Nonlinear wave in a diatomic Toda lattice

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We use a method based on Fourier analysis to obtain wave-train solutions for monatomic as well as diatomic lattices with exponential nearest-neighbor interaction. The monatomic solution obtained by our method agrees with the solutions of Toda. The nonlinear dispersion relation found by us for the diatomic chain goes over to known results in the appropriate limits. The effect of nonlinearity on the dispersion curves is illustrated for various mass ratios.

I. INTRODUCTION

Recently nonlinear wave propagation aroused considerable interest in almost all branches of Physics.[1-6] In 1967 M. Toda[7] proposed a nonlinear lattice with exponential interaction between neighboring particles which admits one-soliton, two-soliton, or N-soliton as well as cnoidal or wave-train solutions. Both numerically[8] and analytically[9] this system is known to be completely integrable whereas Siegel's work[10] suggests that the Toda Hamiltonian with almost every perturbation is nonintegrable. However, the Kolmogorov-Arnold-Moser (KAM) theorem[11] along with various computer experiments[12] ensures that for a wider range of perturbation, the system Hamiltonian may remain at least near integrable. In this connection the work of Casati and Ford[13] is very interesting in the sense that it discusses the near integrable behavior as well as the stochastic transition of the Toda Hamiltonian with two different masses via computer experiments. All these considerations indicate that although the unequal mass Toda lattice is, in general, nonintegrable particular solutions might exist which represents stable orbits surrounded by an arbitrarily close stochastic band. In this paper we demonstrate a method for applying a Fourier series technique to monatomic as well as diatomic Toda lattices, and the new result is a particular closed-form elliptic-function solution for the diatomic Toda chain.

The organization of the paper is as follows. In Sec. II we examine the monatomic Toda lattice by a Fourier series method and obtain solutions which agree with those of Toda. In Sec. III we solve the diatomic Toda lattice equations and also show that in different limits our results agree with known results. In Sec. IV we conclude that the present method is more suitable for problems of exponential lattices because the diatomic chain is more amenable to solution by this method compared to other methods.

II. MONATOMIC EXPONENTIAL LATTICE

The equation of motion for the one-dimensional lattice of particles with nearest-neighbor interactions can be written as

$$m_n \frac{d^2 y_n}{dt^2} = -\phi_n'(y_n - y_{n-1}) + \phi_{n+1}'(y_{n+1} - y_n) , \qquad (2.1)$$

where m_n and y_n stand for the mass and displacement of the nth particle, ϕ_n is the interaction potential between the nth and the $(n-1)$th particles, and prime stands for the derivative.

We consider exponential interaction of the following form[14]:

$$\phi(r) = \frac{a}{b} \exp(-br) + ar + \text{const} . \qquad (2.2)$$

Here, a and b are constants. In the limit $b \to 0$, ab with finite, Eq. (2.2) reduces to harmonic interaction with spring constant $k = ab$. With $b \to \infty$, keeping ab finite, Eq. (2.2) corresponds to the system of hard spheres.

The equation of motion for a monatomic lattice can now be put in the following form[15]:

$$m\ddot{s}_n/(1 + \dot{s}_n) = s_{n-1} + s_{n+1} - 2s_n , \qquad (2.3)$$

where

$$\dot{s}_n = -\partial\phi(r_n)/\partial r_n = \exp(-r_n) - 1 , \quad r_n = y_n - y_{n-1}$$

where m is the mass at every lattice point and the constants a and b are put equal to 1, as retaining a and b will in no way affect the form of the solutions, except the appearance of these constants in the final expressions. Equation (2.3) can now be written as

$$m\ddot{s}_n - (s_{n-1} + s_{n+1} - 2s_n) = \dot{s}_n(s_{n-1} + s_{n+1} - 2s_n). \tag{2.4}$$

We seek a periodic solution of the form

$$s_n = \sum_{j=-\infty}^{+\infty} b_j \exp[i2\pi j(n/\lambda + \nu t)], \tag{2.5}$$

where λ is the wavelength, ν is the frequency, and $b_j s$ are the j-dependent coefficients to be determined. Substituting Eq. (2.5) in Eq. (2.4),

$$-(2\pi\nu)^2 m \sum_{j=-\infty}^{+\infty} j^2 b_j \exp\left[i2\pi j\left(\frac{n}{\lambda} + \nu t\right)\right] - \sum_{j=-\infty}^{+\infty} b_j \exp\left[i2\pi j\left(\frac{n}{\lambda} + \nu t\right)\right]\left[\exp\left(\frac{i2\pi j}{\lambda}\right) + \exp\left(-\frac{i2\pi j}{\lambda}\right) - 2\right]$$
$$= i2\pi\nu \sum_{j=-\infty}^{+\infty} j b_j \exp\left[i2\pi j\left(\frac{n}{\lambda} + \nu t\right)\right] \sum_{p=-\infty}^{+\infty} b_p \exp\left[i2\pi p\left(\frac{n}{\lambda} + \nu t\right)\right]\left[\exp\left(\frac{i2\pi p}{\lambda}\right) + \exp\left(-\frac{i2\pi p}{\lambda}\right) - 2\right]. \tag{2.6}$$

Multiplying both sides by $\exp[-i2\pi(n/\lambda + \nu t)]$ and intergrating over a time period Eq. (2.6) reduces to

$$-m(2\pi\nu)^2 b_1 + 2\left(1 - \cos\frac{2\pi}{\lambda}\right) b_1 = -i2\pi\nu \sum_{j=-\infty}^{+\infty} 2j\left(1 - \cos\frac{2\pi}{\lambda}(1-j)\right) b_j b_{1-j}. \tag{2.7}$$

Here we have used the relation

$$\int_{-T/2}^{+T/2} \exp\left[i2\pi(j-k)\left(\frac{n}{\lambda} + \nu t\right)\right] dt = \frac{1}{\nu}\delta jk.$$

