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Wave packets and localized pulses — a dual approach

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It is shown that an interesting duality exists for wave packets and localized pulses as their representations in $(\vec{x},t)$ and $(\vec{k},\omega)$ spaces are compared. A wave packet in $(\vec{x},t)$ space has a slowly varying envelope. This corresponds to a narrow spectrum in $(\vec{k},\omega)$ space. On the other hand, a localized pulse in $(\vec{x},t)$ space corresponds to a slowly tapering spectrum in $(\vec{k},\omega)$ space. The analysis of wave-packet propagation is usually carried out by means of ray theory. It is well known that ray tracing, in spite of its limitations, provides a powerful tool for the analysis of wave-packet propagation in dispersive, weakly inhomogeneous media. Similarly, it is shown here that localized pulses in inhomogeneous, weakly dispersive media, can be analyzed, using the concepts of dual-dispersion equation, dual-ray tracing, and group slowness. Hamilton's equations of geometrical optics are the Euler-Lagrange equations of the variational form known as Fermat's principle. In an analogous manner the dual Fermat principle is introduced here, being equivalent to the dual-ray equations. The method proposed here facilitates the analysis of localized pulses in inhomogeneous, weakly dispersive media. The dual-ray tracing provides a clue to the way in which the spectrum of a pulse in $(\vec{k},\omega)$ space, hence its shape in $(\vec{x},t)$ space, change in the course of propagation.

I. INTRODUCTION AND SUMMARY

Certain structural properties of waves and their spectra have been known for a long time, in the context of Fourier transforms. But although the Fourier transform is a powerful tool, it cannot properly be used when the constitutive parameters are described in a mixed form, containing variables belonging to both the original and the transform spaces. For a restricted class of problems, namely quasi-plane waves in dispersive, weakly inhomogeneous media, ray theory provides an adequate approximation. The price for using ray theory is the loss of information regarding wave amplitude and polarization, although intensity can be salvaged, to some extent. For ray theory we define a space $(\vec{x}, t; \vec{k}, \omega)$, essentially $(\vec{x}, t)$ space in which $\vec{k}, \omega$ are slowly varying parameters, and a corresponding $(\vec{k}, \omega; \vec{x}, t)$ space. Their interesting structural properties prompted the study of the present dual procedure, in which the roles of these two spaces are exchanged. Consequently, as shown subsequently, a consistent dictionary develops, as in the following examples:

Ray Theory	Dual-Ray Theory
$\vec{x}$	$\vec{k}$
t	ω
$\vec{k}$	$\vec{x}$
ω	t
Wave packet	Dual wave packet
Narrow spectrum	Localized pulse
Group velocity	Group slowness
Dispersion equation $F(\vec{k}, \omega; \vec{x}, t) = 0$	Dual-dispersion equation $G(\vec{x}, t; \vec{k}, \omega) = 0$
Fermat principle	Dual-Fermat principle

Once the dual structure is clearly defined, the physical import can be discussed. In essence, ray tracing describes the transfer of energy in space, for dispersive, weakly inhomogeneous media. Dual-ray tracing describes the transfer of energy for localized pulses in inhomogeneous, weakly dispersive media. It provides general information concerning the evolution of the pulse and its associated spectrum. In this introductory section we present the fundamental dual structure, summarize the ray theory, and present (without proof) the dual concepts. Then the physical implications are discussed. It is hoped that this mode of presentation will provide the reader with an overall view, without the need of going through the detailed derivations, given later.

A. The fundamental duality

Before any attempt is made to investigate the physical contents of the problem at hand, the formalism on which the discussion is based must be displayed. As an example of a physical model, consider the Maxwell equations in space $(\vec{x}, t)$, in a sourceless and lossless domain,[1]

$$\nabla \times \vec{H} - \partial\vec{D}/\partial t = 0\,, \qquad \nabla \times \vec{E} + \partial\vec{B}/\partial t = 0\,, \tag{1}$$

where $\vec{E} = \vec{E}(\vec{x}, t)$, etc., depend on $\vec{x}, t$. The Fourier transform of (1) is given by

$$\vec{k} \times \vec{\mathcal{H}} + \omega\vec{\mathfrak{D}} = 0\,, \qquad \vec{k} \times \vec{\mathcal{E}} - \omega\vec{\mathfrak{B}} = 0\,, \tag{2}$$

where $\vec{\mathcal{E}} = \vec{\mathcal{E}}(\vec{k}, \omega)$ is the transform of $\vec{E}(\vec{x}, t)$, etc. The two sets (1) and (2) are equivalent through the Fourier transformation. As usual, constitutive relations are necessary to render (1) or (2) determinate. Let us presume that the constitutive relations are given, by means of a model, or as experimental data in the form

$$\vec{d}=\epsilon(\vec{k},\omega;\vec{x},t)\cdot\vec{e}\,,\qquad \vec{b}=\mu(\vec{k},\omega;\vec{x},t)\cdot\vec{h}\,, \tag{3}$$

where ϵ, μ are dyadics and different symbols are now used for the fields. Strictly speaking, these fields, being a function of the mixture $\vec{k},\omega,\vec{x},t$ cannot be properly used in conjunction with either (1) or (2). However, for a special class of problems the argument can proceed in a meaningful way. Let us consider the special case where ϵ, μ vary slowly in $(\vec{x},t)$ (in the sense described below), i.e., the medium is assumed to be practically constant over a distance and a time on the order of a wavelength, a period, respectively. This is consistent with defining solutions of (1) describing locally and instantaneously quasi-plane waves, of the form

