Exactly solvable nonlinear model for a Smith-Purcell free-electron laser

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We present a model for a Smith-Purcell free-electron laser, including nonlinear effects, that can be solved in closed form. The model is shown to obey the pendulum equation. The average electron efficiency calculated by means of this model is compared with that obtained from an exact numerical calculation and from a simpler "phase-space" model for a wide range of physical parameters.

In recent years, efforts have been made to extend the usefulness of laser, maser, and microwave tube technologies to make accessible a greater portion of the electromagnetic spectrum. These efforts have spawned a variety of devices based on electron beams interacting synchronously with periodic structures, known collectively as *free-electron lasers*.[1–3] In this Communication we discuss a device in this category referred to in the literature as a Smith-Purcell free-electron laser or *orotron*[4] (also referred to as a *ledatron*[5] or *diffraction radiation generator*[6]), present a model for its operation that includes crucial nonlinear effects, and perform calculations related to the experiment described in the preceding paper.[7]

The laser consists of three basic elements: an electron beam, a metal grating over which it passes, and a resonant cavity around the beam and grating. Its operation is based on the Smith-Purcell effect.[8,9] Electrons in the beam induce image charges in the grating teeth; since these charges are, in effect, bound by the metal surface, they radiate energy by virtue of their constrained motion. Because the grating can absorb momentum from the beam, the ordinary resonant condition for beam-radiation interaction, $\omega = \vec{k}\cdot\vec{v}$, becomes $\omega = (\vec{k}+\vec{K})\cdot\vec{v}$, where ω is the frequency of the radiation, $\vec{k}$ is its wave vector, $\vec{v}$ is the beam velocity, and $\vec{K} = 2\pi n\hat{y}/l$ is a "reciprocal lattice vector" for a grating of period l in the y direction, where n is an integer. This equation can be rewritten as $n\lambda = l(c/v - \cos\theta)$, for a beam velocity $\vec{v} = v\hat{y}$, where λ is the radiation wavelength and θ is the angle between $\vec{k}$ and $\hat{y}$. In this form, the equation is referred to as the Smith-Purcell condition.[8]

The Smith-Purcell effect is essentially a one-electron effect, and indeed the orotrons that have been constructed to date[4–7] utilize low-density beams. The energy supplied to the radiation field is thus largely kinetic energy of the electrons, and plasma effects are negligible; it is therefore possible to treat the motion of a single electron and then average over an ensemble of initial conditions for this electron to get the behavior of the beam. A linearized theroretical treatment of the orotron (based on the collisionless Boltzmann equation) can be found in Ref. 8. For the purposes of this Communication, we note that the crucial figure of merit to be extracted from any model of the electronic motion is the *electron efficiency*, defined by considering an electron arriving at one end of the grating at time $t = t_0$ with velocity v_0 and exiting the interaction region with velocity v_f. The efficiency for that electron is given by the fractional change in its kinetic energy or, in the nonrelativistic regime,

$$\eta(t_0) = 1 - \frac{v_f(t_0)^2}{v_0^2} \quad . \tag{1}$$

To get $\eta(t_0)$, we assume that there exists a static magnetic field along the direction of electron motion that is sufficiently strong to constrain the electron motion to one dimension, which we call y. Then Newton's second law becomes

$$m\ddot{y} = -eE(y,t) \quad , \tag{2}$$

where E is the y component of the experimentally observed[7] mode of the grating-resonator system, which is given by (suppressing the dependence on the coordinates transverse to the electron motion—see Ref. 9)

$$E(y,t) = E_0 e^{-y^2/w^2}\cos\omega t \sum_{n=0}^{\infty} a_n \cos nKy \quad , \tag{3}$$

where w is a width parameter determined by the cavity dimensions, a_n are coefficients characteristic of the slow-wave structure, and $K = 2\pi n/l$. By making suitable approximations, we may convert Eq. (2) into a form for which a closed-form solution is possible, even in the nonlinear regime. To this end, we replace the Gaussian in Eq. (3) by a rectangle whose amplitude is equal to $0.751E_0$ and whose effective width is $\bar{L} = 1.98w$ (as determined by requiring a best fit of the rectangle to the Gaussian in a least-squares sense). Further, we consider the initial velocity v_0 near the Smith-Purcell condition for vertical radiation on the fundamental ($n = 1$), $v_0 \simeq \omega l/2\pi = \omega/K$, and we discard all Fourier components in Eq. (3) except the fundamental and make the approximation