Converting the summation over j into two separate summations we write Eq. (2.7) as

$$-m(2\pi\nu)^2 b_1 + 2b_1\left(1 - \cos\frac{2\pi}{\lambda}\right) = -2i(2\pi\nu)\sum_{j=1}^{\infty} j\left[b_j b_{1-j}\left(1 - \cos\frac{2\pi(1-j)}{\lambda}\right) - b_{-j} b_{j+1}\left(1 - \cos\frac{2\pi(1+j)}{\lambda}\right)\right]. \tag{2.8}$$

s_n can be expressed either in cosine or in sine series. In order to get it in terms of a sine series, we choose

$$b_{-j} = -b_j. \tag{2.9}$$

Hence,

$$-m(2\pi\nu)^2 + 2(1 - \cos 2\pi/\lambda) = \sum_{j=1}^{\infty} j\left[\frac{b_j b_{j+1}}{b_1}\left(1 - \cos\frac{2\pi(j+1)}{\lambda}\right) - \frac{b_j b_{j-1}}{b_1}\left(1 - \cos\frac{2\pi(j-1)}{\lambda}\right)\right]. \tag{2.10}$$

Equation (2.10) is a nonlinear equation. To solve it we linearize $b_j b_{j-1}$ in the difference form as follows:

$$\begin{aligned} b_j b_{j-1}/b_1 &= 2\pi\nu A(c_{j-1} - c_j), \\ b_{j+1} b_j/b_1 &= 2\pi\nu A(c_j - c_{j+1}), \end{aligned} \tag{2.11}$$

where the coefficient c_j depends on j and A is a constant. They are to be determined. The factor $2\pi\nu$ appears in the expression to preserve the time reversal symmetry. The right-hand side of Eq. (2.10) is equal to the following:

$$\sum_{j=1}^{\infty} j\, \frac{b_j b_{j+1}}{b_1}\left(1 - \cos\frac{2\pi(j+1)}{\lambda}\right) - \frac{b_j b_{j-1}}{b_1}\left(1 - \cos\frac{2\pi(j-1)}{\lambda}\right)\Bigg]$$
$$= 2i(2\pi\nu)^2 \sum_{j=1}^{\infty} jA\left[(c_{j-1} - c_j)\left(1 - \cos\frac{2\pi(j-1)}{\lambda}\right) - (c_j - c_{j+1})\left(1 - \cos\frac{2\pi(j+1)}{\lambda}\right)\right]$$
$$= 2i(2\pi\nu)^2 A \sum_{j=1}^{\infty}\left[-jc_j\left(2 - 2\cos\frac{2\pi}{\lambda}\cos\frac{2\pi j}{\lambda}\right) + jc_{j-1}\left(1 - \cos\frac{2\pi(j-1)}{\lambda}\right) + jc_{j+1}\left(1 - \cos\frac{2\pi(j+1)}{\lambda}\right)\right]$$
$$= 2i(2\pi\nu)^2 A \sum_{j=1}^{\infty}\left[-2jc_j\left(1 - \cos\frac{2\pi}{\lambda}\cos\frac{2\pi j}{\lambda}\right) + (j+1)c_j\left(1 - \cos\frac{2\pi j}{\lambda}\right) + (j-1)c_j\left(1 - \cos\frac{2\pi j}{\lambda}\right)\right]$$
$$= -2i(2\pi\nu)^2 A 2\left(1 - \cos\frac{2\pi}{\lambda}\right)\sum_{j=1}^{\infty} jc_j \cos\frac{2\pi j}{\lambda}.$$

With this simplication Eq. (2.10) becomes

$$\frac{m(2\pi\nu)^2}{4\sin^2\pi/\lambda} - 1 = 2i(2\pi\nu)^2 A \sum_{j=1}^{\infty} jc_j \cos\frac{2\pi j}{\lambda}. \qquad (2.12)$$

Now we can use the following identity[16]:

$$\frac{1}{\mathrm{sn}^2 u} = \frac{\pi^2}{4K^2}\csc^2\frac{\pi u}{2K} + \frac{K-E}{K} - \frac{2\pi^2}{K^2}\sum_{n=1}^{\infty} n\frac{q^{2n}}{1-q^{2n}}\cos\frac{n\pi u}{K}, \qquad (2.13)$$

where $q=\exp(-\pi K'/K)$, $K(k)$ and $E(k)$ are complete eliptic integrals of first and second kind, respectively, sn is the Jacobian elliptic function, $K' = K(k')$, and $k'=(1-k^2)^{1/2}$. With $u=2K/\lambda$ we rearrange Eq. (2.13) as

$$\frac{1}{4\sin^2\pi/\lambda} = \frac{K^2}{\pi^2}\left(\frac{1}{\mathrm{sn}^2 2K/\lambda} - 1 + \frac{E}{K}\right) + 2\sum_{j=1}^{\infty} j\frac{q^{2j}}{1-q^{2j}}\cos\frac{2\pi j}{\lambda}. \qquad (2.14)$$

Substituting Eq. (2.14) in Eq. (2.12) we obtain

$$m(2\pi\nu)^2\left(\frac{1}{\mathrm{sn}^2 2K/\lambda} - 1 + \frac{E}{K}\right) - 1 + 2m(2\pi\nu)^2 \times \sum_{j=1}^{\infty} j\frac{q^{2j}}{1-q^{2j}}\cos\frac{2\pi j}{\lambda} = 2iA(2\pi\nu)^2\sum_{j=1}^{\infty} jc_j\cos\frac{2\pi j}{\lambda}. \qquad (2.15)$$

Equation (2.15) will be satisfied if

$$c_j = q^{2j}/(1-q^{2j}) \quad \text{for } j=1,2,\ldots,\infty \qquad (2.16)$$

and

$$A=-im, \qquad (2.17)$$

$$m(2K\nu)^2 = \left(\frac{1}{\mathrm{sn}^2 2K/\lambda} - 1 + \frac{E}{K}\right)^{-1}. \qquad (2.18)$$

We observe that

$$c_j + c_{-j} = -1. \qquad (2.19)$$

This relation will be very useful while solving the diatomic lattice.