$$\vec{E}=\vec{E}_0(\vec{x},t)e^{i\theta(\vec{x},t)}\,, \tag{4}$$

etc. The slow variation implies that the amplitudes, e.g., $\vec{E}_0$, are kept constant with respect to time and space differentiation $\partial/\partial t$, $\partial/\partial\vec{x}$. Also, to comply with plane wave theory, θ satisfies

$$\partial\theta/\partial\vec{x}=\vec{k}\,,\qquad \partial\theta/\partial t=-\omega\,. \tag{5}$$

Consequently, it is now possible to define (3) as operational expressions in $(\vec{x},t)$ space,

$$\vec{D}=\epsilon(-i\partial/\partial\vec{x},i\partial/\partial t;\vec{x},t)\cdot\vec{E}\,,\qquad \vec{B}=\mu(-i\partial/\partial\vec{x},i\partial/\partial t;\vec{x},t)\cdot\vec{H}\,, \tag{6}$$

which can be substituted into (1). The definition of constitutive operators has been considered before.[2,3] Substitution of (4)-(6) in (1) yields

$$\vec{k}\times\vec{H}_0+\omega\epsilon(\vec{k},\omega;\vec{x},t)\cdot\vec{E}_0=0\,,\qquad \vec{k}\times\vec{E}_0-\omega\mu(\vec{k},\omega;\vec{x},t)\cdot\vec{H}_0=0\,, \tag{7}$$

constituting a set of six scalar homogeneous algebraic equations for the unknowns $\vec{E}_0,\vec{H}_0$. The condition of solubility requires that the determinant of (7) vanishes, yielding a scalar function

$$F(\vec{k},\omega;\vec{x},t)=0\,, \tag{8}$$

referred to as the dispersion equation. This is the starting point for ray theory.

In the present study the dual situation is investigated. Let us replace (3) by constitutive relations in inhomogeneous media which are weakly dependent on $\vec{k},\omega$, written in the form

$$\vec{d}=\epsilon(\vec{x},t;\vec{k},\omega)\cdot\vec{e}\,,\qquad \vec{b}=\mu(\vec{x},t;\vec{k},\omega)\cdot\vec{h}\,. \tag{9}$$

Again, (9) does not in general fit any of the forms (1) or (2). Let us assume special solutions of (2), which are analogous to (4) and constitute spectra of the form

$$\vec{\mathcal{E}}(\vec{k},\omega)=\vec{\mathcal{E}}_0(\vec{k},\omega)e^{i\varphi(\vec{k},\omega)}\,, \tag{10}$$

etc., with

$$\partial\varphi/\partial\vec{k}=\vec{x}\,,\quad \partial\varphi/\partial\omega=-t\,, \tag{11}$$

and with a similar stipulation of slow variation which specifies $\vec{\mathcal{E}}_0$ as a constant with respect to $\vec{k},\omega$ differentiation. In view of (9)-(11), we now have in $(\vec{k},\omega)$ space the operational forms

$$\vec{\mathfrak{D}}=\epsilon(-i\partial/\partial\vec{k},i\partial/\partial\omega;\vec{k},\omega)\cdot\vec{\mathcal{E}}\,,\qquad \vec{\mathfrak{B}}=\mu(-i\partial/\partial\vec{k},i\partial/\partial\omega;\vec{k},\omega)\cdot\vec{\mathcal{H}}\,. \tag{12}$$

Substituting (12) into (2) yields

$$\vec{k}\times\vec{\mathcal{H}}_0+\omega\epsilon(\vec{x},t;\vec{k},\omega)\cdot\vec{\mathcal{E}}_0=0\,,\qquad \vec{k}\times\vec{\mathcal{E}}_0-\omega\mu(\vec{x},t;\vec{k},\omega)\cdot\vec{\mathcal{H}}_0=0\,, \tag{13}$$

which should be compared to (7). Before we can proceed, the factors $\vec{k},\omega$ in (13) must be dealt with. A step is now taken which will be justified later on. We differentiate (13) with respect to ω, exploiting the slow variation of $\vec{\mathcal{H}}_0$, $\vec{\mathcal{E}}_0$, ϵ, μ with respect to $\vec{k},\omega$. Defining the group slowness as

$$\frac{d\vec{k}}{d\omega}=\frac{\partial t}{\partial\vec{x}}=\vec{s}\,, \tag{14}$$

(13) becomes

$$\vec{s}\times\vec{\mathcal{H}}_0+\epsilon\cdot\vec{\mathcal{E}}_0=0\,,\qquad \vec{s}\times\vec{\mathcal{E}}_0-\mu\cdot\vec{\mathcal{H}}_0=0\,, \tag{15}$$

which is a set of six homogeneous equations in $\vec{\mathcal{H}}_0$, $\vec{\mathcal{E}}_0$, prescribing that the determinant of (15) vanishes. Hence we obtain

$$\overline{G}(\vec{s},\vec{x},t;\vec{k},\omega)=0\,, \tag{16}$$

which in view of $\vec{s}$ (14) is still a differential equation. In order to have the counterpart of $F=0$ (8), the integral of (16) is considered

$$G(\vec{x},t;\vec{k},\omega)=0\,, \tag{17}$$

and henceforth referred to as the dual-dispersion equation. The integration of (16) is not always necessary, but in any case, the derivation of the dual-dispersion equation (17) is much simpler than a direct solution of the Maxwell equations (1).