$\cos Ky \cos\omega t \simeq \frac{1}{2}\cos(Ky-\omega t)$; i.e., we neglect the backward-traveling component of the standing wave (these approximations are based on results of "exact" calculations, to be discussed later). Finally, we shift the origin of time by $(K\bar{L}-\pi)/2\omega$ and the origin of position by $\bar{L}/2$ and introduce the new independent variable $\phi = Ky - \omega t$ to obtain

$$\ddot{\phi} + \frac{e\bar{E}K}{m}\sin\phi = 0 \quad , \tag{4}$$

for $0 \leq y \leq \bar{L}$, where $\bar{E} = 0.375 a_1 E_0$. This is the pendulum equation. Together with the initial conditions $y(t_0) = 0$ and $\dot{y}(t_0) = v_0$, or $\phi_0 \equiv \phi(t_0) = -\omega t_0$ and $\dot{\phi}_0 \equiv \dot{\phi}(t_0) = K v_0 - \omega$ [i.e., an electron arrives at one end of the grating $(y=0)$ at time t_0 with velocity v_0], this equation is exactly solvable in terms of elliptic functions.

At a time t_f determined by the electron motion, the electron reaches the end of the grating $y = \bar{L}$. Then, if we know $y(t;t_0,v_0)$ [or equivalently $\phi(t;t_0,v_0)$], then t_f can be determined as the solution of $\bar{L} = y(t_f;t_0,v_0)$ [or $K\bar{L} - \omega t_f = \phi(t_f;t_0,v_0)$] and v_f is then equal to $\dot{y}(t_f;t_0,v_0)$ {or $v_f = [\omega + \dot{\phi}(t_f;t_0,v_0)]/K$}. Then $\eta(t_0)$ is determined from Eq. (1). The ensemble average for the beam then yields the device efficiency (denoted by η without arguments):

$$\eta = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt_0\, \eta(t_0) \quad . \tag{5}$$

The solutions of Eq. (4) can be either periodic or aperiodic, corresponding to the usual picture of a pendulum swinging back and forth or executing full 360° rotations. The former motion corresponds to electrons that move, on the average, with the phase velocity of the wave and are "trapped" in wave troughs, while the latter describes electrons that are "above" the wave crests and are relatively unaffected by the wave. To see this more clearly, we define the quantities $\Delta = K v_0/\omega - 1$, $\lambda^2 = eE_0K/m\omega^2$, and $M^2 = \Delta^2/4 + \lambda^2\sin^2(\phi_0/2)$. Then the motion is aperiodic for $|\Delta| > |2\lambda\cos(\phi_0/2)|$ and periodic for $|\Delta| < |2\lambda\cos(\phi_0/2)|$. In the aperiodic case, the velocity can be written as

$$\dot{y} = \frac{\omega}{K}\left\{1 + \frac{\Delta\,\mathrm{dn}(M\omega t|\lambda^2/M^2) - \lambda^2/M \sin\phi_0\,\mathrm{sn}(M\omega t|\lambda^2/M^2)\,\mathrm{cn}(M\omega t|\lambda^2/M^2)}{1 - (\lambda^2/M^2)\sin^2(\phi_0/2)\,\mathrm{sn}^2(M\omega t|\lambda^2/M^2)}\right\} , \tag{6}$$

