersion relation becomes

$$[(\omega+i\nu_+)^2-k^4](\omega^2+2i\omega\nu_i-k^2)-2Wk^4=0\,, \quad (5.4)$$

where $\nu_+=\nu_e(0)+\nu_e(k)$. (We have written this for the one-dimensional case.) The dispersion relation given above is well known[1] for the stability of a plasma driven by a spatially homogeneous, *coherent* pump of amplitude $E_0\cong\sqrt{W}$. Here, however, we are considering a fluctuating pump with $\langle E_0\rangle=0$ and coherence time $\nu_e(0)^{-1}$, and (5.4) is only a first approximation to the dispersion relation in this case. In the case of zero damping the solutions of this equation are trivial. It is useful to view this as a standard coupled mode problem which shows mode repulsion near $k=1$ where the unperturbed dispersion curves cross (i.e., $k=k^2$). The complete solution as a function of k is shown in Figs. 1 and 2 for a value of $W=1.0$. The two positive-frequency branches of this dispersion relation are distorted from the unperturbed dispersion curves. The most dramatic effect is that for $k<\sqrt{2W}$ the damping decrement $\Gamma^M(k)$ of one of these branches can become negative indicating instability; this is the familiar modulational instability (MI) of a narrow Langmuir spectrum centered at $q=0$. (This

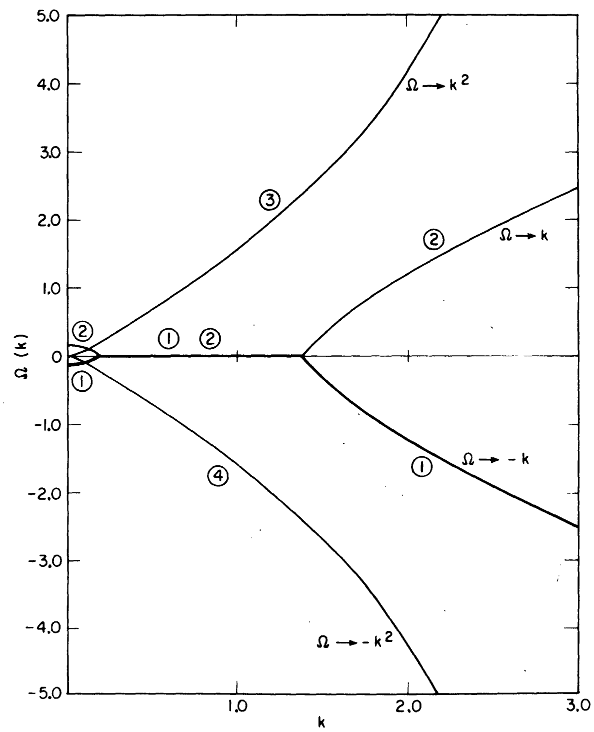

FIG. 1. Real parts of the four poles which are present in $\hat{R}^{nn}(k,\omega)$ after one iteration of the DIA, with ν_e (pump)$=\frac{1}{8}$, $\nu_e(k)=\frac{1}{4}$, $\nu_i(k)=k/4$, and $W=1.0$.

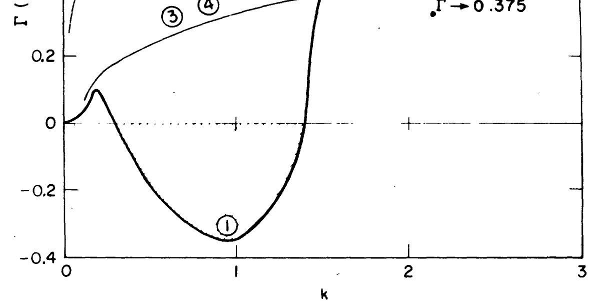

FIG. 2. Imaginary parts of the four poles which are present in $\hat{R}^{nn}(k,\omega)$ after one iteration of the DIA, with the same parameter values as Fig. 1.

instability is often called the oscillating two-stream instability but we will call all such instabilities modulational instabilities.) The narrow-band approximation for a spectrum of half-width Δk centered at $q=0$ is valid as long as $k \gg \Delta k$. More detailed consideration of the finite bandwidth case is given in Sec. IX.

The various roots or branches of $\hat{R}^{nn}_{(1)}(k,\omega)^{-1}=0$ asymptote to the unperturbed frequencies: Branches marked 3 and 4 asymptote to $\pm k^2$ as $\Omega^{(3,4)}(k) \to \pm(k^4+2W)^{1/2}$, with $\Gamma^{(3,4)}(k)=\nu_+$, for $k \gg \sqrt{W}$. The residues at these poles approach zero for $k \gg \sqrt{W}$ as $Z^{(3,4)}(k) \cong W/k^8$. A listing of the residues of the various poles of $\hat{R}^{nn}_{(1)}$ is found in Table I. The branches labeled 1 and 2 have $\Omega^{(1)}(k) = -\Omega^{(2)}(k)$. For the range $0.2 < k < 1.4$ for the parameters chosen [$W=1$, $\nu_e(0)=0.125$, $\nu_e(k) = 0.250$, $\nu_i(k)=0.25k$], $\Omega^{(1)}(k)=\Omega^{(2)}(k)=0$. Branch 1 is the modulationally unstable branch whose growth rate can be calculated in the adiabatic approximation, $\Gamma^{(1)}(k) \ll k$, as

$$\Gamma^{(1)}(k)=\nu_+ - k(2W-k^2)^{1/2} . \tag{5.5}$$

The marginally stable mode $\Gamma^{(1)}(k)=0$ always satisfies the adiabatic condition; the value of $W = W_c(k)$ at which a particular mode k becomes marginally stable is

$$W_c(k)=\tfrac{1}{2}(k^2+\nu_+^2/k^2) . \tag{5.6a}$$

Therefore the value of $k=k_m$ at which $W_c(k)$ is minimized is $k_m^2=\nu_+$, and the minimum threshold condition is

$$W=\nu_+=\nu_e(0)+\nu_e(k) \quad \text{at} \quad k_m=\sqrt{W} \tag{5.6b}$$

at which $\Omega^{(1)}(k)=\Gamma^{(1)}(k)=0$.

The modes 1 and 2 are asymptotic to the value $\pm k$ for $k \gg \sqrt{W}$ and the residues approach $\pm i(2k)^{-1}$ in this limit. Therefore these modes go continuously into the unperturbed modes for $k \gg \sqrt{W}$. This is the weak-turbulence limit where dispersion dominates nonlinearity. The theory presented here makes a smooth transition between the low-k regime of strong nonlinearity to the large k regime where weak-turbulence theory is valid.

Next we investigate the Langmuir response $R^{\phi\phi}$ in the first iterative approximation: In (5.2) we set $N^\nu(\vec{k})=0$ (see discussion above), again use $W^\nu(\vec{k})=W(\vec{k})$, $\omega^\nu(k)=k^2$, and $\gamma^\nu(k)=\nu_e(k)$, and replace $\hat{R}^{nn}$ by its unperturbed value $(-\omega^2-2i\omega\nu_i+k^2)^{-1}$.

The resulting equation for $R^{\phi\phi}(\vec{k},\omega)$ is

$$\left(-i(\omega-k^2)+\nu_e(k)+i\int d\vec{q}\,\frac{W(\vec{q})(\hat{k}\cdot\hat{q})^2(\vec{k}-\vec{q})^2}{[\omega-q^2+i\nu_e(q)]^2+2i\nu_i(\vec{k}-\vec{q})[\omega-q^2+i\nu_e(q)]-(\vec{k}-\vec{q})^2}\right)R^{\phi\phi}_{(1)}(\vec{k},\omega)=0 . \tag{5.7}$$

We again first investigate the narrow-band pumping spectrum limit $W(q)=(2\pi)^d\delta^{(d)}(q)W$; the resulting dispersion relation for the poles of $R^{\phi\phi}_{(1)}(k\omega)$ is (for $d=1$)

$$iR^{\phi\phi}_{(1)}(k,\omega)^{-1}=\omega-k^2+i\nu_e - \frac{k^2W}{(\omega+i\nu_e)^2+2i(\omega+i\nu_e)\nu_i-k^2}=0 . \tag{5.8}$$

The three roots of this cubic dispersion relation are shown for a typical case in Figs. 3 and 4; these parameters are the same as for the ion response $\hat{R}^{nn}_{(1)}(k,\omega)$ of Figs. 1 and 2.

We note that the unstable branch for $R^{\phi\phi}_{(1)}$ has a growth rate $-\gamma_{(1)}(k)$ which is smaller than the growth rate $-\Gamma_{(1)}(k)$ shown in Fig. 2 for $\hat{R}^{nn}$; the unstable regions of k are also different for $R^{\phi\phi}$ and $\hat{R}^{nn}$. In the next sections we will show by two different numerical methods that the growth rates of $R^{\phi\phi}$ and $\hat{R}^{nn}$ and the unstable region of k are the same in the complete DIA.

The frequencies of the various branches or poles again are asymptotic to k^2 or k. Root 3 is asymptotic to $\omega^{(3)}=k^2$, $\gamma^{(3)}(k)=\nu_e(k)$ for $k \gg \sqrt{W}$ and its residue $z^{(3)}(k)$ is asymptotic to unity; see Table II. Therefore branch 3 continuously connects to the free Langmuir mode of weak-turbulence theory at high k. Branches 1 and 2 are asymptotic to $\omega^{(1,2)}=\mp k$ at high k with residues tending to zero. Branch 1 is the modulationally unstable one and has negative frequency in the unstable range $0.2 < k < 0.8$.