B. Recapitulation of ray theory

The outline given above provides the basis for both ray theory and dual-ray theory. Ray theory has been discussed by many authors[3-5] on various levels of mathematical rigor and physical applications. Our aim here is to outline in a straightforward way the derivation of Hamilton's equations of geometric optics. The dual-ray theory will be considered in the following subsection.

The starting point for ray theory is the dispersion Eq. (8). Reinstating the definition (5), this becomes a differential equation on θ

$$F(\partial\theta/\partial\vec{x}, -\partial\theta/\partial t; \vec{x}, t) = 0 , \tag{18}$$

usually referred to as the eikonal equation. Ray theory is a method for solving (18), hence only θ in (4) can be found. The intensity EE^* can be derived by using energy conservation arguments, but information concerning polarization is lost, in general. In spite of the limitations of ray theory it is a powerful tool for understanding the propagation of wave packets in dispersive, slowly varying inhomogeneous media, for which in general the full wave solution is not available.

Inasmuch as (18) is in general a complicated nonlinear equation, it is transformed into a set of first-order differential equations which is easier to integrate. This is the familiar method of characteristics. Instead of solving $F=0$ (18) directly, we shall consider $dF=0$, whose solution is $F=$const, and with proper initial conditions becomes $F=0$. Expanding dF and dividing throughout by $dt\neq 0$, $\partial F/\partial\omega\neq 0$, we obtain

$$\frac{\partial F/\partial\vec{k}}{\partial F/\partial\omega}\cdot\frac{d\vec{k}}{dt}+\frac{d\omega}{dt}+\frac{\partial F/\partial\vec{x}}{\partial F/\partial\omega}\cdot\frac{d\vec{x}}{dt}+\frac{\partial F/\partial t}{\partial F/\partial\omega}=0 . \tag{19}$$

For reasons discussed later, the group velocity is defined as

$$\vec{v}=\frac{d\vec{x}}{dt}=-\frac{\partial F/\partial\vec{k}}{\partial F/\partial\omega} . \tag{20}$$

Substitution of (20) into (19) prescribes

$$\frac{d\vec{k}}{dt}=\frac{\partial F/\partial\vec{x}}{\partial F/\partial\omega} , \qquad \frac{d\omega}{dt}=-\frac{\partial F/\partial t}{\partial F/\partial\omega} . \tag{21}$$

Thus, the system (20) and (21), satisfying (19), is tantamount to $F=0$ when proper initial conditions are applied to the solution. Defining

$$\theta(\vec{x},t)=\int_{P_0(\vec{x}_0,t_0)}^{P(\vec{x},t)}[\vec{k}\cdot d\vec{x}-\omega dt] , \tag{22}$$

the phase θ of the wave (4) can be reconstructed, hence finally (18) can be considered solved. The line integral (22) is independent of the integration path provided[6]

$$\frac{\partial}{\partial\vec{x}}\times\vec{k}=0 , \qquad \frac{\partial\vec{k}}{\partial t}+\frac{\partial\omega}{\partial\vec{x}}=0 . \tag{23}$$

Note that the first equation (23) is the Snell law of refraction.

We are now ready to discuss the physical implications of this mathematical model. The solution of (20), (21) provides paths $\vec{x}(t)$ on which the slowly varying $\vec{k}(t)$, $\omega(t)$, characterizing the wave packet, are known. The direction of the group velocity is tangential to the path $\vec{x}(t)$, indicating the direction of energy flow. Hence, although the detailed structure of the amplitude is lost, the convergence (or divergence) of rays indicates the way their intensity grows (or decreases).

C. Dual-ray theory

Substituting (11) into (17), it becomes clear that $G=0$ is a differential equation on φ. Its solution describes the argument of the spectrum (10) of a localized pulse. Conventional ray theory is not involved in the detailed description of the wave packet's envelope [$\vec{E}_0$ in (4)]. At best, the evolution of $|\vec{E}_0|$ can be inferred from the behavior of the rays. Similarly, in (10) the only assumption is that the power spectrum $\vec{\mathcal{E}}\cdot\vec{\mathcal{E}}^*$ is very wide, i.e., changing slowly as a function of $\vec{k}$, ω. Dual-ray theory describes the behavior of φ in inhomogeneous, weakly dispersive media. This provides a clue to the way an initially sharp pulse will deteriorate in such a medium. Using the dictionary given above and following the argument that led to (20) and (21), the dual-ray equations are obtained:

$$\frac{d\vec{k}}{d\omega}=-\frac{\partial G/\partial\vec{x}}{\partial G/\partial t}=\vec{s} , \qquad \frac{d\vec{x}}{d\omega}=\frac{\partial G/\partial\vec{k}}{\partial G/\partial t} , \qquad \frac{dt}{d\omega}=-\frac{\partial G/\partial\omega}{\partial G/\partial t} , \tag{24}$$

where the group slowness $\vec{s}$ has already been defined in (14). More detail will be supplied in a subsequent section.