Equations (2.15)–(2.18) with linearizing expression (2.11) determine b_j as follows:

$$c_{j-1} - c_j = q^{2j-2}/(1-q^{2j-2}) - q^{2j}/(1-q^{2j}) = [q^j/(1-q^{2j})][q^{j-1}/(1-q^{2j-2})]/[q/(1-q^2)]. \qquad (2.20)$$

This expression suggests that b_j is equal to

$$b_j = -im2\pi\nu q^j/(1-q^{2j}). \qquad (2.21)$$

We can therefore verify that $b_{-j}=-b_j$. Now,

$$s_n = \sum_{j=-\infty}^{+\infty} b_j \exp\left[i2\pi j\left(\frac{n}{\lambda}+\nu t\right)\right],$$

$$\dot{s}_n = i2\pi\nu\sum_{j=-\infty}^{\infty} jb_j\exp\left[i2\pi_j\left(\frac{n}{\lambda}+\nu t\right)\right] = i2\pi\nu\sum_{j=1}^{\infty} 2jb_j\cos 2\pi j\left(\frac{b}{\lambda}+\nu t\right). \qquad (2.22)$$

Using the expression for b_j, we get

$$\dot{s}_n = 2m(2\pi\nu)^2\sum_{j=1}^{\infty} j\frac{q^j}{1-q^{2j}}\cos 2\pi j\left(\frac{n}{\lambda}+\nu t\right)$$

or

$$\dot{s}_n(\equiv e^{-r_n}-1) = m(2K\nu)^2[\mathrm{dn}^2 2(n/\lambda+\nu t)K - E/K]. \qquad (2.23)$$

In writing Eq. (2.23) we make use of the identity[16,7]

$$\mathrm{dn}^2 u - \frac{E}{K} = \frac{2\pi^2}{K^2}\sum_{j=1}^{\infty} j\frac{q^j}{1-q^{2j}}\cos\frac{\pi j u}{K}. \qquad (2.24)$$

Our solution and dispersion relation for a monatomic exponential lattice given, respectively, by equations (2.23) and (2.18) agree with that of Toda[17] which had been obtained in an alternate method by comparing Eq. (2.3) with the following relation[2]:

$$\frac{Z''(u)}{(1/\mathrm{sn}^2 v) - 1 + E/K + Z'(u)} = Z(u+v) + Z(u-v) - 2Z(u), \qquad (2.25)$$

where $Z(u)$ is a periodic function known as the Jacobian zeta function.

III. DIATOMIC EXPONENTIAL LATTICE

The equation of motion in the case of a diatomic lattice with masses m_1 and m_2 at the even and odd sites can be written as

$$m_1\frac{d^2 y_{2n}}{dt^2} = \exp(-r_{2n}) - \exp(-r_{2n+1}), \qquad (3.1a)$$

$$m_2\frac{d^2 y_{2n-1}}{dt^2} = \exp(-r_{2n-1}) - \exp(-r_{2n}), \qquad (3.1b)$$

where $r_{2n}=y_{2n}-y_{2n-1}$. Multiplying the first equation by m_2 and the second equation by m_1 and then subtracting one from the other, we have

$$m_1 m_2\frac{d^2 r_{2n}}{dt^2} = (m_1+m_2)\exp(-r_{2n}) - m_1\exp(-r_{2n-1}) - m_2\exp(-r_{2n+1}).$$

Similarly,

$$m_1 m_2\frac{d^2 r_{2n-1}}{dt^2} = (m_1+m_2)\exp(-r_{2n-1}) - m_1\exp(-r_{2n}) - m_2\exp(-r_{2n-2}).$$

Using the notation of Sec. II, that is,

$$\dot{s}_{2n} = \exp(-r_{2n}) - 1,$$

we rewrite the above equation as

$$-m_1 m_2 \ddot{s}_{2n}/(1+\dot{s}_{2n}) = (m_1+m_2)s_{2n} - m_1 s_{2n-1} - m_2 s_{2n+1} \quad (3.2a)$$

$$-m_1 m_2 \ddot{s}_{2n-1}/(1+\dot{s}_{2n-1}) = (m_1+m_2)s_{2n-1} - m_2 s_{2n-2} - m_1 s_{2n} \quad (3.2b)$$

or as

$$-m_1 m_2 \ddot{s}_{2n} + m_1 s_{2n-1} + m_2 s_{2n+1} - (m_1+m_2)s_{2n} = \dot{s}_{2n}(m_1+m_2)s_{2n} - m_1 s_{2n-1} - m_2 s_{2n+1}, \quad (3.3a)$$

$$-m_1 m_2 \ddot{s}_{2n-1} + m_1 s_{2n} + m_2 s_{2n-2} - (m_1+m_2)s_{2n-1} = \dot{s}_{2n-1}(m_1+m_2)s_{2n-1} - m_2 s_{2n-2} - m_1 s_{2n}. \quad (3.3b)$$

We look for periodic solutions with different sets of coefficients $\{a_j\}$ and $\{b_j\}$ in the following form:

$$s_{2n} = \sum_{j=-\infty}^{\infty} a_j \exp\left[i 2\pi j\left(\nu t + \frac{2n}{\lambda}\right)\right], \quad (3.4a)$$

and

$$s_{2n-1} = \sum_{j=-\infty}^{\infty} b_j \exp\left[i 2\pi j\left(\nu t + \frac{2n-1}{\lambda}\right)\right]. \quad (3.4b)$$

Substituting Eqs. (3.4) in Eq. (3.3a), multiplying throughout by $\exp[-i2\pi(\nu t + 2n/\lambda)]$, and then integrating both sides over a time period we obtain

$$[m_1 m_2 (2\pi\nu)^2 - m_1 - m_2]a_1 + e(1)b_1 = i2\pi\nu(m_1+m_2)\sum_{j=-\infty}^{\infty} j a_j a_{1-j} - i2\pi\nu \times \sum_{j=-\infty}^{\infty} j a_j b_{1-j} e(1-j). \quad (3.5a)$$