while in the periodic case,

$$\dot{y} = \frac{\omega}{K}\left\{1 + \frac{\Delta\,\mathrm{cn}(\lambda\omega t|M^2/\lambda^2) - \lambda\sin\phi_0\,\mathrm{sn}(\lambda\omega t|M^2/\lambda^2)\,\mathrm{dn}(\lambda\omega t|M^2/\lambda^2)}{1 - \sin^2(\phi_0/2)\,\mathrm{sn}^2(\lambda\omega t|M^2/\lambda^2)}\right\} , \tag{7}$$

where sn, cn, and dn are the usual Jacobi elliptic functions.[10] These expressions show that the electron motion depends on ϕ_0 and on the parameter $\epsilon = 2\lambda/\Delta$. We can discuss the two limiting cases:

(a) $|\epsilon| \ll 1$. This corresponds to either a weak field or a large deviation from the Smith-Purcell condition. In this case, the motion is aperiodic for all values of ϕ_0, and indeed $\dot{y}$ reduces simply to v_0 as $\epsilon \to 0$.

(b) $|\epsilon| \gg 1$. This is the strongly nonlinear regime. It also corresponds to a beam that nearly satisfies the Smith-Purcell condition and hence is the case of nearly complete trapping in the sense that if $\Delta = 0$ $(\epsilon = \infty)$ an electron executes periodic motion for any initial condition ϕ_0. In this case, the solution becomes

$$\dot{y} = \frac{\omega}{K}\left\{1 - \lambda\sin\phi_0\frac{\mathrm{sn}(\lambda\omega t|\sin^2(\phi_0/2))}{\mathrm{dn}(\lambda\omega t|\sin^2(\phi_0/2))}\right\} , \tag{8}$$

unless ϕ_0 is very close to zero. It is not hard to show that in this case the efficiency approaches zero.

In general, the evaluation of the average efficiency is hampered by the implicit nature of the dependence of $\eta(t_0)$ on t_0. The integral in Eq. (5) is performed numerically, and η shows a characteristic oscillatory behavior with field. If these oscillations are ignored, a simple description of the average efficiency can be given. Assume that only the trapped electrons are effective in contributing to η and that these arrive at the end of the grating with the phase velocity of the wave, ω/K, regardless of their time of departure.

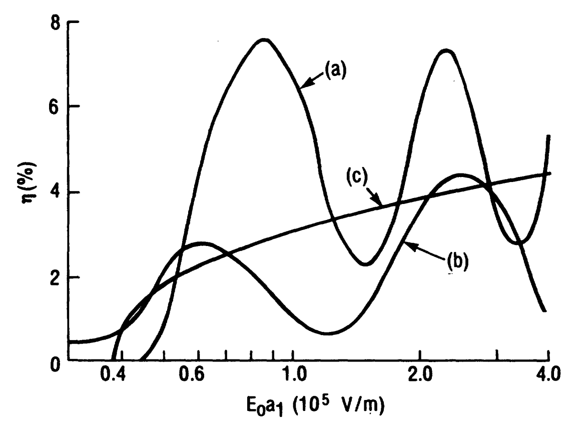

FIG. 1. Average electron efficiency vs electric field amplitude a_1E_0 for (a) "exact" Gaussian model, (b) pendulum model, and (c) phase-space model. The electron velocity is $v_0 = 3.08 \times 10^7$ m/s.

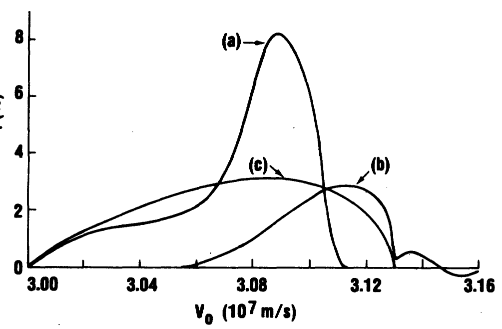

FIG. 2. Average electron efficiency vs electron velocity v_0 for (a) "exact" Gaussian model, (b) pendulum model, and (c) phase-space model. The electric field amplitude is $a_1E_0=10^5$ V/m ($\bar{E}=3.75\times10^4$ V/m).