We are now in the position to make a few observations regarding the physics of the problem.

As opposed to the full wave solution, and similar to conventional-ray theory, dual-ray theory is capable of describing the effects of a specific medium, but cannot supply the detailed description of a specific pulse. Knowing $G=0$ amounts to having a function $t=t(\vec{x};\vec{k},\omega)$ which for a special instant t_1 describes a surface $t_1=t(\vec{x};\vec{k},\omega)$ on which the pulse is found in $\vec{x}$ space. The weak dependence on $\vec{k},\omega$ introduces some ambiguity as to the exact location of the pulse. Although different spectral components propagate at different slowness value, as long as the dispersion is weak we expect to be able to locate the pulse within reasonable bounds. The counterpart of $G=0$ (17) is $F=0$ (8) which for a given frequency, characterizing a wave packet, describes a surface $\omega_1=\omega(\vec{k};\vec{x},t)$ in $\vec{k}$ space. The pertinent $\vec{k}$ is located on this surface. The weak inhomogeneity in $\vec{x},t$ affects the value of $\vec{k},\omega$ as prescribed by the ray equations (21). Similarly, solving the dual-ray equations (24) describes how $\vec{x},t$ change for various $\vec{k},\omega$, in a given medium. For conventional-ray tracing (20), the tangent to the ray defines the group velocity, indicating the direction of energy flow. Similarly, the tangent to $\vec{k}(\omega)$ defines the direction of the group slowness vector, indicating the direction in which the energy of various spectral components is moving. Consequently the dual rays describe the fission of the pulse in $\vec{x}$ space. The convergence of the dual rays in the vicinity of some value ω points to the fact that the energy density for this spectral part is increasing, i.e., $|\vec{\mathcal{S}}_0|$ (10) is peaking. Usually this is associated with broadening of the pulse in $\vec{x}$ space. This is another way of looking at the problem of pulse fission in $\vec{x}$ space.

II. DUAL STRUCTURE OF WAVE PACKETS AND LOCALIZED PULSES

The overview presented above needs more elaboration. In particular, we have introduced the key concepts of group velocity and group slowness without establishing their physical origin. A wave packet in homogeneous media is defined by a superposition of plane waves, of the form

$$f(\vec{x},t)=\int_{-\infty}^{\infty} d^3k\, d\omega\, \delta(F)\bar{f}(\vec{k},\omega)e^{i\vec{k}\cdot\vec{x}-i\omega t}\,, \qquad (25)$$

where eventually the infinite range of integration will be restricted to a narrow band spectrum. Here $\delta(F)$ (in the sense of the Dirac δ function) indicates that the pertinent dispersion equation $F(\vec{k},\omega)=0$, [i.e., (8) independent of $\vec{x},t$] is satisfied. The method used here is an extension of a technique given by Stratton.[1] It follows that (25) is actually a threefold integration and can be written as

$$f(\vec{x},t)=\int_{-\infty}^{\infty} d^3k\,\bar{f}(\vec{k},\omega[\vec{k}])e^{i\vec{k}\cdot\vec{x}-i\omega(\vec{k})t}\,. \qquad (26)$$

With t considered to be a parameter, (26) defines a three-dimensional Fourier transform and $\bar{f}e^{-i\omega(\vec{k})t}$ is the spectrum of the wave in question. Assuming a narrow band spectrum, the dispersion equation $\omega(\vec{k})$ can be expanded about a central value $\vec{k}_0$, keeping only the leading terms

$$\omega(\vec{k})=\omega_0(\vec{k}_0)+\frac{\partial\omega}{\partial\vec{k}_0}\cdot(\vec{k}-\vec{k}_0)\,. \qquad (27)$$

Substituting (27) into (26) yields

$$f(\vec{x},t)=e^{i\vec{k}_0\cdot\vec{x}-i\omega_0 t}\int d^3k\,\bar{f}e^{i(\vec{k}-\vec{k}_0)\cdot(\vec{x}-\partial\omega/\partial\vec{k}_0 t)}\,, \qquad (28)$$

displaying, as in (4), the carrier plane wave and the amplitude (or envelope, or modulation), represented by the integral (28). The amplitude remains constant on the path $\vec{x}-\vec{v}t=\text{const}$, where

$$\vec{v}=\frac{d\vec{x}}{dt}=\frac{\partial\omega}{\partial\vec{k}}=-\frac{\partial F/\partial\vec{k}}{\partial F/\partial\omega}\,, \qquad (29)$$

justifying the definition of the group velocity (20) for weakly inhomogeneous media.