Similarly, Eq. (3.3b) becomes

$$a_1 e(-1) + [m_1 m_2 (2\pi\nu)^2 - m_1 - m_2]b_1 = i2\pi\nu(m_1+m_2)\sum_{j=-\infty}^{\infty} j b_j b_{1-j} - i2\pi\nu \times \sum_{j=-\infty}^{\infty} j b_j a_{1-j} e(j-1), \quad (3.5b)$$

where $e(j) = m_1 \exp(-i2\pi j/\lambda) + m_2 \exp(i2\pi j/\lambda)$, $e(-j) = e^*(j)$, and * stands for complex conjugation. Equations (3.5) can be put in the following matrix form:

$$\begin{bmatrix} m_1 m_2 (2\pi\nu)^2 - m_1 - m_2 & e(1) \\ e(-1) & m_1 m_2 (2\pi\nu)^2 - m_1 - m_2 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix}, \quad (3.6)$$

where M and N stand for the right-hand expression for Eqs. (3.5a) and (3.5b), respectively. We now proceed to solve this matrix equation for the coefficients a_j and b_j. The coefficients thus obtained satisfy the above equation, and this is shown[18] explicitly in Appendix E.

Let us consider the following eigenvalue problem:

$$\begin{bmatrix} \sigma - m_1 - m_2 & e(1) \\ e(-1) & \sigma - m_1 - m_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0, \quad (3.7)$$

where $\sigma = m_1 m_2 (2\pi\nu)^2$. The eigenvalues can be written as

$$\sigma_\pm = (m_1+m_2) \pm [(m_1+m_2)^2 - 4m_1 m_2 \sin^2 2\pi/\lambda]^{1/2}. \quad (3.8)$$

Corresponding eigenvectors are

$$\begin{bmatrix} a_+ \\ b_+ \end{bmatrix}, \quad \begin{bmatrix} a_- \\ b_- \end{bmatrix}. \quad (3.9)$$

These eigenfunctions are orthogonal, and after normalization the elements are given by the following expressions:

$$a_+ = a_- = 1/\sqrt{2}, \quad (3.10a)$$

$$-b_+ = b_- = (1/\sqrt{2})L^*, \quad (3.10b)$$

where

$$L = [e(1)/e(-1)]^{1/2}. \quad (3.10c)$$

We then expand

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

of Eq. (3.6) in terms of the complete set of eigenfunctions as

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \alpha \begin{bmatrix} a_+ \\ b_+ \end{bmatrix} + \beta \begin{bmatrix} a_- \\ b_- \end{bmatrix} \quad (3.11)$$

with α and β as expansion coefficients.

Substituting Eq. (3.11) in Eq. (3.6) and then multiplying both sides by

$$[a_+^* b_+^*],$$

we get the following equation:

$$\{m_1 m_2 (2\pi\nu)^2 - m_1 - m_2 - [(m_1+m_2)^2 - 4m_1 m_2 \sin^2 2\pi/\lambda]^{1/2}\}\alpha = a_+^* M + b_+^* N. \quad (3.12a)$$

Similarly, multiplying by

$$[a_-^* b_-^*],$$

we have

$$\{m_1 m_2 (2\pi\nu)^2 - m_1 - m_2 + [(m_1+m_2)^2 - 4m_1 m_2 \sin^2 2\pi/\lambda]^{1/2}\}\beta = a_-^* M + b_-^* N. \quad (3.12b)$$

Here, use has been made of the orthogonality of the eigenvectors. M and N contain nonlinear terms in the coefficients a_j and b_j. Our linearization procedure of the monatomic lattice (Sec. II) suggests the following choices:

$$a_j a_{j-1} = 2\pi\nu A(c_{j-1} - c_j), \quad c_j + c_j = -1 \tag{3.13a}$$

$$a_{-j} = -a_j, \tag{3.13b}$$

$$b_j b_{j-1} = 2\pi\nu B(d_{j-1} - d_j), \quad d_j + d_{-j} = -1 \tag{3.14a}$$

$$b_{-j} = -b_j, \tag{3.14b}$$

$$b_j a_{j-1} = 2\pi\nu D(g_{j-1} - g_j), \tag{3.15a}$$

$$g_j + g_{-j} = -1, \tag{3.15b}$$

where A, B, D are constants and c_j, d_j, g_j depend on j and have to be determined. We know that

$$M = i2\pi\nu(m_1 + m_2)\sum_{j=-\infty}^{\infty} j a_j a_{1-j} - i2\pi\nu \sum_{j=-\infty}^{\infty} j a_j b_{1-j} e(1-j), \tag{3.16a}$$

$$N = i2\pi\nu(m_1 + m_2)\sum_{j=-\infty}^{\infty} j b_j b_{1-j} - i2\pi\nu \sum_{j=-\infty}^{\infty} j b_j a_{1-j} e(j-1). \tag{3.16b}$$

Expressions for various sums contained in M and N are given below (detailed evaluations are worked out in the Appendix A):

$$\sum_{j=-\infty}^{\infty} j a_j a_{1-j} = -2\pi\nu A c_0, \tag{3.17a}$$

$$\sum_{j=-\infty}^{\infty} j b_j b_{1-j} = -2\pi\nu B d_0, \tag{3.17b}$$

$$\sum_{j=-\infty}^{\infty} j b_j a_{1-j} e(j-1) = 2\pi\nu D\left(-(m_1+m_2)g_0 + 2i(m_1+m_2)\sum_{j=1}^{\infty} g_j \sin 2\pi j/\lambda + \sum_{j=1}^{\infty} 2j g_j[e(-1) - e(0)]\right), \tag{3.17c}$$

$$\sum_{j=-\infty}^{\infty} j a_j b_{1-j} e(1-j) = 2\pi\nu D\left((m_1+m_2)(1+g_0) - 2i(m_1+m_2)\sum_{j=1}^{\infty} g_j \sin 2\pi j/\lambda + \sum_{j=1}^{\infty} 2j g_j[e(1) - e(0)]\right). \tag{3.17d}$$