The cubic dispersion relation (5.8) can be reduced to a quadratic if we ask for the marginally stable mode $\gamma^{(1)}(k)=0$. We then find (for $\nu_e \gg \nu_i$ and $\nu_e=\text{const}$) that for this mode

$$\omega^{(1)}(k)=-\tfrac{1}{3}(k^2+\nu_e^2)^{1/2} . \tag{5.9}$$

The value of $W=W_c(k)$ which makes the mode k marginally stable is

$$W_c(k)=\frac{2}{9}\left(1+\frac{4\nu_e^2}{k^2}\right)(k^2+\nu_e^2)^{1/2} . \tag{5.10}$$

The value of k corresponding to the minimum threshold is

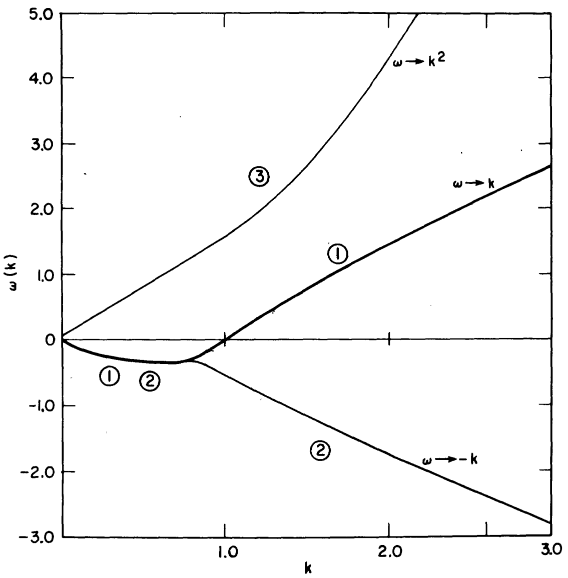

FIG. 3. Real parts of the poles which are present in $R^{\phi\phi}(k,\omega)$ after one iteration of the DIA, with ν_e(pump) $=\frac{1}{8}$, $\nu_e(k)=\frac{1}{4}$, $\nu_i(k)=k/4$, and $W=1.0$.

$$k_m^2=2(1+\sqrt{3})\nu_e^2=(1.64)^2W_c^2(k_m)\,, \tag{5.11}$$

and the minimum threshold is

$$W_c(k_m)=1.423\nu_e\,. \tag{5.12}$$

For the case $\nu_e(k)=\nu_e(0)$ we see that this threshold is 30% lower that the modulational threshold calculated from $\hat{R}^{nn}_{(1)}$ and the most unstable mode k_m scales differently in this case. A renormalized dispersion relation for Langmuir waves has been considered by Tsytovich (for $\nu_i=0$). We can

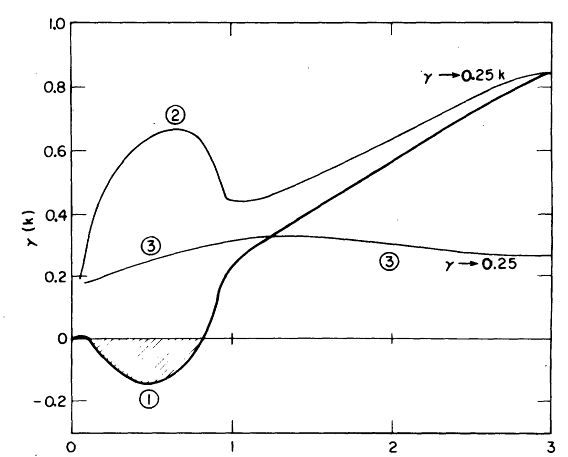

FIG. 4. Imaginary parts of the three poles which are present in $R^{\phi\phi}(k,\omega)$ after an iteration of the DIA with the same parameters as Fig. 3.

TABLE I. Residues for $\hat{R}^{nn}$ after one iteration (poles are labeled 1, 2, 3, 4 as in Figs. 1 and 2).

k	Z_1	Z_2	$Z_3=Z_4^*$
0.2	5.3	−0.22	$-0.23+i(0.18)$
0.6	0.50	−0.29	$-0.11+i(0.34)$
1.0	0.41	−0.36	$-0.022+i(0.16)$
1.4	1.0	−1.0	$-0.0018+i(0.069)$
1.8	$-i(0.44)$	$i(0.44)$	$0.0019+i(0.026)$
2.2	$-i(0.30)$	$i(0.30)$	$0.0013+i(0.0092)$
2.6	$-i(0.23)$	$i(0.23)$	$0.00060+i(0.0036)$
3.0	$-i(0.19)$	$i(0.19)$	$0.00022+i(0.0014)$

make contact with his results by assuming

$$|[\omega+i\nu_e(0)]^2|\,,\quad |[\omega+i\nu_e(0)]\nu_i|\ll k^2\,.$$

Under these conditions (5.8) has the solution

$$\omega=k^2-W-i\nu_e(0) \tag{5.13}$$

corresponding to a shifted Langmuir mode. This limiting case is valid if $k^2-W\ll k$ and $\nu_i k^2-W\ll k^2$; these conditions fail for $2k>(\sqrt{1+4W}-1)$ and $2k>(\sqrt{1+4W}+1)$. It is easily seen from (5.8) that any root with frequency ω_k must have $\omega_k\to 0$ as $k\to 0$ since $\lim_{k\to 0}[\nu_e(0)/k]\to\infty$; this behavior is seen in Fig. 3. The approximation leading to (5.13) also misses the unstable root of $R^{\phi\phi}_{(1)}(k,\omega)^{-1}=0$, which is the modulational response of the Langmuir field.

VI. CONTINUED-FRACTION REPRESENTATION FOR A NARROW-BAND PUMPING SPECTRUM

The considerations of Sec. V make it imperative that we develop a convergent scheme for the pole representation of the response functions which goes beyond the first iteration. We need to understand in a more complete way how the DIA can be used to generate approximations for the response functions which are sums of simple poles in ω (or exponentials in time). It is particularly important to know if the modulational pole which arises in

TABLE II. Residues for $R^{\phi\phi}$ after one iteration (poles are labeled 1, 2, 3 as in Figs. 3 and 4).

k	z_1	z_2	z_3
0.2	$0.31-i(0.069)$	$0.40+i(0.12)$	$0.29-i(0.051)$
0.6	$0.34-i(0.30)$	$0.32+i(0.29)$	$0.34+i(0.010)$
1.0	$0.99-i(0.34)$	$-0.45+i(0.29)$	$0.46+i(0.047)$
1.4	$0.46-i(0.086)$	$-0.10+i(0.021)$	$0.64+i(0.066)$
1.8	$0.24-i(0.056)$	$-0.048+i(0.0077)$	$0.81+i(0.049)$
2.2	$0.12-i(0.024)$	$-0.027+i(0.0038)$	$0.91+i(0.026)$
2.6	$0.067-i(0.014)$	$-0.017+i(0.0021)$	$0.95+i(0.012)$
3.0	$0.034-i(0.007)$	$-0.012+i(0.0013)$	$0.97+i(0.007)$

the first iteration maintains its identity in higher iterations and if the iteration scheme converges or if any new unstable or weakly damped poles arise in higher approximations.

We shall first consider the one-dimensional model of a zero-bandwidth fluctuating pump spectrum which has the advantage of avoiding complications associated with branch discontinuities of $\Sigma(\vec{k},\omega)$. This model consists of a dynamically fluctuating pump at $k=0$ and only infinitesimal fluctuations at other wave numbers k. The more general case of narrow but finite-bandwidth pump will be discussed in Secs. VII–IX. In contrast to the zero-bandwidth case the question of finite density fluctuations must be addressed in the case of finite bandwidth. The first iterative approximation for the response functions in this model was discussed in Sec. V. From Eqs. (5.1) and (5.2) we can write the complete DIA response equations in this model as

$$\{-\omega^2-2i\omega\nu_i(k)+k^2-ik^2W\,[R^{\phi}(\vec{k},\omega+i\nu_e(0))-R^{\phi^\dagger}(\vec{k},\omega+i\nu_e(0))]\}R^n(\vec{k},\omega)=1\,, \tag{6.1}$$

$$[-i(\omega-k^2)+\nu_e(k)-iW\,k^2R^n(\vec{k},\omega+i\nu_e(0))]R^{\phi}(\vec{k},\omega)=1\,, \tag{6.2}$$

where now we use the notation $R^{\phi}(\vec{k},\omega)$ for $R^{\phi\phi}(\vec{k},\omega)$, $R^{\phi^\dagger}$ for $R^{\phi^\dagger\phi^\dagger}$, and R^n for $\hat{R}^{nn}$.

Consider a schematic representation for the above equations:

$$R(\omega)=\frac{1}{R_0^{-1}(\omega)-\Sigma(R,\omega)}\,. \tag{6.3}$$

Define a sequence of continued-fraction approximations to R by

$$R_{l+1}(\omega)=\frac{1}{R_0^{-1}(\omega)-\Sigma(R_l,\omega)}\,; \tag{6.4}$$

$R_{l=0}(\omega)=R_0(\omega)=$ response for the linear normal modes. For example,

$$[R_0^n(k\omega)]^{-1}=-\omega^2-2i\nu_i\omega+k^2\,,$$
$$[R_0^{\phi}(k\omega)]^{-1}=-i(\omega-k^2)+\nu_e\,. \tag{6.5}$$

From the form of Σ as given by the DIA, it is clear that each $R_l(\omega)$ is a meromorphic function of ω, with the number of poles increasing with l. *For a given value of l,*

$$R_l^{\phi}(\omega)=i\sum_{j=1}^{N_l^{\phi}}\frac{z_l(j)}{\omega-s_l(j)}\,, \tag{6.6}$$

$$R_l^{n}(\omega)=i\sum_{j=1}^{N_l^{n}}\frac{Z_l(j)}{\omega-S_l(j)}\,, \tag{6.7}$$

where $z_l(j)$ is the residue at the complex pole

$$s_l(j)=\omega_l(j)-i\gamma_l(j)\,,$$

and $Z_l(j)$ is the residue at the complex pole

$$S_l(j)=\Omega_l(j)-i\Gamma_l(j)\,.$$

Let $\vec{s}_l$ represent the collection of poles of R_l^{ϕ} and $\vec{S}_l$ represent the collection of poles of R_l^n, and let $\vec{z}_l$ and $\vec{Z}_l$ represent the corresponding residues. These poles and residues implicitly depend upon k.