We are now ready for the definition of the dual concepts, the dual wave packet and the group slowness. The counterpart of (25), for inhomogeneous nondispersive media, is a spectrum given in the form

$$g(\vec{k},\omega)=\int d^3x\, dt\, \delta(G)\bar{g}(\vec{x},t)e^{i\vec{k}\cdot\vec{x}-i\omega t}\,, \qquad (30)$$

where $\delta(G)$ signifies that $G=0$ (17), corresponding to $t=t(\vec{x})$ is satisfied. Let us consider a localized pulse such that the integrand in (30) is significant in the vicinity of a central value $\vec{x}_0$, $t_0=t(\vec{x}_0)$ only. Corresponding to (27) we now have

$$t(\vec{x})=t_0+\frac{\partial t}{\partial\vec{x}_0}\cdot(\vec{x}-\vec{x}_0)\,. \qquad (31)$$

Substituting (31) in (30), we obtain

$$g(\vec{k},\omega)=e^{i\vec{k}\cdot\vec{x}_0-i\omega t_0}\int d^3x\,\bar{g}(\vec{x},t[\vec{x}])e^{i(\vec{x}-\vec{x}_0)\cdot(\vec{k}-\partial t/\partial\vec{x}_0\omega)}\,, \qquad (32)$$

displaying the same structure as (10). This specifies the spectrum of the localized pulse as a dual-wave packet, possessing an amplitude which remains constant on the path $\vec{k}-\vec{s}\omega=\text{const}$, where $\vec{s}$ the group slowness, has been defined in (14). Because of the localized pulse structure in (32), i.e., small interval of integration, the amplitude of the dual wave packet tapers off slowly as a function of $\vec{k}, \omega$. This justifies the assumptions we made on $\mathscr{E}_0$ (10).

The present section is more than an exercise in Fourier transforms, because the physical properties are included via the dispersion equation $F(\vec{k}, \omega) = 0$ [or equivalently $\omega(\vec{k})$], and the dual-dispersion equation $G(\vec{x}, t) = 0$ [or equivalently $t(\vec{x})$]. In (25)–(29) this prescribes values $\vec{k}, \omega$ which satisfy $F = 0$. Similarly in (30)–(32), for a given time t_0 the pulse is spread in an interval $\vec{x}_0 \pm \Delta\vec{x}$ about a central value $\vec{x}_0$, such that $G(\vec{x}_0, t_0) = 0$ is satisfied. There is a slight difference between the two cases which should be mentioned. Since $G = 0$ (17) is the integral of (16), there exists a constant of integration which permits an arbitrary choice for the initial location of the pulse.

III. THE DUAL-DISPERSION EQUATION AND RAY TRACING

A. Theory

The manner in which the group slowness concept appears as the counterpart of the group velocity suggests that ray tracing as well has its analog. This is the dual-ray tracing theory. It describes the propagation of localized pulses in inhomogeneous weakly dispersive media. Dual-ray theory, like conventional ray theory, rather than dealing with specific signals, describes the effects induced by the medium at hand. A cursory outline has been given above, and necessary additional detail is supplied here.

Starting with the constitutive relations (9) and assuming solutions of the form (10), with properties (11) and slow variation of the amplitude, the operational forms (12) are obtained. This yields (13) which still contains the rapidly varying factors $\vec{k}, \omega$, which must be properly eliminated. The argument presented in the former section for inhomogeneous nondispersive media led to the concept of group slowness $\vec{s} = \partial t/\partial\vec{x}$, and described the amplitude of the dual-wave packet as constant on the line $\vec{k} - \vec{s}\omega = \text{const}$ in $\vec{k}$ space. This corresponds to $\vec{s} = d\vec{k}/d\omega$. These results are adopted for inhomogeneous weakly dispersive media, justifying the use of (14) in (13). Finally this leads to the definition of the dual-dispersion equation (17). The transition to (24) is now derived in more detail.

Reinstating (11) in $G = 0$ (17) yields the analog of (18):

$$G(\partial\varphi/\partial\vec{k}, -\partial\varphi/\partial\omega; \vec{k}, \omega) = 0 , \tag{33}$$

whose solution is sought. Following the same line of argument as for the conventional ray-tracing theory, the counterpart of (19) is obtained

$$\frac{\partial G/\partial\vec{x}}{\partial G/\partial t}\cdot\frac{d\vec{x}}{d\omega} + \frac{dt}{d\omega} + \frac{\partial G/\partial\vec{k}}{\partial G/\partial t}\cdot\frac{d\vec{k}}{d\omega} + \frac{\partial G/\partial\omega}{\partial G/\partial t} = 0 , \tag{34}$$

whose solution, with proper initial conditions, satisfies (33). Using (14) in (34) prescribes (24) as the set of dual-ray equations equivalent to (33). Similarly to (22) $\varphi(\vec{k}, \omega)$ is represented as a line integral

$$\varphi(\vec{k}, \omega) = \int_{P_0(\vec{k}_0, \omega_0)}^{P(\vec{k}, \omega)} (\vec{x}\cdot d\vec{k} - t\,d\omega) , \tag{35}$$

and stipulating the uniqueness conditions

$$\frac{\partial}{\partial\vec{k}} \times \vec{x} = 0 , \qquad \frac{\partial\vec{x}}{\partial\omega} + \frac{\partial t}{\partial\vec{k}} = 0 , \tag{36}$$

(11) is satisfied. Hence φ can be reconstructed and (33) can be considered solved. By inspection of (23) and (36) it is noticed that $(\partial/\partial\vec{k}) \times \vec{x} = 0$ prescribes a dual Snell law of refraction for the dual rays in $\vec{k}$ space.