Using the expressions (3.17),

$$M = i(2\pi\nu)^2\left(\xi - 2D[e(1) - e(0)]\sum_{j=1}^{\infty} j g_j \cos\frac{2\pi j}{\lambda}\right), \tag{3.18a}$$

$$N = i(2\pi\nu)^2\left(\eta - 2D[e(-1) - e(0)]\sum_{j=1}^{\infty} j g_j \cos\frac{2\pi j}{\lambda}\right), \tag{3.18b}$$

where

$$\xi = -(m_1+m_2)(Ac_0 + Dg_0 + D) + 2i(m_1 - m_2)D\sum_{j=1}^{\infty} g_j \sin\frac{2\pi j}{\lambda}, \tag{3.18c}$$

and

$$\eta = (m_1+m_2)(-Bd_0 + Dg_0) - 2i(m_1 - m_2)D\sum_{j=1}^{\infty} g_j \sin\frac{2\pi j}{\lambda}. \tag{3.18d}$$

Using expressions (3.10) and Eqs. (3.18) we get, after some algebraic simplification,

$$a_+^* M + b_+^* N = i(2\pi\nu)^2\frac{1}{\sqrt{2}}(\xi - L\eta) + i(2\pi\nu)^2 DP\sum_{j=1}^{\infty} j g_j \cos\frac{2\pi j}{\lambda}, \tag{3.19a}$$

and

$$a_-^* M + b_-^* N = i(2\pi\nu)^2\frac{1}{\sqrt{2}}(\xi + L\eta) + i(2\pi\nu)^2 DQ\sum_{j=1}^{\infty} j g_j \cos\frac{2\pi j}{\lambda}, \tag{3.19b}$$

where

$$P = \sqrt{2}L[e(-1) - e(0)] - \sqrt{2}[e(1) - e(0)], \tag{3.19c}$$

$$Q = -\sqrt{2}L[e(-1) - e(0)] - \sqrt{2}[e(1) - e(0)]. \tag{3.19d}$$

Using Eqs. (3.19) in Eqs. (3.12),

$$\left[m_1 m_2(2\pi\nu)^2 - m_1 - m_2 - \left((m_1+m_2)^2 - 4m_1 m_2 \sin^2\frac{2\pi}{\lambda}\right)^{1/2}\right]\alpha = i(2\pi\nu)^2\frac{1}{\sqrt{2}}(\xi - L\eta) + i(2\pi\nu)^2 DP\sum_{j=1}^{\infty} j g_j \cos\frac{2\pi j}{\lambda}, \tag{3.20a}$$

$$\left[m_1 m_2(2\pi\nu)^2 - m_1 - m_2 + \left((m_1+m_2)^2 - 4m_1 m_2 \sin^2\frac{2\pi}{\lambda}\right)^{1/2}\right]\beta = i(2\pi\nu)^2\frac{1}{\sqrt{2}}(\xi + L\eta) + i(2\pi\nu)^2 DQ\sum_{j=1}^{\infty} j g_j \cos\frac{2\pi j}{\lambda}. \tag{3.20b}$$

The dispersion relation for the diatomic exponential lattice can now be obtained from Eqs. (3.20). This should reduce to that of the monatomic case (Eq. 2.18) when $m_1 = m_2$ and also to that of the harmonic diatomic dispersion in the appropriate limit. These considerations lead us to make the following choices:

$$\xi - L\eta = 0\,, \tag{3.21a}$$

$$\xi + L\eta = 0\,, \tag{3.21b}$$

that is,

$$\xi = \eta = 0\,, \tag{3.21c}$$

$$P/\alpha = Q/\beta\,, \tag{3.21d}$$

and

$$g_j = q^{2j}/(1-q^{2j})\,. \tag{3.21e}$$

Now Eqs. (3.20) yield a dispersion relation of the following form:

$$(2K\nu)^2 = \frac{(m_1+m_2) \pm [(m_1+m_2)^2 - 4m_1 m_2 \sin^2 2\pi/\lambda]^{1/2}}{4m_1 m_2 \sin^2\pi/\lambda (1/\mathrm{sn}^2 2K/\lambda - 1 + E/K)}\,, \tag{3.22}$$

with

$$D = -i\, 8 m_1 m_2 (\beta/Q) \sin^2\pi/\lambda\,. \tag{3.23}$$

When m_1 is put equal to m_2 in Eq. (3.22), the dispersion relation reduces to that of a monatomic expression, namely,

$$m(2K\nu)^2 = \left(\frac{1}{\mathrm{sn}^2 2K/\lambda} - 1 + \frac{E}{K}\right)^{-1}.$$

For $k \ll 1$ (i.e., $E/K = 1$, $\mathrm{sn}^2 2K/\lambda \sim \sin\pi/\lambda$, $K = \pi/2$), Eq. (3.22) takes the following form:

$$(2\pi\nu)^2 = (1/m_1 + 1/m_2) \pm [(1/m_1 + 1/m_2)^2 - (4/m_1 m_2)\sin^2 2\pi/\lambda]^{1/2}\,.$$

This agrees with the dispersion relation of an harmonic diatomic chain.