The sequence of continued-fraction approximations to R^{ϕ} and R^n may be viewed as recursion relations for the parameters N, $\vec{s}$, $\vec{S}$, $\vec{z}$, and $\vec{Z}$. While the recursion for N is trivial,

$$N_{l+1}^{\phi}=N_l^n+1\,,\quad N_{l+1}^n=2N_l^{\phi}+2\,, \tag{6.8}$$

those for the poles and residues must be determined numerically. In Figs. 5 and 6, the first

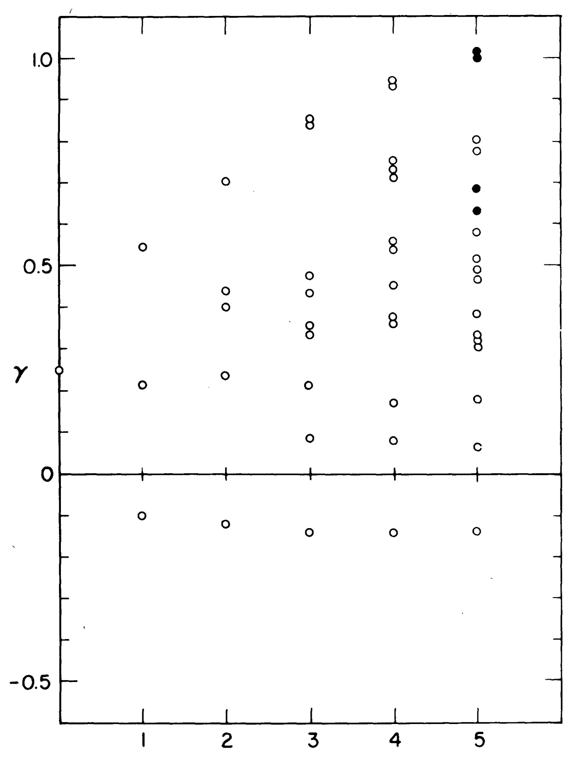

FIG. 5. Decay rate of the poles which are present in $R^{\phi\phi}(k=0.3)$ after 0, 1, 2, 3, 4, and 5 iterations of the DIA. The solid circles represent a pair of poles with oppositely signed frequencies. Pump and response mode parameters as in Fig. 3. Note the rapid convergence of the unstable pole.

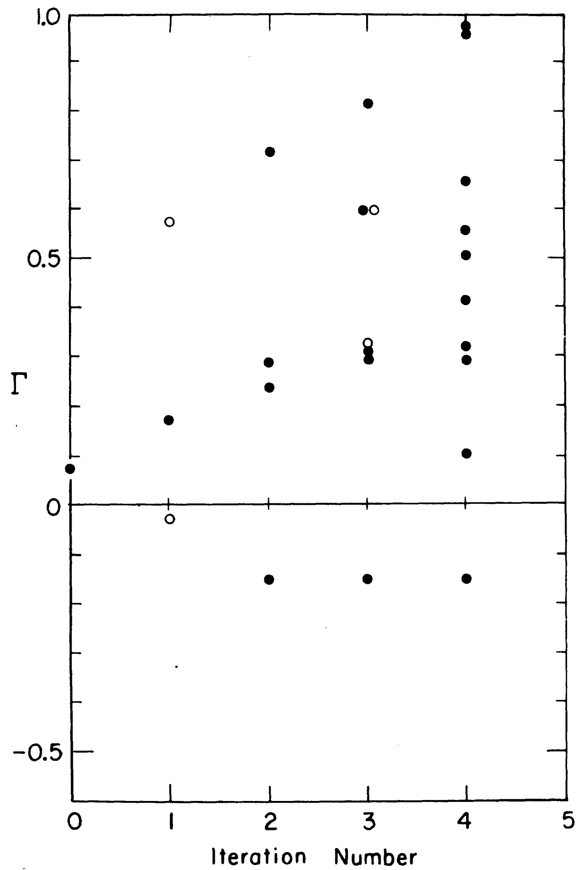

FIG. 6. Decay rate of the poles which are present in $R^{nn}(k=0.3)$ after 0, 1, 2, 3, and 4 iterations of the DIA. Symbols and parameters as in Fig. 5.

five iterations for γ and four iterations for Γ, respectively, are displayed for the case of an unstable wave number. These data are shown for k fixed at $k=0.3$ for the same values of the free parameters as used in Figs. 1–4. It is apparent that the unstable pole is insensitive to higher iterations, and that the two response functions share a common growth rate. When this rate is compared to the numerical integration of the DIA for large times (i.e., see Fig. 9), excellent agreement is obtained; the direct numerical integration procedure is described in Sec. VII. The unstable poles generated in the first and second iterations for $R^{\psi}(k,\omega)$ are shown in Figs. 7 and 8 again for the same set of parameters. These figures show a large change in the unstable mode dispersion in going from $l=1$ to 2. The negative frequency associated with the unstable pole of $R_l^{\psi}(k,\omega)$, shown explicitly for $l=1$ in Fig. 3 and $l=2$ in Fig. 7, persists as the iteration sequence is carried to convergence. In addition, several stable poles with negative frequencies are found. [Note that

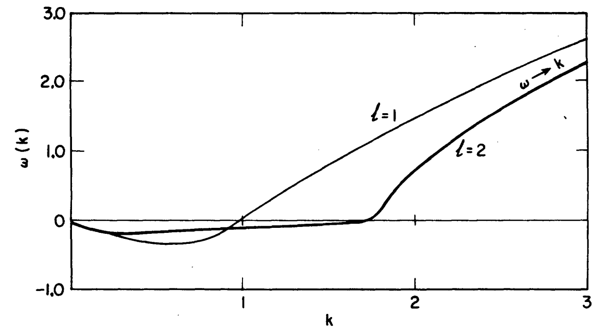

FIG. 7. Comparison of the real parts of the modulationally unstable poles of $R^{\psi\psi}(k,\omega)$ in the first and second iterations. The parameter values are the same as in Fig. 3.

the group velocities of the modes in the negative frequency region are very small, consistent with the expected absolute nature of the modulational instability.] This iterative procedure is more efficient than numerical integration, and it also provides new insights.

In the limit of ν_e(pump) → 0, the ensemble-averaged growth rates for the density and electric-field linear response functions must coincide. This is because for a time-independent pump, a straightforward eigenvalue analysis of the equations of motion for n, u, ψ, and $\psi^\dagger$ yields the result that if an unstable eigenmode exists, it has

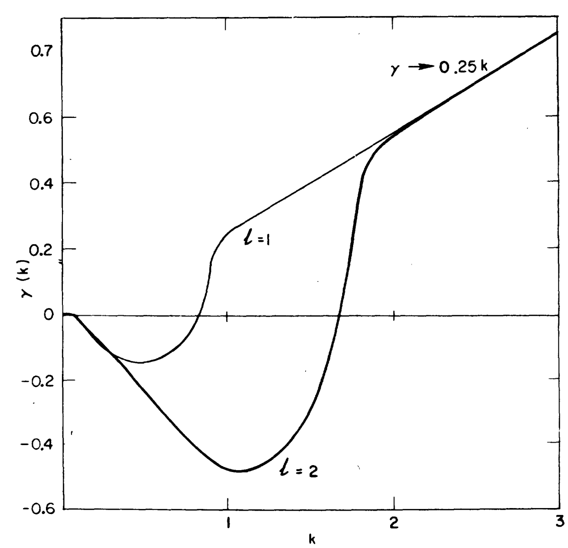

FIG. 8. Comparison of the imaginary parts of the (modulationally) unstable poles of $R^{\psi\psi}(k,\omega)$ in the first and second iterations. The parameter values are the same as in Fig. 3.

a nonzero projection on all of n, u, ψ, and $\psi^\dagger$. Therefore, the linear response functions all have the same growth rate for each realization of the pump ensemble, and thus the same ensemble-averaged growth rate. In the limit $\nu_e(\text{pump}) \to \infty$, it can be shown that the DIA is exact. Empirically, we have found that the DIA always predicts the same growth rates for the density and electric-field responses. If we can extrapolate back from infinite values of $\nu_e(\text{pump})$ to large values, but still small enough so that n and ψ are still unstable, the growth rates for R^n and R^ψ must be the same to the extent that the DIA is correct.

For another set of parameters, which yield stable responses at all k, the decay rates for three particular branches of R^n which are present after four iterations are plotted as a function of k in Fig. 9. Two of these branches are associated with sound and Langmuir waves at large k, while the third almost becomes unstable for an intermediate range of k. Also displayed is a graph consisting of the longest-lived (i.e., smallest Γ) mode at each k. Since this mode is sometimes on one of the above mentioned three branches and sometimes on another, the graph has a discontinuous slope. Essentially the same graph is obtained by integrating the DIA for long times (see Sec. VII) and noting the rate of exponential decay of the modulus of R^n. The uneven character of this graph is nicely explained by the continued-fraction representation.

From these two examples, the first displaying an unstable range of k, we conclude that even though the weak-turbulence approximation is valid for large k (large compared to $\sqrt{W}$), there are poles present at lower k which are complicated hybrids of sound, Langmuir, and modulational-like responses.

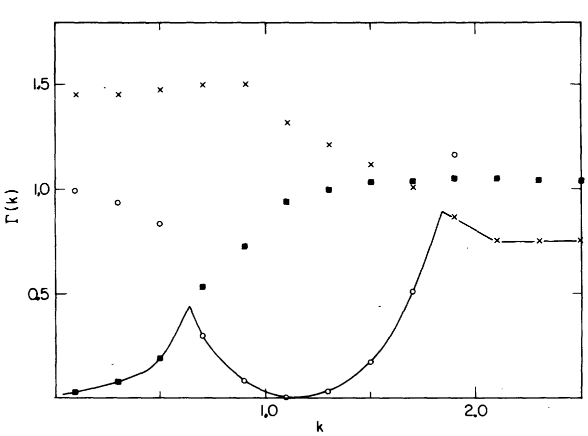

FIG. 9. Damping rates for three poles of $\hat{R}^{nn}$ after four iterations, with $\nu_e(\text{pump}) = \frac{3}{4}$, $\nu_e(k) = \frac{1}{4}$, $\nu_i(k) = k/4$, and $W = 1.0$. The curve follows the least damped pole as a function of k.

Convergence of the continued-fraction representation

One model example for which the sequence $R_1, R_2, \ldots$ converges to the R of the DIA is discussed in Ref. 27. Since we are unaware of any general results of this kind, let us focus on the simpler question: Does the continued-fraction sequence of response functions converge to anything? For many specific cases of the narrow-band pump model which we have considered, the answer seems to be affirmative. As the number of poles increases with iteration, the longest-lived (and the possibly unstable) poles which are present seem to converge after a few iterations. Another feature which is observed is that some poles seem to bifurcate into a cluster of nearby poles, perhaps a signal that a localized continuum or branch discontinuity is present in the limiting response function.

Since the numerical evaluation of the complex roots of a polynomial whose degree is larger than 40 (especially when the roots are not well separated) begins to take a noticeable amount of computer time, there is a practical limit to a relatively small number of iterations. However, even though there is an appearance of convergence after only a few iterations, it is important to understand how to perform a large number of iterations in an approximate fashion. The reasons are twofold.

(1) If a multimode pump were used instead of a single mode, then after one iteration the number of poles in R^ψ at a fixed wave number would be $1 + 2N$, where N is the number of pump modes, and this would be prohibitive if N were larger than 20, and increasingly more so after two and three iterations.

(2) For the analogous continued-fraction representation of the full DIA (i.e., determine both response *and* covariance functions), there is likely to be a slower convergence because of transients in the fluctuation spectra.