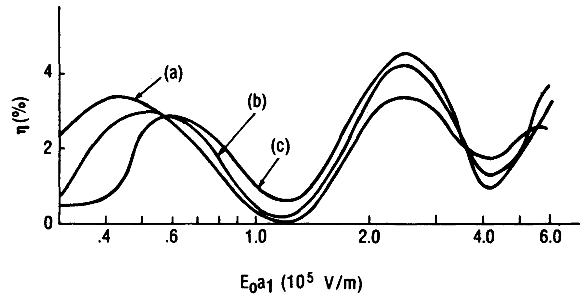

FIG. 3. Efficiency vs electric field amplitude for a variety of electron velocities in the pendulum model: (a) $v_0=3.06\times10^7$ m/s, (b) $v_0=3.07\times10^7$ m/s, and (c) $v_0=3.08\times10^7$ m/s.

Then η depends only on how many electrons are trapped, i.e., the available phase space, and is given explicitly by

$$\eta=\left[1-\frac{(\omega/K)^2}{v_0^2}\right]F$$

$$=\frac{2}{\pi}\left[1-\left(\frac{\omega}{Kv_0}\right)^2\right]\cos^{-1}\left[\frac{1}{\lambda}\left(\frac{Kv_0}{\omega}-1\right)\right]\ , \quad (9)$$

where F is the fraction of electrons that are trapped.

A detailed numerical study has been made of the efficiency η for both the exact and the approximate models of the electron motion. The results are presented in Figs. 1–3 for an orotron with the following physical parameters, appropriate for the orotron experiment at Harry Diamond Laboratories[7]: grating period $l=0.4$ mm ($K=1.57\times10^4$ m^{-1}); effective grating length $\bar{L}=19.8$ mm ($w=10$ mm); cavity center frequency $f=75$ GHz ($\omega=4.7\times10^{11}$ s^{-1}); and wave phase velocity $\omega/K=3\times10^7$ m/s. In Fig. 1, we compare the calculated efficiencies for (a) the model with a Gaussian envelope [E given by Eq. (3), solved by direct numerical integration], (b) the pendulum model, and (c) the "phase-space" calculation, Eq. (9), as a function of the electric field, a_1E_0; the beam velocity is 3.08×10^7 m/s, and so $\Delta=0.0267$. On this plot, the parameter λ ranges from 0 to 0.07; hence, the range of values of ϵ corresponding to mostly trapped to entirely untrapped electrons is well covered. Curve (a), the Gaussian envelope model, seems to exhibit much stronger oscillations than curve (b), the pendulum model. This behavior suggests that the Gaussian tail is more important than one might expect. Apart from its lack of oscillatory behavior, curve (c), the phase space model, gives reasonable quantitative agreement with the pendulum curve.

Figure 2 is a plot of η against v_0 for an electric field $\bar{E}=3.75\times10^4$ V/m ($a_1E_0=10^5$ V/m) for the same three models. Again, the Gaussian yields a higher efficiency. The oscillations in E are present also in the v_0 plot for the pendulum model, again reflecting the dependence of η on λ/Δ. This duality is not preserved in the more exact treatment. Again, the phase-space curve gives the correct order-of-magnitude behavior. In Fig. 3, both $\bar{E}$ and v_0 are varied for the pendulum model; for high fields, a weak dependence of the oscillation period on v_0 is evident. As a further test of the model's validity, we examined the dependence of the velocity v_f on ϕ_0 for the pendulum model and for the standing-wave form of E, $E\propto\cos Ky\cos\omega t$. It was found that the two models agreed extraordinarily well. In particular, both exhibit a striking near discontinuity for that initial condition for which the motion goes from trapped to untrapped. Further, we demonstrated that the solution is not sensitive to the addition of higher harmonics in Eq. (3). A similar pendulum model has been discussed[11–13] for the Stanford free-electron-laser experiment with a wiggler magnet.

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