Using the Eqs. (3.10), (3.13)–(3.15), (3.21), and (3.23) and going through very cumbersome algebraic calculations, we get the following expressions[18] (we have calculated the expressions in Appendices B, C and D):

$$a_j = Xq^j/(1-q^{2j})\,, \tag{3.24a}$$

with

$$X = i\,2\pi\nu(4m_1 m_2 \sin^2\pi/\lambda)/[e(-1) - e(0)]\,, \tag{3.24b}$$

$$b_j = Yq^j/(1-q^{2j})\,, \tag{3.25a}$$

with

$$Y = i\,2\pi\nu(4m_1 m_2 \sin^2\pi/\lambda)/[e(1) - e(0)]\,, \tag{3.25b}$$

$$g_j = c_j = d_j = q^{2j}/(1-q^{2j})\,, \tag{3.26}$$

$$A = (X^2/2\pi\nu)(q/1-q^2)\,, \tag{3.27a}$$

$$B = (Y^2/2\pi\nu)(q/1-q^2)\,, \tag{3.27b}$$

$$D = -2\pi\nu\left(\frac{q}{1-q^2}\right)\frac{16 m_1^2 m_2^2 \sin^4\pi/\lambda}{[e(-1)-e(0)][e(1)-e(0)]}\,. \tag{3.27c}$$

Finally, we obtain the solution of the diatomic exponential lattice as follows:

$$s_{2n} = \sum_{j=-\infty}^{\infty} a_j \exp[i\,2\pi j(\nu t + 2n/\lambda)]$$

$$= a_0 - 2\pi\nu \times \frac{4m_1 m_2 \sin^2\pi/\lambda \sum_{j=1}^{\infty} 2q^j/(1-q^{2j}) \sin 2\pi j(\nu t + 2n/\lambda)}{m_1 \exp(i\,2\pi/\lambda) + m_2 \exp(-i\,2\pi/\lambda) - m_1 - m_2}\,,$$

$$\dot{s}_{2n} = \frac{(2K\nu)^2 4m_1 m_2 \sin^2\pi/\lambda}{m_1 + m_2 - m_1 e^{i2\pi/\lambda} - m_2 e^{-i2\pi/\lambda}} \times \left[\mathrm{dn}^2 2\left(\nu t + \frac{2n}{\lambda}\right)K - \frac{E}{K}\right]. \tag{3.28}$$

Here, use is made of the relation[7]

$$\mathrm{dn}^2(2xK) = \frac{2\pi^2}{K^2}\sum_{j=1}^{\infty} j\,\frac{q^j}{1-q^{2j}}\cos 2\pi x j + \frac{E}{K}\,. \tag{3.29}$$

Similarly,

$$\dot{s}_{2n-1} = (2K\nu)^2 \frac{4m_1 m_2 \sin^2\pi/\lambda}{m_1 + m_2 - m_1 e^{-i2\pi/\lambda} - m_2 e^{i2\pi/\lambda}} \times \left[\mathrm{dn}^2 2\left(\nu t + \frac{2n-1}{\lambda}\right)K - \frac{E}{K}\right]. \tag{3.30}$$

It is of interest to see that when m_1 is equal to m_2, Eqs. (3.33) and (3.36) reduce to that of the monatomic case, namely,

$$\dot{s}_n = m(2K\nu)^2[\mathrm{dn}^2 2(\nu t + n/\lambda)K - E/K]\,.$$

IV. CONCLUSIONS

The method we develop here is more general than the method used by Toda. Its merit lies in the fact that both monatomic and diatomic exponential lattices can be studied with the help of this technique. In the method used by Toda an equation connecting Jacobian zeta functions of the form (2.25) is necessary for obtaining the wave-train solution. We could not find suitable equations of the form of (2.25) to compare with Eqs. (3.2a) and (3.2b) of the diatomic lattice. So we conclude that though Toda's intuitive method succeeds in the case of a monatomic lattice, its extension to other complicated systems is not easy.

The dispersion relation (3.22) of the diatomic lattice not only reduces to the monatomic expression (2.18) when $m_1 = m_2$, but also goes over to the harmonic case in the limit $k \to 0$. The nonlinear dispersion curves (Fig. 1) preserve the general harmonic characteristic; that is, with increase of the mass ratio, the separation between the acoustic and optical branches widens. But they differ from the harmonic curves as regards their shapes.

Even though the cnoidal solutions of the monatomic Toda lattice are particular solutions,[14] they reveal something of the general solution, as this Hamiltonian represents an integrable system. This is not true in the case of a diatomic Toda lattice. Our solution is strictly a particular solution which, for certain mass ratios, is an unstable periodic orbit lying in a stochastic sea. Moreover, even when this orbit is stable in the sense of KAM, it is surrounded by an arbitrarily close stochastic band.[19]

Lastly, it is of interest to see how the Fourier series converges. It has been known since the time of Poincare[20] that a Fourier series diverges in general, though it may converge for a particular solution. Further, this fact is exploited by Eminheizer, Hellman, and Montroll[21] in obtaining particular solutions for some nonintegrable systems. Very elaborately they have demonstrated a method for avoiding divergencies of the Fourier series by assuming a single frequency expansion. In light of their work, we observe that our Fourier method represents a convergent series, as we take a single frequency expansion and seek a particular periodic solution, not a general one, in the form of elliptic functions.

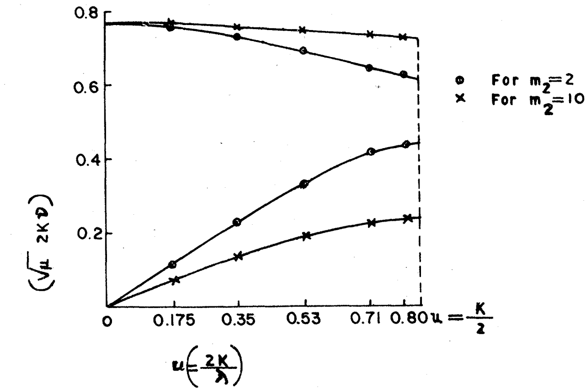

FIG. 1. Dispersion of diatomic wave train ($k = 0.5$, $m_1 = 1$).