We shall now describe a method of consolidating the poles present at a given level of iteration for the previously discussed single-mode pump case. The sequence of (iteration, consolidation), ..., ensures that the number of poles ceases to change after a finite number of steps into this sequence, although the numerical value of the residues, frequencies, and decay rates may change. This sequence is a new iterative procedure, where each step is a composite of a primitive iteration followed by a consolidation.

Given a collection of poles $\vec{s}$ and residues $\vec{z}$ for

R^ψ, order them according to the following prescription.

(a) Place the unstable poles first, with the larger growth rates before the smaller growth rates.

(b) The stable poles come second, with an internal order determined by a descending sequence of values for $|z(j)|/\gamma(j)$, with the index j running through the stable poles.

If the number of poles (dimension of $\vec{s}$ and $\vec{z}$) N, is larger than a fixed cutoff n, take the last $(N-n+1)$ of them and consolidate these into a single pole whose complex value is their weighted mean, with $|z(j)|$ as the weighting factor, $n \leq j \leq N$, and whose residue is the complex valued sum of their residues.

It is not necessary to explicitly consolidate the poles in R^n, since their number at the next iteration is determined by the number of poles in R^ψ at the current iteration. For example, if it is decided that a maximum of $n=5$ poles of R^ψ will be carried over from one iteration to the next, after two primitive iterations, there are five poles in R^ψ and eight poles in R^n. The third primitive iteration produces 9 poles in R^ψ and 12 poles in R^n. The nine poles of R^ψ are consolidated into five. The fourth iteration again produces 12 poles in R^n but 13 poles in R^ψ. Consolidation of R^ψ back to five poles, and further iteration now produces a pattern of pole numbers which repeats indefinitely. This is illustrated below and the horizontal arrow denotes consolidation:

Iteration	0	1	2	3	4	5	
N^ψ	1	3	5	$9\to 5$	$13\to 5$	$13\to 5$	$\cdots$
N^n	2	4	8	12	12	12	

For the specific parameter case whose results are displayed in Fig. 9, the application of the above consolidated iteration procedure always converged, with the longest-lived pole within one percent or so of the pole inferred from the long-time behavior of the numerically integrated DIA. The convergence of this pole was very quick, showing less than a one percent change after four iterations. The remaining poles converged more slowly, but all poles remained essentially unchanged after ten consolidated iterations.

How sensitively does the consolidated iteration procedure depend upon the cutoff parameter n? For a wide variety of pump parameter choices, we have found that as n increases, the results obtained come into closer agreement with the long-time behavior of the DIA (see Fig. 10).

Another important feature is the retention of all of the poles in R^n which follow from the consolidation of R^ψ. It has been found that an explicit consolidation of both R^ψ and R^n leads to inferior results.

If instead of limiting the number of poles in R^ψ, and explicit limit and consolidation were put on R^n with no explicit consolidation of R^ψ, inferior results are again obtained. Since many of the poles in R^n come in pairs symmetric about the imaginary axis, the twelve poles which appear in R^n when $n=5$ is effectively a six pole description, and therefore the level of detail in the pole representation is roughly comparable between R^n and R^ψ. When R^n is explicitly consolidated, it has a much coarser description than R^ψ.

The success of consolidated iteration is not so satisfactory when there are unstable modes. Even though a small number of primitive iterations shows evidence of converging towards growth rates which are in close agreement with the DIA, the application of subsequent consolidated iterations leads to growth rates which can differ appreciably from the actual DIA value. Sometimes there is a lack of convergence with a growth rate which changes from one iteration to the next. However, as n is increased the level of change decreases, and on occasion convergence is regained. For the stable case convergence always appears to occur. This is the case of main interest for the steady-state solution of the DIA.

VII. INTEGRATION OF DIA FOR RESPONSE FUNCTIONS

The task of numerically integrating the DIA equations is greatly simplified if the fluctuation functions (covariances) are given. Since the linear response functions are retarded, the calculation of $\partial R(t_1-t_3)/\partial t_1$, for $t=t_1-t_3 \geq 0$, only requires a knowledge of $C(\tau)$ and $R(\tau)$, $0\leq \tau \leq t$. In the time homogeneous regime now being con-

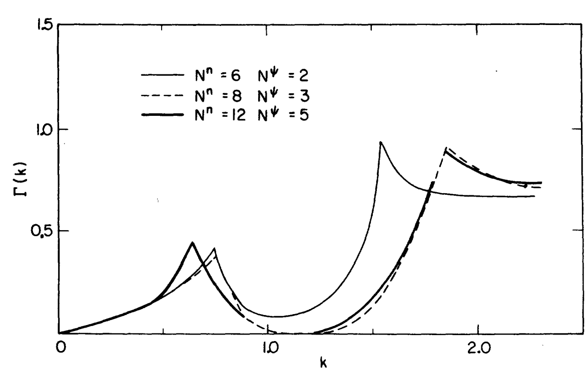

FIG. 10. Consolidated iteration damping rates for the least damped pole of R^{nn}. The solid curve is essentially the same as in Fig. 9.

sidered, $\partial C(t)/\partial t$ depends on the entire time history of $C(\tau)$, $-\infty < \tau \leq t$, and if not given, must be determined either in an iterative fashion, or the time homogeneous regime must be considered as a large-time asymptotic limit of the full initial value problem where the covariances are specified at some time t_0.

We have evaluated the response functions for a range of wave numbers, k, given the electric-field fluctuations at zero wave number, with no fluctuations at any other $k \neq 0$. This may be regarded as a fluctuating version of the (coherent) modulational instability. Just as a spatially uniform electric field is an exact solution to the Zakharov equations (unforced, or forced and damped at $k=0$), it is easy to show that our model which has a white-noise driver and linear damping at $k=0$, with no driving or initial value fluctuations or mean field at $k \neq 0$, is an exact solution of the DIA. This solution is trivial and coincides with that of the linear Langevin equation:

$$\frac{\partial E(0)}{\partial t} + \nu_e(0)E(0) = \xi(t)\,, \tag{7.1}$$

$$\langle \xi(t) \rangle = 0\,, \qquad \langle \xi(t)\xi^\dagger(t') \rangle = S_0^{EE^\dagger}(0)\delta(t-t')\,. \tag{7.2}$$

$S_0^{EE^\dagger}(0)$, a positive number, is the strength of the linear noise for the electric field at $k=0$. The solution is

$$\langle E(0) \rangle = 0\,, \qquad C^{EE^\dagger}(t,t') = \langle E(0,t)E^\dagger(0,t') \rangle = [S_0^{EE^\dagger}(0)/2\nu_e(0)]\exp[-\nu_e(0)\,|t-t'\,|]\,. \tag{7.3}$$

For long times, the numerical solutions of the DIA for R_{kt}^{ψ} and R_{kt}^{n} are well represented by either exponential growth or decay, possibly modulated by oscillation about the value $R=0$. Typically, the growth rates and frequencies for R^{ψ} and R^{n} are closely related, with obvious differences only arising for values of k which border on a range of stable wave numbers. In Fig. 11 R_0^n and R^n are graphed. After a transient behavior for small values of time, which is reminiscent of the response functions for the strictly linear Langmuir and sound waves, both R^{ψ} and R^{n} grow exponentially with the same rate. In Fig. 12 the common growth rate is shown as a function of k, and it is compared with the growth rates obtained by a Monte Carlo simulation, to be discussed in Sec. VIII. The squared modulus, $k^2(\hat{R}^{nn})^2 + (R^{uu})^2$, showing exponential growth which is periodically modulated, is shown in Fig. 13.

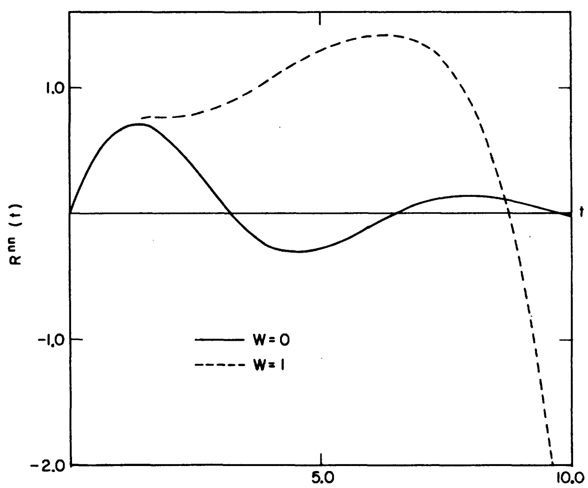

FIG. 11. The evolution of $R^n(k=1,t)$ as determined by DIA, with $\nu_e(\text{pump})=\frac{1}{8}$, $\nu_e(k)=\frac{1}{4}$, $\nu_i(k)=k/4$.

Modulational instability for nonzero bandwidth pumps

There is a special class of solutions to the Zakharov equations (and to the DIA) for which the Langmuir wave function ψ has but a single wave vector $\vec{k}$. If the nonzero initial conditions and forcing are limited to $\psi(\vec{k})$ [and correspondingly to $\psi^\dagger(-\vec{k})$], then the evolution of ψ will be strictly linear and the density n will not evolve.

The stability of this solution has been discussed in Secs. V and VI for the case $k=0$. This case may be pathological because every mode $k \neq 0$, in the linear limit, is only coupled to the pump, $\psi(k=0)$.

Rather than study the case of a single-mode pump with $k \neq 0$, we shall now consider a narrowband, discrete pump centered about $k=0$. The purpose of this study, as before, is to understand the DIA predictions for the linear response func-

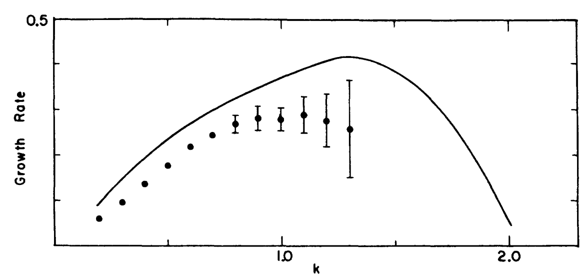

FIG. 12. Growth rate of modulational instability for $\nu_e(\text{pump})=\frac{1}{8}$, $\nu_e(k)=\frac{1}{4}$, $\nu_i(k)=k/4$, and $W=1.00$. The curve represents the rate determined by a long-time integration of the DIA, while the circles are from a Monte Carlo simulation.