B. Examples

The solution of (20) and (21), or (24) can be conveniently obtained using numerical methods (e.g., the Runge-Kutta method[7]). Even though the present study is theoretical, a few simple examples are presented to highlight the new method and its physical import. In (15) let us consider an isotropic, inhomogeneous, and nondispersive medium, such that $\mu = \mu_0 = \text{const}$ and $\epsilon = \epsilon(\vec{x}, t)$, where μ_0, ϵ are scalars. Consequently, (16) is derived in the form

$$\overline{G} = \det(\vec{s} \times \vec{s} \times \tilde{I} - \epsilon\mu_0\tilde{I}) = 0 , \tag{37}$$

where $\tilde{I}$ is the idemfactor dyadic. For transverse one-dimensional fields (37) reduces to

$$\vec{s} = \frac{dt}{dx} = \pm[\mu_0\epsilon(x, t)]^{1/2} . \tag{38}$$

The simplicity of (38) is somewhat misleading since for the full wave solution (1) with arbitrary $\epsilon(\vec{x}, t)$ usually leads to complicated differential equations. It is due to the present method of defining localized pulses that (38) is obtained in this simple form. Clearly, for a large variety of cases (38) should be easily integrated, yielding the dual-dispersion equation (17). For example, consider

$$\epsilon(x, t) = \epsilon_0(1 + \alpha x)^2 , \tag{39}$$

in the domain $\epsilon > 0$ with α an arbitrary real parameter. The integration yields

$$G = \mp ct + A + x + \frac{\alpha}{2}x^2 = 0, \quad c = (\mu_0\epsilon_0)^{-1/2} , \tag{40}$$

and for $x=0$ at $t=0$ the constant A vanishes. From (38) and (40) it is clear that the pulse slows down as the dielectric constant increases. An interesting example is provided by the case of a "material wave", where a change in the dielectric parameter is propagating through the medium with a velocity u, according to

$$s=\frac{dt}{dx}=\pm\frac{1}{c}f(x-ut)\,. \tag{41}$$

A simple instructive example is provided by $f=U(x-ut)$, where U is a unit step function with its discontinuity at $x=ut$. As long as the localized pulse is in the region $x>ut$ its location is governed by $G=x\pm ct=0$, i.e., it is not affected by discontinuities in regions not occupied by the pulse. The effects introduced by gradients in ϵ can be studied in terms of the example $f=1+[\alpha(x-ut)]^2$. At $x=ut$ the pulse moves according to $x\pm ct=0$, but as soon as it leaves this dielectric well s increases and the pulse is slowed down. Eventually it will move with the velocity u of the material wave. A simpler case of the above class is provided by a linear function $f=1+\alpha(x-ut)$ with $\alpha>0$, $x\geqslant ut$, and only propagation in the positive x direction is considered. For this case we solve the differential equation $ct'+\alpha ut=1+\alpha x$. The homogeneous solution is $t=Ae^{-(\alpha u/cx)}$, the particular solution is $t=x/u+B$, $A,B=\text{const}$, hence for x large enough, such that the exponential is negligible, we have

$$G=x-ut+\text{const}=0\,. \tag{42}$$

Again, the pulse will slow down to the velocity of propagation of the material wave. The practical implementation of the phenomenon might be very complicated but the way it has been derived from a simple example demonstrates the potential of the new method.

For the above examples (24) is trivial, $dt/dw=0$, $d\vec{x}/d\omega=0$ simply indicating the absence of dispersion effects. Weak dispersion will be introduced by modifying (39) to the form

$$\epsilon=\epsilon_0(1+\alpha x)^2(1+\beta\omega)^2\,, \tag{43}$$

where β is a small real constant. Accordingly,

$$G=\mp ct+x(1+\alpha x/2)(1+\beta\omega)\,, \tag{44}$$

and (24) prescribe

$$\frac{dt}{d\omega}=\pm\beta x(1+\alpha x/2)/c\,, \tag{45}$$

displaying the change of t as a function of ω along the dual ray defined by (24):

$$s=\frac{dk}{d\omega}=\pm(1+\alpha x)(1+\beta\omega)/c\,. \tag{46}$$

Integrating (46) yields the equation of the dual ray

$$k=\pm(1+\alpha x)\omega(1+\beta\omega/2)/c\,. \tag{47}$$

For a given location x, (45) prescribes that t, the time when the pulse exists there, is proportional to $\beta\omega$. Hence different spectral components propagate at different slowness rates [as demonstrated by (46) as well]. For large x, (45) shows that the time effect grows, displaying the fission of the pulse as it propagates in the medium. The effect can be decreased by having smaller β, but even in a homogeneous medium defined by $\alpha=0$ the effect will persist. Finally, a somewhat oversimplified example will be considered for demonstrating the construction and evaluation of dual rays. Consider a homogeneous dispersive medium whose dual-dispersion equation is given by

$$G=\vec{x}\cdot\vec{x}-[c(\vec{k})]^2t^2=0\,,\qquad c(\vec{k})=c(1+\gamma k^2),\quad \gamma>0\,, \tag{48}$$

where $c=(\mu_0\epsilon_0)^{-1/2}$ and γ is small enough such that $\gamma k^2\ll 1$ for the range of k^2 considered here, otherwise the assumption of weak dispersion is violated. From (24) we have

$$\frac{d\vec{k}}{d\omega}=\frac{\vec{x}}{t[c(\vec{k})]^2}\approx\frac{\vec{x}}{tc^2}\,,\qquad \frac{dt}{d\omega}=0\,,\qquad \frac{d\vec{x}}{d\omega}=\frac{2tc\gamma\vec{k}}{c(\vec{k})}\approx 2t\gamma\vec{k}\,. \tag{49}$$