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APPENDIX A: DERIVATION OF EQUATIONS (3.17)

$$\sum_{j=-\infty}^{\infty} j a_j a_{1-j} = \sum_{j=1}^{\infty} j a_j a_{1-j} + \sum_{j=-1}^{-\infty} j a_j a_{1-j}$$

$$= \begin{cases} -\sum_{j=1}^{\infty} j a_j a_{j-1} + \sum_{j=1}^{\infty} j a_j a_{j+1}, & \text{since } a_{-j} = -a_j \\ -2\pi\nu A \sum_{j=1}^{\infty} j(c_{j-1} - c_j) + 2\pi\nu A \sum_{j=1}^{\infty} j(c_j - c_{j+1}), & \text{using Eq. (3.13)} \end{cases} \tag{A1}$$

$$= 2\pi\nu A\left[2\sum_{j=1}^{\infty} j c_j - \sum_{j=0}^{\infty}(j+1)c_j - \sum_{j=2}^{\infty}(j-1)c_j\right]$$

$$= 2\pi\nu A\left[2\sum_{j=1}^{\infty} j c_j - c_0 - \sum_{j=1}^{\infty}(j+1)c_j - \sum_{j=1}^{\infty}(j-1)c_j\right] = -2\pi\nu A c_0 .$$

Similarly,

$$\sum_{j=-\infty}^{\infty} j b_j b_{1-j} = -2\pi\nu B d_0 . \tag{A2}$$

Now,

$$\sum_{j=-\infty}^{\infty} j b_j a_{1-j}\left(m_1 e^{i(1-j)2\pi/\lambda} + m_2 e^{-i(1-j)2\pi/\lambda}\right) = 2\pi\nu D\left(\sum_{j=1}^{\infty} j(g_j - g_{j-1})\left(m_1 e^{-i(j-1)2\pi/\lambda} + m_2 e^{i(j-1)2\pi/\lambda}\right)\right.$$
$$\left. + \sum_{j=1}^{\infty} -j(g_{-j} - g_{-j-1})\left(m_1 e^{i(j+1)2\pi/\lambda} + m_2 e^{-i(j+1)2\pi/\lambda}\right)\right). \tag{A3a}$$

Here we have used Eq. (3.15a).

Taking into account the following identity which can be written on the basis of Eq. (3.15b) as

$$g_{-j}-g_{-j-1}=g_{j+1}-g_j\,,$$

we write the right-hand side of (A3a) in the following form:

$$2\pi\nu D\,\sum_{j=1}^{\infty} j\,g_j(m_1e^{-i(j-1)2\pi/\lambda}+m_2e^{i(j-1)2\pi/\lambda})-\sum_{j=0}^{\infty}(j+1)g_j(m_1e^{-i(j2\pi/\lambda)}+m_2e^{i(j2\pi/\lambda)})$$

$$+\sum_{j=1}^{\infty} j\,g_j(m_1e^{i(j+1)2\pi/\lambda}+m_2e^{-i(j+1)2\pi/\lambda})-\sum_{j=1}^{\infty}(j-1)g_j(m_1e^{ij2\pi/\lambda}+m_2e^{-ij2\pi/\lambda})$$

$$=2\pi\nu D\left(-g_0(m_1+m_2)+2i(m_1-m_2)\sum_{j=1}^{\infty}g_j\sin\frac{j2\pi}{\lambda}+2(m_1e^{i2\pi/\lambda}+m_2e^{-i2\pi/\lambda}-m_1-m_2)\sum_{j=1}^{\infty}j\,g_j\cos\frac{j2\pi}{\lambda}\right).$$

Hence,

$$\sum_{j=-\infty}^{\infty} jb_j a_{1-j}e(j-1)=2\pi\nu D\left(-g_0(m_1+m_2)+2i(m_1-m_2)\sum_{j=1}^{\infty}g_j\sin\frac{j2\pi}{\lambda}+2[e(-1)-e(0)]\sum_{j=1}^{\infty}jg_j\cos\frac{j2\pi}{\lambda}\right). \tag{A3b}$$

Similarly,

$$\sum_{j=-\infty}^{\infty} j\,a_jb_{1-j}e(1-j)=\sum_{j=-\infty}^{\infty}(1-j)b_ja_{1-j}e(j)$$

$$=2\pi\nu D\left((m_1+m_2)(1+g_0)-2i(m_1-m_2)\sum_{j=1}^{\infty}g_j\sin\frac{j2\pi}{\lambda}+2[e(1)-e(0)]\sum_{j=1}^{\infty}jg_j\cos\frac{j2\pi}{\lambda}\right). \tag{A4}$$

APPENDIX B: EXPRESSIONS FOR a_j AND b_j

Using expression (3.21e)

$$2\pi\nu D(g_{j-1}-g_j)=2\pi\nu D\,\frac{1-q^2}{q}\left(\frac{q^j}{1-q^{2j}}\right)\left(\frac{q^{j-1}}{1-q^{2j-2}}\right).$$

Comparison of this with Eq. (3.15) suggests the form of a_j and b_j as

$$a_j=X\,\frac{q^j}{1-q^{2j}}\,, \tag{B1a}$$

$$b_j=Y\,\frac{q^j}{1-q^{2j}}\,. \tag{B1b}$$

However, from Eqs. (3.11), (3.10), and (3.21d)

$$a_1=\alpha a_+ +\beta a_- =\frac{1}{\sqrt2}(\alpha+\beta)$$

$$=\frac{1}{\sqrt2}\,\alpha\left(1+\frac{Q}{P}\right)=\frac{1}{\sqrt2}\beta\left(1+\frac{P}{Q}\right), \tag{B2a}$$

$$b_1=\alpha b_+ +\beta b_- =\frac{1}{\sqrt2}L^*(\beta-\alpha)$$

$$=\frac{1}{\sqrt2}\,L^*\left(\frac{Q}{P}-1\right)\alpha=\frac{1}{\sqrt2}\,L^*\left(1-\frac{Q}{P}\right)\beta. \tag{B2b}$$

From Eqs. (B1a) and B2a)

$$a_1=X\,\frac{q}{(1-q^2)}=\frac{1}{\sqrt2}\left(\frac{Q+P}{P}\right)\alpha\,, \tag{B3a}$$

$$b_1=Y\,\frac{q}{(1-q^2)}=\frac{1}{\sqrt2}\,L^*\left(\frac{Q-P}{P}\right)\alpha. \tag{B3b}$$