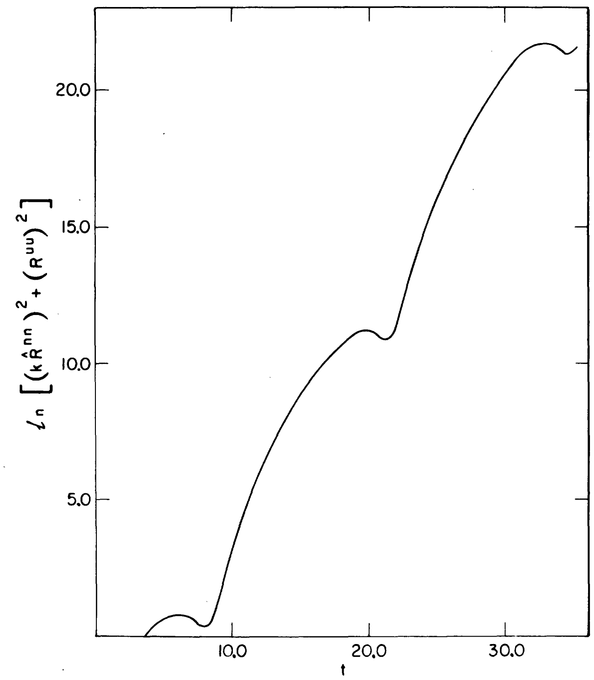

FIG. 13. The growth in time of the modulus of the density response function as calculated by the DIA for the case described in Fig. 11.

tions without having to cope with the complexity associated with self-consistently determined fluctuation functions.

There is, however, a technical difficulty in using nonself-consistent fluctuation functions. Since the response and fluctuation functions are not, together, a solution of the DIA, what dynamic system should be used as a basis for comparison?

Consider the following pump model equations:

$$\frac{\partial \psi(k)}{\partial t} = -ik^2\psi(k) - \nu_e(k)\psi(k) - i(p/k)n(p)\phi(q)\,,$$

$$\frac{\partial u(k)}{\partial t} = -2\nu_i(k)u(k) + n(k) - pq[\psi(p)\phi^\dagger(q) + \phi(p)\psi^\dagger(q)]\,,$$

$$\frac{\partial n(k)}{\partial t} = -k^2 u(k)\,,$$

where $p+q=k$, summation over p is implied. The field ϕ obeys a linear equation of motion with a Gaussian random source chosen to make ϕ's two-point fluctuation functions the same as a specified $C^{\phi\phi^\dagger}(k,t)$, with k ranging over the various modes in the pump. It can be shown that the linear response functions for ψ, u, and n, as predicted by the DIA applied to this specific model, are the same as those predicted by the DIA applied to the Zakharov equations, when the C^{nn} term in $\Sigma^{\psi\psi}$ [see Eq. (2.35)] is omitted. When the Zakharov equations are linearized about one of the single-mode pump solutions, they reduce to the above pump model with the substitution of the single-mode pump for ϕ. For this pump model, we shall sometimes refer to ψ, u, and n as *response modes*, and the field ϕ as being constituted of *pump modes*.

The results of integrating the DIA response function equations for an unstable pump model are found in Fig. 14. As the pump bandwidth goes to zero, the growth rate of a representative, linearly unstable response mode approaches the growth rate determined by a single-mode pump located at $k=0$. While in principle, an infinite number of discrete response modes are coupled together by a multimode pump (even if the pump has a finite number of modes), after the number of modes of a finitely truncated response band (centered about the particular response mode of interest) exceeds the effective number of modes in the pump, the growth rate at the center of the response band is insensitive fo further increases in response bandwidth (see Figs. 15 and 16).

However, a Monte Carlo simulation of the multimode pump model revealed that even in the narrow pump-band limit, the growth rate was greater than that found for a single-mode, $k=0$, pump. When single-mode and multimode pump models are compared, W is kept the same, and the linear damping rate is the same for all modes in the single or multimode pumps. The linear oscillation frequency of a pump mode at wave number k is k^2, though for a narrow-band pump centered

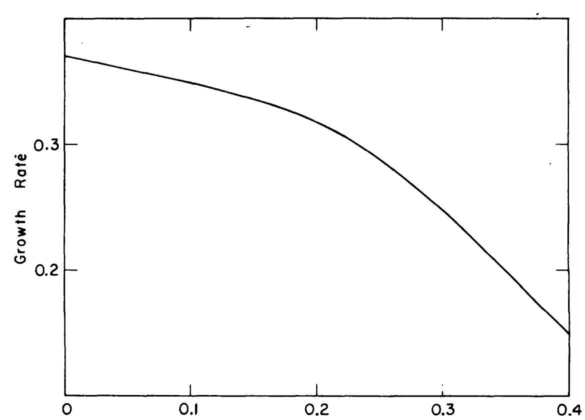

FIG. 14. DIA growth rate of $R^n(k=0.9)$ as a function of pump spectral width, Δk, with a Lorentzian envelope, $W=1$, $\nu_e(\text{pump})=\frac{1}{8}$, and 13 pump modes. The 17 response modes are centered about $k=0.9$, with $\nu_e(k)=\frac{1}{4}$ and $\nu_i(k)=k/4$. The spacing between modes in the pump, and between modes in the response band equal $\Delta k/2$.

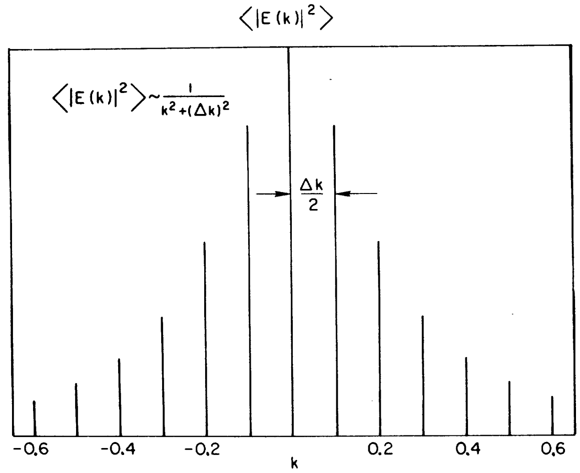

FIG. 15. Discrete pump spectrum with Lorentzian envelope for $\Delta k = 0.2$, with 13 modes.

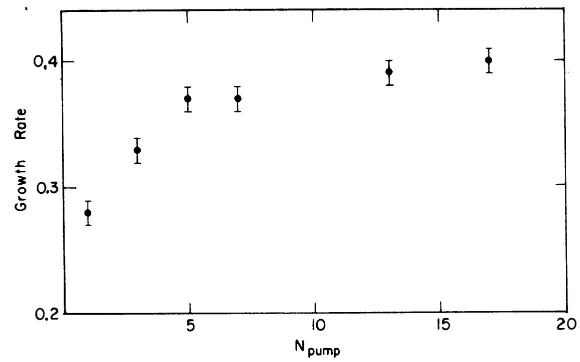

FIG. 17. Monte Carlo growth rate of density at $k = 0.9$, with $\nu_e(k) = \frac{1}{4}$, $\nu_i(k) = k/4$, as a function of the number of pump modes for a top hat spectrum of halfwidth 0.1, with $W = 1$ and $\nu(\text{pump}) = \frac{1}{8}$. The spacing between modes when there are N pump modes is $0.2/(N-1)$.

about $k = 0$, these frequencies are essentially zero.

As the number of modes in the pump is increased, keeping the pump width and W fixed, the linear response growth rate increases monotonically, saturating after there are about 15 modes in the pump (see Fig. 17).

The insensitivity of the DIA predicted growth rate to the number of pump modes in the narrow-band limit is consistent with a smooth dependence of the DIA response function upon its wave number argument. However, in the actual pump model dynamics, the addition of statistically independent pump modes changes the overall pump statistics. For example, even though the fluctuating pump strength, $\hat{W}$,

$$\hat{W} = \sum_k |E_k|^2$$

has a mean value W which is independent of the number of pump modes, the fluctuations in $\hat{W}$ go to zero as the number of pump modes increases.

A priori, it is hard to know what effect the change in pump statistics will have on a growth rate. It is observed in the multimode pump, that the increase in the mean growth rate (compared to that of the single-mode pump) is accompanied by a lower level of fluctuations in the growth rate. Since in a laboratory plasma, the smallest wave number allowed is small compared to the fundamental unit of wavelength in the Zakharov equations, a pump model with many pump modes seems to be more realistic than a model which only has a few modes. Coincidentally, the DIA growth rate lies roughly halfway between that of the Monte Carlo simulation of the multimode and that of the single-mode pump.

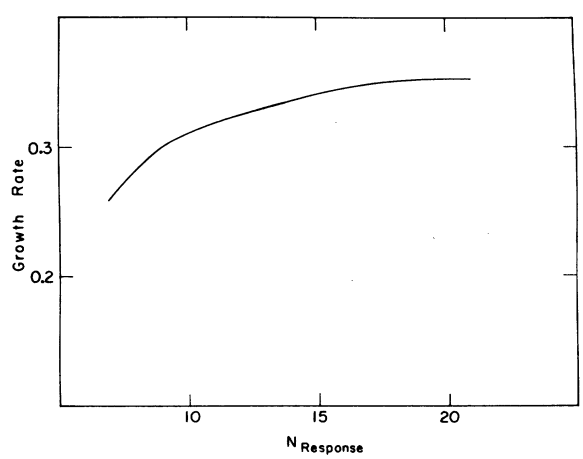

FIG. 16. DIA growth rate of density response function as a function of the number of response modes centered about $k = 0.9$, with $\nu_e(k) = \frac{1}{4}$, $\nu_i(k) = k/4$, for a 13 mode Lorentzian pump spectrum, with $W = 1$, $\Delta k = 0.1$, and $\nu_e(\text{pump}) = \frac{1}{8}$.

VIII. MONTE CARLO SIMULATION

The simulation of a Langevin equation on a computer involves three operations at every time step of a finite difference discretization of the original differential equation.

(1) Use a standard pseudorandom number generator to obtain two independent random numbers ξ_1, ξ_2 from a distribution which is uniform between zero and one.

(2) Convert these into a pair of random numbers independently sampled from a zero mean, unit variance, Gaussian distribution ζ_1 and ζ_2, by application of the well-known Box-Muller transformation:

$$\zeta_1 = (-2\ln\xi_1)^{1/2}\cos(2\pi\xi_2)\,,$$
$$\zeta_2 = (-2\ln\xi_1)^{1/2}\sin(2\pi\xi_2)\,.$$

(3) Advance the finite difference equations using, in place of the white-noise source terms, ζ_1, ζ_2 (and other independent samples if required), each augmented in strength by a factor which varies inversely with $\sqrt{dt}$, where dt is the time step.