The central position of the pulse is given by values $\vec{x},t$ satisfying (48) for some mean value of k^2. According to (49), $t=\text{const}$, and $\vec{x}$ is affected by the dual-ray tracing. Let us check a few values of $\vec{k}$ presumably existing in the spatial spectrum of the pulse at time t. From the approximate expressions in (49) it is clear that the dual ray starting with an initial value $\vec{k}$ which is parallel to $\vec{x}$ will be a straight line. For an initial $\vec{k}$ oblique with respect to the initial $\vec{x}$ the incremental $d\vec{k}/d\omega$, $d\vec{x}/d\omega$ change the directions of $\vec{k}$ and $\vec{x}$ such that they gradually move towards some common direction. It follows that the rays (i.e., the segments $d\vec{k}$ put end to end) diverge less as ω increases. This demonstrates the way fission of the pulse takes place as various spectral components move in different directions. The convergence of the dual rays also suggests that the amplitudes $\vec{\mathcal{E}}_0(\vec{k},\omega)$ (10) increase as ω increase. More concrete examples require machine computation.

IV. THE DUAL-FERMAT PRINCIPLE

The restricted Fermat principle for time-independent media is stated in many textbooks on optics and electromagnetic theory (see, for example, Jones[8]). Verbally stated, it says that the travel time of wave packets between two fixed points in space is stationary (maximum, minimum, or inflection). The generalized form, applicable to time varying media as well, has been given by Synge,[9] in variational form

$$\delta \int_{P_a}^{P_b} (\vec{k}\cdot d\vec{x} - \omega dt) = 0\,, \tag{50}$$

where $P_a(\vec{x}_a, t_a), P_b(\vec{x}_b, t_b)$ are fixed endpoints. Since $\vec{k}, \omega, \vec{x}, t$ are interrelated through the dispersion equation (8), the integrand (50) must be augmented, using a Lagrange multiplier function λ,

$$\delta \int_{P_a}^{P_b} \left(\vec{k}\cdot\frac{d\vec{x}}{d\tau} - \omega\frac{dt}{d\tau} + \lambda(\tau)F(\vec{k}, \omega; \vec{x}, t)\right) d\tau = 0\,, \tag{51}$$

where τ is an integration parameter. Of course, the addition of $F=0$ into (51) does not change the value of the integrand, but derivatives are affected. The associated Euler-Lagrange equations of (51) are obtained as

$$\frac{d\vec{x}}{d\tau} = -\lambda(\tau)\frac{\partial F}{\partial \vec{k}}, \quad \frac{d\vec{k}}{d\tau} = \lambda(\tau)\frac{\partial F}{\partial \vec{x}}, \quad \frac{dt}{d\tau} = \lambda(\tau)\frac{\partial F}{\partial \omega}, \quad \frac{d\omega}{d\tau} = -\lambda(\tau)\frac{\partial F}{\partial t}. \tag{52}$$

Dividing by $dt/d\tau$ we obtain the Hamilton equations (20) and (21) without additional assumptions. The remaining equation $dt/d\tau = \lambda(\partial F/\partial\omega)$ is not relevant to ray tracing although its physical interpretation is of considerable interest, as shown later.

Exploiting the formal resemblance of (20), (21), and (24), suggests that (24) too be derivable from a variational principle. Using the dictionary developed above, we obtain

$$\delta \int_{P_a}^{P_b} \left(\vec{x}\cdot\frac{d\vec{k}}{d\Omega} - t\frac{d\omega}{d\Omega} + \Lambda(\Omega)G(\vec{x}, t; \vec{k}, \omega)\right) d\Omega = 0\,, \tag{53}$$

where $P_a(\vec{k}_a, \omega_a)$, $P_b(\vec{k}_b, \omega_b)$ are fixed endpoints. The Euler-Lagrange equations of (53) are then given by

$$\frac{d\vec{k}}{d\Omega} = -\Lambda(\Omega)\frac{\partial G}{\partial \vec{x}}, \quad \frac{d\vec{x}}{d\Omega} = \Lambda(\Omega)\frac{\partial G}{\partial \vec{k}}, \quad \frac{d\omega}{d\Omega} = \Lambda(\Omega)\frac{\partial G}{\partial t}, \quad \frac{dt}{d\Omega} = -\Lambda(\Omega)\frac{\partial G}{\partial \omega}. \tag{54}$$