Step (3) is consistent with the fact that

$$\langle\zeta^2\rangle \sim dt\,,$$

where

$$\zeta = \int_0^{dt} \xi(s)ds \tag{8.1}$$

if ξ is white noise.

The expectation value of any dynamic quantity can be defined in terms of an ensemble average over initial conditions, with the white noise forcing independently chosen for each member of the ensemble. However, if the correlation information desired is translationally invariant in time, then a time average could be taken. This is not necessarily the same as the ensemble average, but the possible differences will be ignored for our purposes.

Even though a growth rate is derived from an unstable initial condition which has not yet evolved into a steady state, if the equations of motion are linear in the unstable modes (i.e., the modes at wave number k in our pump model), then the growth rate will, in the mean, be independent of time.

The precise method that we have used to "measure" a mean growth rate is the following.

(a) Advance the finite difference equations, driven by a "pseudo-white noise" force.

(b) Record the time at which an unstable mode reaches a local maximum in modulus.

(c) The next time a local maximum is obtained which exceeds the previous one, calculate a growth rate by dividing the natural logarithm of their ratio by the time difference between them.

(d) Repeat steps (a)–(c) for a long time and average.

The growth rates obtained in this way are compared with those obtained from the DIA in Fig. 12. This operationally defined growth rate should presumably be compared to the equal-time behavior of the covariances $C^{\psi\psi^\dagger}_{kt,t} = \langle|\psi(k,t)|^2\rangle$ and $C^{nn}_{kt,t} = \langle|n(k,t)|^2\rangle$; however, the relationship between this operational time-averaged growth rate and the ensemble-averaged quantities is obscure. Empirically, comparisons such as Fig. 12 convince us that they are closely related.

In Secs. VI and VII we determined the time behavior of the *response* functions as a function of their *difference* time variables. The growth of $R^{\psi\psi}_{kt,t'}$, with respect to $t-t'$, sets a lower bound on the growth of $C^{\psi\psi^\dagger}_{kt,t}$ as a function of t. This is because, in general, the covariance equation can be written as

$$C^{\psi\psi^\dagger}_{kt,t} = |R^{\psi\psi}_{kt,t_0}|^2 C^{\psi\psi^\dagger}_{kt_0,t_0} + \int dt' \int dt'' R^{\psi\psi}_{kt,t'}(R^{\psi\psi}_{kt,t''})^* S^{\psi\psi^\dagger}_{kt',t''}\,, \tag{8.2}$$

and S is positive definite. Because of the uncertainty in the operationally defined equal-time growth rates, we cannot use this bounding property at present to ascertain the accuracy of our calculations. In Sec. X we discuss a particular case for which the response growth rates and equal-time covariance growth rates differ.

One question which we would like to, but cannot with assurance, answer is: Do the growth rates of the density and electric-field linear response functions, at a given wave number k, coincide? The DIA solutions found in Secs. VI and VII predicted that these growth rates were equal. In Fig. 18 the two growth rates, as operationally defined above, are displayed as a function of the linear damping, ν_e, of the single-mode pump at $k=0$, for a response wave number $k=1$. These rates differ, but seem to approach each other for small and large values of ν_e (pump).

Before analyzing these results, let us look in detail at the dynamics:

$$\begin{aligned}
\frac{\partial n(k)}{\partial t} &= -k^2 u(k)\,,\\
\frac{\partial u(k)}{\partial t} &= -2\nu_i(k)u(k) + n(k)\\
&\quad -ik[E_0^\dagger\psi(k) + E_0\psi^\dagger(k)]\,,\\
\frac{\partial\psi(k)}{\partial t} &= -\nu_e(k)\psi(k) - ik^2\psi(k) + E_0\frac{n(k)}{k}\,,\\
\frac{\partial\psi^\dagger(k)}{\partial t} &= -\nu_e(k)\psi^\dagger(k) + ik^2\psi^\dagger(k) - E_0^\dagger\frac{n(k)}{k}\,,
\end{aligned} \tag{8.3}$$

where $E_0(t)$ is a zero-mean, Gaussianly distributed random variable, whose covariance is given in Eq. (7.3). For the particular case in Fig. 18, $\langle|E_0(t)|^2\rangle = 1.0$. If $\psi(k)$ has a modulus which is growing without bound for large time, since $E_0(t)$ is typically of order unity, it follows that $n(k)$ is also growing without bound in modulus and its order of magnitude must be the same as $\psi(k)$. This is confirmed by a direct analysis of the Monte Carlo data, independent of the initial conditions for ψ, n, and u. If we can assume that E_0 is strictly bounded (not true for the Gaussian E_0), then this assertion can be proven. Integrate the

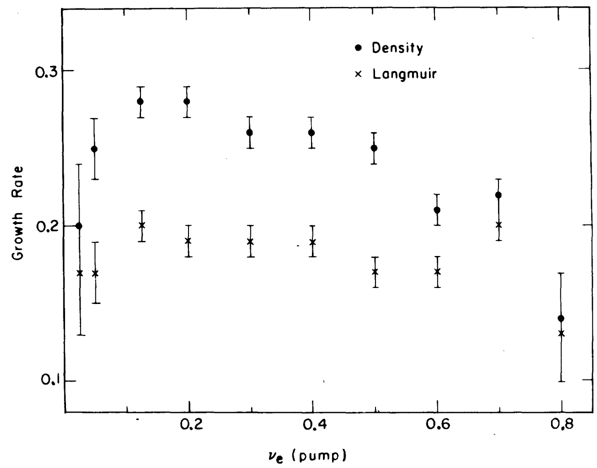

FIG. 18. Monte Carlo growth rate, as a function of pump decay rate, for density and Langmuir fields at $k=1.0$, with $\nu_e(k)=\frac{1}{4}$ and $\nu_i(k)=k/4$; single-mode pump at $k=0$, with $W=1$.

equations of motion for n, u, and ψ between $t=0$ and $t=T$, where T is very large so that n, u, and ψ at T are much bigger in modulus than their initial values. The equation of motion for ψ has $\exp[\nu_e(k)+ik^2]t$ as an integrating factor. This implies that

$$e^{\nu_e T}|\psi(T)|<T|n(T)|e^{\nu_e T},$$

where factors of order unity [such as the maximum value of $E_0(t)$] have been ignored. Similar considerations for $\psi^\dagger$ yield

$$|\psi^\dagger(T)|<T|n(T)|.$$

The equations of motion for u and n have an exponential integrating factor too, which yields

$$|u(T)|+|n(T)|<[|\psi(T)|+|\psi^\dagger(T)|]T.$$

If $|u|$, $|n|$, $|\psi|$, and $|\psi^\dagger|$ grow exponentially with T, the factors of T in the above inequalities are irrelevant which then force $|n|$ and $|\psi|$ and $|\psi^\dagger|$ to be the same order of magnitude.

IX. RANDOM GAUGE TRANSFORMATIONS

Since the $n(k=0)$ component of the density field is time independent, an initial value ensemble which assigned, for example, a zero-mean Gaussian distribution to $n(k=0)$, would maintain this distribution as it evolved in time according to the Zakharov equations. For a realization of the ensemble [with $n(k=0)\equiv n(0)$] a given Langmuir mode $\psi(k,t)$ has a time evolution described by the additional phase factor $\exp[-in(0)t]$ in comparison with the evolution of a realization with $n(0)=0$. The evolution of the density, on the other hand, is unaffected.

Let us call such a transformation

$$\psi(k,t)\to\{\exp[-in(0)t]\}\psi(k,t), \tag{9.1}$$

a gauge transformation. The ensemble average over $n(0)$ produces the following changes:

$$R^{\psi\psi}_{kt,t'}\to R^{\psi\psi}_{kt,t'}\exp(-\tfrac{1}{2}n_0^2|t-t'|^2), \tag{9.2}$$

$$C^{\psi\psi^\dagger}_{kt,t'}\to C^{\psi\psi^\dagger}_{kt,t'}\exp(-\tfrac{1}{2}n_0^2|t-t'|^2), \tag{9.3}$$

where $n_0^2=\langle n^2(0)\rangle$. The effect of an ensemble of gauge transformations (a random gauge transformation) has the important property that the density-density correlation functions are invariant, as well as the *equal-time* Langmuir correlation functions. However, for unequal times the Langmuir response function has gained a decaying factor which will turn a growing or unstable $R^{\psi\psi}$ into a decaying time behavior as $|t-t'|\to\infty$. This was in distinction to the zero-bandwidth case considered in Sec. VI in which $\hat{R}^{nn}$ and $R^{\psi\psi}$ had equal growth rates in the unstable case. This example also illustrates that $C^{\psi\psi}_{kt,t}=\langle|\psi(k,t)|^2\rangle$ can grow with t while the linear response function $R^{\psi\psi}_{kt,t'}$ decays as a function of $|t-t'|$.

The DIA does not transform properly under a random gauge transformation. Under such a transformation of ensembles, $C^{nn}_{kt,t'}$ acquires a component at $k=0$ and the DIA expressions for the self-energy and nonlinear noise source transform according to

$$\Sigma^{\psi\psi}_{kt,t'}\to\Sigma^{\psi\psi}_{kt,t'}-n_0^2R^{\psi\psi}_{kt,t'}, \tag{9.4}$$

$$S^{\psi\psi^\dagger}_{kt,t'}\to S^{\psi\psi^\dagger}_{kt,t'}+n_0^2C^{\psi\psi^\dagger}_{kt,t'}. \tag{9.5}$$

Numerical integration of the DIA response function equations for the zero-bandwidth pump model shows that a formerly stable $R^{\psi\psi}(k)$ is made more stable by this addition to $\Sigma^{\psi\psi}$, which is expected from the argument above, but that this also occurs to $\hat{R}^{nn}(k)$. Thus, we have evidence that the DIA is not invariant, in its prediction for $\hat{R}^{nn}$, under a random gauge transformation, and we expect that this is also true for both equal-time covariances.

This is reminiscent of the noninvariance of the DIA, as applied to the Navier-Stokes equation, under a random Galilean transformation.[11] A more closely related example is the noninvariance of the DIA, as applied to the random Schrödinger equation, under a random gauge transformation.[28]

How bad is this failure of the Zakharov DIA and what is its physical significance? The first of these questions is largely unanswered, but we are aware of one specific way in which it differs from the Navier-Stokes case. For the latter, it can be shown that a random Galilean transformation has no effect upon the nonlinear noise of the DIA so that the level of fluctuations is spuriously reduced; while in the Zakharov DIA it can be shown that if the following fluctuation dissipation approximation[29]

$$C^{\psi\psi^\dagger}_{kt,t'} = C^{\psi\psi^\dagger}_{kt,t} R^{\psi\psi}_{kt,t'} \tag{9.6}$$

is invoked, then at equal times, $t=t'$, the DIA equation for $(\partial/\partial t)C^{\psi\psi^\dagger}_{t,t'}$ is unchanged for all wave numbers. Since the two-time behavior of the correlation functions is noninvariant, and since the DIA involves time histories over all possible two-time differences, there is bound to be an effect on $C^{\psi\psi^\dagger}_{t,t}$; but at least it is an indirect one.

The physical significance of a random gauge transformation is connected with the adiabatic approximation, $n(x) = -|\nabla\psi(x)|^2$. This is expected to be valid for $n(k)$, with $k \ll 1$. Therefore, a large peak in the Langmuir wave fluctuation spectrum at long wavelengths (due to the tendency to form a condensate) will lead to a corresponding peak in the density spectrum. The fact that $n(k=0)$ is time independent and can be taken equal to zero is irrelevant because density fluctuations with arbitrarily small but finite wave numbers will be excited in a steady state, although they may be absent at the early stages of modulational instability. Though these density fluctuations are not time independent, they are not likely to be rapidly varying in time compared to modes in a modulationally affected range. Therefore, to the extent that the DIA misrepresents the effect of frozen (time-independent) density fluctuations at infinite wavelength by spuriously damping the two time density correlation functions, it will also overestimate the effect of slowly varying long-wavelength density fluctuations. If initially there are no long-wavelength density fluctuations, it would take a time roughly k^{-1} to build up an adiabatic density response of wavelength k^{-1} (the speed of sound is one in our units). If the growth rate γ^M of the MI satisfies $\gamma_M \gg k$, then the modulational response calculated for $C^{nn}=0$ will have time to develop before C^{nn} has time to attain its adiabatic value.

Multimode Monte Carlo computer simulations to be reported elsewhere show that the Langmuir spectrum broadens in k space as the system approaches a nonlinear saturated steady state. Only the portion of this spectrum for $k \ll 1$ is expected to drive adiabatic density fluctuations for which the noninvariance of the DIA is a problem. Various statistical theories of the adiabatic limit of the Zakharov equations (i.e., the nonlinear Schrödinger equation) will be discussed in another paper.[31]

There are many possible *ad hoc* cures to this noninvariance of the DIA, just as in the Navier-Stokes case. Also, there are systematic Lagrangian history closures[30] which for the Zakharov equations consists in introducing the auxiliary field $\psi(t|t')$,

$$\nabla^2 \frac{\partial\psi(xt|t')}{\partial t} = i\vec{\nabla}\cdot[n(xt)\vec{\nabla}\psi(xt|t')],$$

$$\psi(xt|t) = \psi(x,t),$$

and developing approximations for the generalized Langmuir correlation functions such as $\langle\psi(kt|t')\psi^\dagger(-kt''|t''')\rangle$. Both of these will be discussed in a future publication.

X. RESPONSE FOR BROADBAND SPECTRA AND "QUASILINEAR" SATURATION OF MODULATIONAL INSTABILITY

We next return to the $l=1$ iteration and examine the consequences of a broadband spectrum $W(q)$ in the expressions (5.3) and (5.7). Here we again consider the case $C^{nn}=0$, ignoring for the moment the effect of self-consistent density fluctuations. In Sec. VI this case was treated numerically by direct integration of the DIA as well as in a Monte Carlo simulation in Sec. VII. The integrals in the self-energy terms in (5.3) and (5.7) will generally introduce branch cuts in the complex ω plane so that the response functions will also have branch cut singularities in addition to poles corresponding to the zeros of $R^{-1}(\vec{k},\omega)$. These poles will lie on one or another of the Riemann sheets of the cut function $R(\vec{k},\omega)$. In this section we will confine our attention to a Lorentzian shaped finite-bandwidth pump spectrum in one dimension:

$$W(q) = 2\frac{\Delta k}{q^2+(\Delta k)^2}W. \tag{10.1}$$

This choice will allow us to analytically[32] carry out the q integrations in (5.3) or (5.7), at least approximately. This case also illustrates the physical effect of finite bandwidth and results in a simple but nontrivial analytic structure for the cut function.

We carry out the one-dimensional integration over q in (5.3) using Cauchy's theorem noting that the integrand has poles at $q=\pm i\Delta k$ and at $q=\frac{1}{2}k \pm\frac{1}{2}(\omega+i\nu_+)k^{-1}$. Here we approximate the damping factors that appear by $\nu_e(q)+\nu_e(k-q)\cong\nu_e(0)+\nu_e(k)\equiv\nu_+$. The result for the integral in (5.3) is

$$I_+(k,\omega) = \frac{2k^2}{(\omega+i\nu_+ + 2ik\Delta k)^2 - k^4}$$

for

$$\mathrm{Im}\,\omega > -\nu_+,$$

$$I_-(k,\omega) = \frac{2k^2}{(\omega+i\nu_+ - 2ik\Delta k)^2 - k^4} \tag{10.2}$$

for

$$\mathrm{Im}\,\omega < -\nu_+ .$$

This function has a cut along the line $\mathrm{Im}\,\omega = -\nu_+$ across which the function is discontinuous. The analytic continuation across the cut onto the next Riemann sheet is simply obtained by extending $I_+(k,\omega)$ to $\mathrm{Im}\,\omega < -\nu_+$. The analytically continued function

$$I_+(k,\omega) = \frac{2k^2}{(\omega + i\nu_+ + 2ik\Delta k)^2 - k^4} \tag{10.3}$$

then has only pole singularities. In evaluating $R^{\phi}_{(1)}(k,t)$ by taking the inverse Fourier transform, the analytically continued function $R^{\phi}_{(1)}(k,\omega)_c$ can be used if the contour of integration is closed or deformed onto the second Riemann sheet.

Thus, the analytically continued function $R^{\phi}_{(1)}(\vec{k},\omega)$ in the broadband case differs from the narrow-band case only by an effectively larger damping decrement $\nu_+ + 2k\Delta k$. Therefore, the narrow-band dispersion relation is trivially modified by this replacement and the marginal stability condition is now

$$(\nu_+ + 2k\Delta k) = k(2W - k^2)^{1/2} . \tag{10.4}$$

The minimum threshold for instability is now increased for finite $\Delta k \gtrsim \nu_+^{1/2}$. If $2\Delta k \gg \nu_+^{1/2}$, then the minimum threshold is increased to

$$W_m \cong 2(\Delta k)^2 . \tag{10.5}$$

Or stated conversely, a broadband pump of strength W is stabilized against modulational instability if

$$\Delta k > (W/2)^{1/2} \tag{10.6}$$

(and $2\Delta k \gg \nu_+^{1/2}$), at least in the $l=1$ iteration. Therefore, we see that if there is a dynamical mechanism to spread the spectral width it may be an efficient saturation mechanism for the modulational instability.

Many-mode Monte Carlo calculations based on the noise driven Zakharov equation model discussed in Sec. VIII have been carried out by Hafizi.[17] These indicate that an initially narrow Langmuir spectrum broadens in k space as the system saturates and approaches a turbulent steady state. This saturation by spectral broadening is analogous to the quasilinear flattening of a bump-on-tail velocity distribution as the system stabilizes. In both cases, the spectrum which is the source of the instability is modified in shape as the system approaches nonlinear stability. A quasilinear model for this saturation of the modulational instability will be discussed in another paper. There we will also address the iterative continued-fraction solution for the DIA response functions for the case of the broadband pumping spectrum.

The $l=1$ iteration for $R^{\phi\phi}_{(1)}(k,\omega)$ can also be treated analytically as was the case for $\hat{R}^{nn}_{(1)}$ considered above. This case is somewhat more complicated but leads to qualitatively similar results; we will not present the details here.

XI. SUMMARY AND COMPARISONS

We are guided in our research by the work of Kraichnan and others in the development of such statistical closure schemes for Navier-Stokes turbulence; this work provides a paradigm for this type of research which develops from careful and systematic closure approximations and *detailed* comparisons with numerical simulations and experiment. The subject of Navier-Stokes turbulence is certainly not closed with such problems remaining as the description of intermittency, but we believe it is fair to say that this subject is understood in far greater depth than is the case for strong plasma turbulence.

We are attempting to follow a similar program using the Zakharov equations as a model nonlinear description of the system with no apology for the (many) physical effects[1,2] not included in these equations. We have also limited our detailed results to one dimension in full recognition that higher dimensions have qualitatively different behavior; in one dimension extensive comparison to numerical simulations is possible at a relatively modest cost.

This paper represents a first step toward this goal. The main results can be summarized as follows.

(1) We have developed the equations for the statistical description of the Zakharov equations in the DIA which is a realizable theory which conserves the mean invariants $\langle N\rangle$, $\langle P\rangle$, and $\langle H\rangle$ for zero damping.

(2) This theory represents a significant improvement over weak-turbulence theory since it naturally contains a turbulence-induced modulational instability and the associated strong modification of the response function at small wave numbers ($k \lesssim W^{1/2}$); it continuously connects to weak-turbulence theory at high wave numbers.

(3) The predictions of this statistical theory have been compared satisfactorily with Monte Carlo simulations for the early stages of evolution of the turbulence-induced modulational instability. We adopt the philosophy that no further approximations will be made to the DIA equations until we understand their full consequences; we are therefore faced with serious numerical problems resulting from the long-time histories associated

with oscillatory, long-lived, or unstable responses.

(4) A continued-fraction iteration scheme for the DIA response functions has been introduced. It clearly shows through the sequence of poles (nonlinear normal modes) which it produces at each wavelength a transition from a modulationally unstable range of wave numbers to a range of higher wave numbers, where weak-turbulence theory is regained. This method is more efficient than direct temporal integration of the DIA equations and provides an important tool for the complete self-consistent solution of the full DIA equations which will be treated in another paper.

(5) We have recognized at least two shortcomings of the DIA as applied to the Zakharov equations. First, it is noninvariant to a random gauge transformation; this defect is evident even in the response functions (noninvariance of the modulational growth rate). There is reason to believe that a Lagrangian history type of closure approximation will remedy the random gauge transformation noninvariance. Second, we believe another defect of the DIA is its inability to quantitatively represent the intermittent fluctuations associated with the tendency to collapse. This is likely to be one of the important mechanisms for the dissipation of Langmuir energy. Intermittency cannot be described by any of the known closure techniques.