Again, division by $d\omega/d\Omega$ yields (24) and $d\omega/d\Omega = \Lambda(\partial G/\partial t)$ itself is not material for the dual-ray tracing procedure, although it would be desirable to have some physical interpretation for it. It is therefore natural to call (53) the dual-Fermat principle. This formal construct needs now physical interpretation. For that, we turn back to the restricted Fermat principle for time-independent media, which can be formally stated as

$$\delta \int_{\vec{x}_a}^{\vec{x}_b} \vec{k}\cdot d\vec{x} = 0\,. \tag{55}$$

For this case (21) prescribes a fixed frequency. Dividing (55) by ω yields $\vec{k}/\omega$ in the integrand, which is the reciprocal of the phase velocity. Consequently $(\vec{k}/\omega)\cdot d\vec{x} = dt$ has the dimension of time and (55) is equivalent to $\delta[t(b) - t(a)] = 0$, conforming with the verbal statement at the beginning of this section. Corresponding to (55) we have for frequency independent media

$$\delta \int_{\vec{k}_a}^{\vec{k}_b} \vec{x}\cdot d\vec{k} = 0\,. \tag{56}$$

For this case (24) prescribes $t=\text{const}$ along the dual ray, hence (56) can be written as $\delta\int \vec{x}/t\cdot d\vec{k} = \delta\int d\omega = 0$ or $\delta[\omega(b) - \omega(a)] = 0$. Thus the (restricted) dual-Fermat principle prescribes a stationary frequency difference between two points on the dual ray. To understand this in a more physical way, consider two adjacent points. For fixed $\Delta\vec{k} = \vec{k}_b - \vec{k}_a$ and stationary value of $\Delta\omega = \omega_b - \omega_a$ this means that the slowness function $\vec{s} = d\vec{k}/d\omega$ is stationary. For minimum $\Delta\omega$ this means that the pulse moves at the highest possible speed (subject to $G=0$). This conforms with the original idea of the Fermat principle, prescribing a minimum travel time.

The interpretation of (51), hence also (53) is much more difficult. Consider (51) first. Since the time interval $t(b) - t(a)$ in (50) is fixed we have to look for another quantity to replace the stationary travel time of the restricted Fermat principle. Censor[5,10] proposed the interpretation that in general the proper time[11] (in the relativistic sense) is stationary. For an observer attached to the wave packet the proper time τ is related to laboratory time t by

$$\frac{d\tau}{dt} = (1 - \vec{v}\cdot\vec{v}/c^2)^{1/2} = \frac{1}{\gamma}\,, \tag{57}$$

where $\vec{v}$ is the group velocity of the wave packet. This is motivated by the observation that for fixed dt in (57), $d\tau$ will be minimum for maximum $\vec{v}$. The idea that the wave packet moves with maximum velocity conforms with the minimum travel time concept in the restricted Fermat principle. A plausible interpretation of (50) follows in a consistent way. Writing the integrand (50) in the form

$$-[(\omega - \vec{k}\cdot\vec{v})\gamma]\frac{dt}{\gamma d\tau}\, d\tau = -\Omega d\tau\,, \tag{58}$$

the quantity in brackets is recognized as the relativistic transformation formula for the frequency, hence in the proper frame we have Ω. This quantity will be minimized as $\vec{v}$ is increased. Similarly, the generalized dual-Fermat principle (53) can be interpreted in a way which links it with the restricted form (56). Here again (53) prescribes fixed endpoints such that $\omega(b)-\omega(a)$ = const, and therefore we have to look for a different quantity which is stationary on the dual ray. Motivated by (57), we now define

$$\frac{d\Omega}{d\omega}=\left(1-\frac{\vec{u}\cdot\vec{u}}{c^2}\right)^{1/2}=\frac{1}{\gamma},\qquad \vec{u}=\vec{s}/(\vec{s}\cdot\vec{s}), \tag{59}$$

where $d\Omega$ is the element of proper frequency for an observer attached to the localized pulse. To understand the physical justification for (59) we recall that (57) describes the relativistic time dilatation phenomenon. This is derived by differentiating the time transformation formula $t=\gamma(\tau+\vec{v}\cdot\vec{x}'/c^2)$ holding $\vec{v}\cdot\vec{x}'$ fixed, where $\vec{x}'$ is the position vector in the proper frame. Similarly, differentiating the frequency transformation formula $\omega=\gamma(\Omega+\vec{k}'\cdot\vec{u})$ with $\vec{k}'\cdot\vec{u}$ = const yields (59). We now rewrite the generalized dual-Fermat principle in the form

$$\delta\int\vec{x}\cdot d\vec{k}-t\,d\omega=\delta\int[(\vec{x}\cdot\vec{s}-t)\gamma]d\Omega, \tag{60}$$

and further modify the integrand to obtain

$$\vec{s}\cdot[(\vec{x}-\vec{u}t)\gamma]\,d\Omega=\vec{s}\cdot\vec{x}'d\Omega=\vec{x}'\cdot dk', \tag{61}$$

where $\vec{x}'$ is the position vector and $d\vec{k}'$ is the element of $\vec{k}'$ for an observer attached to the localized pulse. This representation conforms with (56), hence we can finally state that along the dual rays the proper frequency is stationary. Again, from (59) this means that the pulse moves at the highest speed to minimize the travel time.

V. CONCLUSIONS

Propagation of localized pulses in inhomogeneous weakly dispersive media is discussed, using Maxwell's equations of the electromagnetic field as a concrete physical model. It is shown that a duality exists between the present case and conventional ray tracing in dispersive weakly inhomogeneous media. The analysis leads to the definition of dual-ray theory, the group slowness concept, the dual-Fermat principle, etc. At each stage of the discussion the formal analogy helps in choosing the correct definitions and relations, and helps in understanding the physical import of the new concepts. Some simple examples are given, however, for specific applications machine computations are necessary. The present discussion is confined to linear lossless systems. The implications for lossy and nonlinear media will be considered in the future.

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