In spite of these problems we believe that the DIA or a closely related theory can describe the evolution of a weak modulational instability for a Zakharov model with linear damping. The regime of strong Langmuir turbulence may not be adequately described by this type of theory if it is dominated by localized, intermittent structures. These structures are indicated by numerical simulations and are the basis for many *ad hoc* but physically motivated theories.

The next step in this program to be addressed in a second paper is to study the DIA predictions for the self-consistent fluctuation spectra. We have carried out a few mode truncation of the Zakharov equations for which Monte Carlo simulations show a negative correlation between density and energy fluctuations, i.e., $\langle n|E|^2\rangle < 0$, which is qualitatively described by the DIA. A many-mode Monte Carlo code being developed by Hafizi should provide a more realistic standard of comparison and preliminary results also show this negative correlation. The correlation represents the likelihood of Langmuir waves being trapped, at least transiently, in density depressions. This leads us to believe that the conservation (without damping) of the mean Hamiltonian $H = H_0 + n|E|^2$ is especially significant. The complete DIA calculates the MI self-consistently: The turbulently renormalized responses and a turbulent noise source are determined in such a way that the total Hamiltonian is conserved in the mean. Weak-turbulence theory has nothing to say about these matters.

For the example of a linearly damped system driven to modulational instability by externally imposed noise, these simulations show an evolution to a nonlinear steady state in which electrostatic energy is distributed over a broadened, modulationally stable spectrum. The tendency for Langmuir energy to cascade toward lower k is modified in the DIA by the warping of the response function dispersion curves for $k \lesssim W^{1/2}$. Details will be discussed in another paper.

We are extending the continued-fraction scheme to the case of a broadband pumping spectrum. This is an important technical simplification in solving the complete DIA equations in the many-mode case.

At this stage we can make only preliminary and qualitative comparisons with other statistical theories of Langmuir turbulence. In Ref. 24, Tsytovich derives a renormalized dispersion relation for the MI for a broadband pumping spectrum. In this theory there is only one response function—his renormalized dielectric which we identify as R^{nn}—and he carries this out only to what we would call the first iteration. His results differ qualitatively and quantitatively from ours and no comparison with numerical simulation is made. The broadband dispersion relation of Bardwell-Goldman[23] is exactly the first iteration of our $\hat{R}^{nn}$. We have seen that the modulational instability is also found in the Langmuir response $R^{\psi\psi}$ and that the first iteration is usually not accurate but can be systematically improved by simple iteration.

The work of Khakimov and Tsytovich[12] appears to be the most serious attempt to construct a statistical theory of Langmuir turbulence. A detailed formal comparison is difficult since these authors proceed from the Vlasov equation rather than from the Zakharov model, and a comparison of results must await the full self-consistent solution of the DIA. We can make a few preliminary comments: Again, the double average theory of Ref. 12 seems to have only one response function (their $\bar{\epsilon}_k$) but it also contains some fourth-order cummulants (their $G_{k;k_1;k'}$) which are not included in the DIA but which arise in higher-order vertex corrections to the DIA. The complicated nonlinear integral equations of this theory are not solved by systematic means and a number of additional assumptions and approximations are added to the original theory in order to obtain results. We hope to be able to make more detailed comparisons in the future.

In two interesting recent papers[13] Pelletier has considered Langmuir turbulence as a critical phenomenon. The second paper tries to relate an infinite correlation length for the Langmuir fluctuations to the onset of modulational instability. We believe that this effect is due to the cascade of the decay instability rather than the modulational instability. One reason for this conclusion is that the onset of modulational instability typically occurs at a finite wave number ($\sim\sqrt{W}$) as opposed to zero wave number. Also, the modulational instability is a new mode which has no counterpart in the uncoupled system, whereas Pelletier's methods only involve a renormalization of Langmuir and sound waves. Pelletier's main results were for three dimensions but we believe the comments above apply as well to this case as to one dimension. His procedure cannot in any case answer the interestmg problem of how Langmuir wave energy is transferred from the condensate region near $k=0$ to the dissipation region at high k.

ACKNOWLEDGMENTS

We wish to thank George Carnevale, Uriel Frisch, Martin Goldman, Bahman Hafizi, Robert Kraichnan, Dwight Nicholson, and Denis Pesme for helpful discussions. We especially thank Magne Espedal for his collaboration in the early stages of this research.

APPENDIX: THE DIA IN AN UNSYMMETRIC ENSEMBLE

In principle, one must allow for the possibility of initial conditions and/or driving forces which break the symmetry of the fully symmetric ensemble. The form of the DIA in this more general case is displayed below, though spatial homogeneity is still assumed.

A uniform electric field $\vec{E}_0$ is allowed which replaces the Hamiltonian in Eq. (2.4) by

$$H \to H + i\int d\vec{k}[\vec{E}_0\psi^\dagger(-\vec{k}) + \vec{E}_0^\dagger\psi(-\vec{k})]\cdot\vec{k}n(\vec{k}),$$

and the equations of motion (2.10) are supplemented by

$$\frac{\partial\vec{E}_0}{\partial t} = -i\frac{\partial H}{\delta\vec{E}_0^\dagger} = -\int d\vec{k}\,\psi(\vec{k})n(-\vec{k})\vec{k},$$

$$\frac{\partial\vec{E}_0^\dagger}{\partial t} = i\frac{\delta H}{\delta\vec{E}_0},$$

$$\frac{\partial u(\vec{k})}{\partial t} = \frac{\partial u(\vec{k})}{\partial t}\bigg|_{\vec{E}_0=0} - i\vec{k}\cdot[\vec{E}_0^\dagger\psi(\vec{k}) + \vec{E}_0\psi^\dagger(\vec{k})],$$

$$\frac{\partial\psi(\vec{k})}{\partial t} = \frac{\partial\psi(\vec{k})}{\partial t}\bigg|_{\vec{E}_0=0} + \vec{E}_0\cdot\vec{k}n(\vec{k})/k^2.$$

There are actually three different symmetry classes for homogeneous Zakharov turbulence.

(I) The (fully) symmetric case which is discussed in the main body of this paper.

(II) The partially symmetric case which satisfies

$$\langle E\rangle = C^{n\psi} = C^{u\psi} = R^{\psi u} = R^{u\psi} = R^{n\psi} = R^{\psi n} = 0,$$

but

$$C^{\psi\psi} \neq 0, \quad R^{\psi\psi^\dagger} \neq 0.$$

(III) The nonsymmetric case in which none of the functions listed in (II) vanishes.

It can be shown from the form of the DIA equations that if the initial value ensemble belongs to one of the above classes, it will remain in that class for all time. Of course, the addition of nonsymmetric driving forces invalidates these conclusions. We have no reason to believe that (nonequilibrium) Zakharov turbulence will exhibit spontaneous symmetry breaking, though a strong (infrared) Langmuir wave cascade due to the decay instability may have similar features.

When $\vec{E}_0$ is allowed, the DIA for case (I) now includes equations of motion for $R^{\vec{E}_0\vec{E}_0}$ and $C^{\vec{E}_0\vec{E}_0^\dagger}$, which can be formally obtained from those for $R^{\psi\psi}_{\vec{k}}\ldots$ by setting $\vec{k}=0$, and using the tensor self-energy and nonlinear noise:

$$\Sigma^{\vec{E}_0\vec{E}}_{t_1,t_2} = -\int \hat{\vec{k}}\hat{\vec{k}}(R^{\psi\psi}_{\vec{k}t_1,t_2}C^{nn}_{-\vec{k}t_1,t_2} + ik^2C^{\psi\psi^\dagger}_{\vec{k}t_1,t_2}R^{nu}_{-\vec{k}t_1,t_2})d\vec{k},$$

$$S^{\vec{E}_0\vec{E}_0^\dagger}_{t_1,t_2} = \int \vec{k}\vec{k}C^{\psi\psi^\dagger}_{\vec{k}t_1,t_2}C^{nn}_{-\vec{k}t_1,t_2}d\vec{k}.$$

The contributions by $\vec{E}_0$ to the self-energy and nonlinear noise components at $\vec{k}\neq 0$ are

$$\Sigma^{\psi\psi^\dagger}_{\vec{k}t_1,t_2}\Big|_{\vec{E}_0} = -\vec{k}\cdot R^{\vec{E}_0\vec{E}_0}_{t_1,t_2}\cdot\vec{k}C^{nn}_{\vec{k}t_1,t_2}/k^2 - i\vec{k}\cdot C^{\vec{E}_0\vec{E}_0^\dagger}_{t_1,t_2}\cdot\vec{k}R^{nu}_{\vec{k}t_1,t_2}/k^2,$$

$$\Sigma^{un}_{\vec{k}t_1,t_2}\Big|_{\vec{E}_0} = i\vec{k}\cdot R^{\vec{E}_0^\dagger\vec{E}_0^\dagger}_{t_1,t_2}\cdot\vec{k}C^{\psi\psi^\dagger}_{\vec{k}t_1,t_2} + \text{c.c.} + i\vec{k}\cdot C^{\vec{E}_0\vec{E}_0^\dagger}_{t_1,t_2}\cdot\vec{k}R^{\psi^\dagger\psi^\dagger}_{\vec{k}t_1,t_2}/k^2 + \text{c.c.},$$

$$S^{\psi\psi^\dagger}_{\vec{k}t_1,t_2}\Big|_{\vec{E}_0} = \vec{k}\cdot C^{\vec{E}_0\vec{E}_0^\dagger}_{t_1,t_2}\cdot\vec{k}C^{nn}_{\vec{k}t_1,t_2}/k^4,$$

$$S^{uu}_{\vec{k}t_1,t_2}\Big|_{\vec{E}_0} = \vec{k}\cdot C^{\vec{E}_0\vec{E}_0^\dagger}_{t_1,t_2}\cdot\vec{k}C^{\psi^\dagger\psi}_{\vec{k}t_1,t_2} + \text{c.c.}$$

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[18]The Fourier transform $R(\vec{k},\omega)=\int_{-\infty}^{\infty} dt\, e^{i\omega t}R_{\vec{k}}(t)$ is well defined for $\mathrm{Im}\,\omega > -\min(\gamma_k^\nu)$, where γ_k^ν are the fundamental damping rates of the response functions discussed in the text. For stable poles, i.e., $\gamma_k^\nu > 0$, the contour of integration over ω for the inverse transform can be taken to be the real ω axis; for unstable poles $\gamma_k^\nu < 0$ the contour must lie *above* the most unstable pole. This also ensures the causality condition on $R_k(t-t')=R_{kt,t'}$.

[19]Note that Eqs. (3.3) and (3.7) can also be written in the more compact form of Eq. (4.11).

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