Therefore,

$$\frac{X}{Y}=\frac{(Q+P)}{L^*(Q-P)}\,. \tag{B3c}$$

Putting Eqs. (B1) and (3.21e) in Eq. (3.15a)

$$XY=2\pi\nu D\,\frac{(1-q^2)}{q}$$

$$=-i(2\pi\nu)\frac{\alpha}{P}\,8m_1m_2\sin^2\frac{\pi}{\lambda}\,\frac{(1-q^2)}{q}\,. \tag{B4}$$

Substituting the value of α from Eq. (B3a) in Eq. (B4)

$$Y=-\frac{8\sqrt2}{Q+P}\,(i\,2\pi\nu)m_1m_2\sin^2\frac{\pi}{\lambda}\,.$$

Also, using Eq. (B3c)

$$X=-\frac{8\sqrt2}{L^*(Q-P)}\,(i\,2\pi\nu)m_1m_2\sin^2\frac{\pi}{\lambda}\,.$$

However, Eqs. (3.19c) and (3.19d) give

$$Q+P=-2\sqrt2\,(m_1e^{-i2\pi/\lambda}+m_2e^{i2\pi/\lambda}-m_1-m_2)$$

$$=-2\sqrt2\,[e(1)-e(0)]\,, \tag{B5a}$$

and

$$Q-P=-2\sqrt{2}\,L(m_1e^{i2\pi/\lambda}+m_2e^{-i2\pi/\lambda}-m_1-m_2)$$
$$=-2\sqrt{2}\,L[e(-1)-e(0)]\,. \quad \text{(B5b)}$$

Therefore,

$$X=i2\pi\nu\frac{4m_1m_2\sin^2\pi/\lambda}{e(-1)-e(0)}\,, \quad \text{(B6a)}$$

$$Y=i2\pi\nu\frac{4m_1m_2\sin^2\pi/\lambda}{e(1)-e(0)}\,. \quad \text{(B6b)}$$

Hence,

$$a_j=i2\pi\nu\frac{4m_1m_2\sin^2\pi/\lambda}{e(-1)-e(0)}\;\frac{q^j}{(1-q^{2j})}, \quad \text{(B7a)}$$

and

$$b_j=i2\pi\nu\frac{4m_1m_2\sin^2\pi/\lambda}{e(1)-e(0)}\;\frac{q^j}{(1-q^{2j})}\,. \quad \text{(B7b)}$$

APPENDIX C: EXPRESSIONS FOR α, β, A, B, AND D

From Eq. (B3a)

$$\alpha=\frac{\sqrt{2}P}{Q+P}\;\frac{q}{1-q^2}X\,.$$

Using the expressions for X and $Q+P$ from Eqs. (B5) and (B6)

$$\alpha=-(2\sqrt{2})i2\pi\nu\frac{q}{1-q^2}m_1m_2\sin^2\frac{\pi}{\lambda}\times\left(\frac{L}{e(1)-e(0)}-\frac{1}{e(-1)-e(0)}\right), \quad \text{(C1)}$$

$$\beta=\frac{Q}{P}\alpha=2\sqrt{2}\,i2\pi\nu\frac{q}{1-q^2}m_1m_2\sin^2\frac{\pi}{\lambda}\times\left(\frac{L}{e(1)-e(0)}+\frac{1}{e(-1)-e(0)}\right), \quad \text{(C2)}$$

$$D=-8im_1m_2\sin^2\left(\frac{\pi}{\lambda}\right)\frac{\alpha}{P}$$
$$=-2\pi\nu\frac{16m_1^2m_2^2\sin^4\pi/\lambda}{[e(1)-e(0)][e(-1)-e(0)]}\;\frac{q}{(1-q^2)}\,. \quad \text{(C3)}$$

Using Eq. (B1a)

$$a_j\,a_{1-j}=X^2\left(\frac{q^{2j}}{1-q^{2j}}-\frac{q^{2j-2}}{1-q^{2j-2}}\right)\frac{q}{(1-q^2)}\,.$$

Comparing the right-hand side of this equation with Eqs. (3.13),

$$c_j=\frac{q^{2j}}{(1-q^{2j})},\quad 2\pi\nu A=X^2\frac{q}{(1-q^2)}\,,$$

where,

$$A=\frac{X^2}{2\pi\nu}\;\frac{q}{(1-q^2)}\,. \quad \text{(C4)}$$

Similarly,

$$d_j=\frac{q^{2j}}{(1-q^{2j})}\,,$$

and

$$B=\frac{Y^2}{2\pi\nu}\;\frac{q}{(1-q^2)}\,. \quad \text{(C5)}$$

APPENDIX D: EXPRESSIONS FOR a_0 AND b_0

Setting $j=1$ in Eqs. (3.13)–(3.15) we get

$$a_1a_0=2\pi\nu A(c_1-c_0)\,, \quad \text{(D1a)}$$

$$b_1b_0=2\pi\nu B(d_1-d_0)\,, \quad \text{(D1b)}$$

$$a_0b_1=2\pi\nu D(g_1-g_0)\,, \quad \text{(D1c)}$$

$$a_1/b_1=A(c_1-c_0)/D(g_1-g_0)$$
$$=X/Y=(Q+P)/L^*(Q-P)\,. \quad \text{(D1d)}$$

Here we have used expressions (D1a), (D1c), and B(3). Now,

$$L^*(Q-P)A\left(\frac{q}{1-q^2}-c_0\right)=D(Q+P)\left(\frac{q}{1-q^2}-g_0\right),$$

or

$$\frac{q}{1-q^2}-c_0=\frac{q}{1-q^2}-g_0\,.$$

Therefore,

$$c_0=g_0\,. \quad \text{(D2)}$$

Using Eq. (D2) we get from Eq. (3.21c)

$$Ag_0+Dg_0+D=2iD\frac{m_1-m_2}{m_1+m_2}\sum_{j=1}^{\infty}g_j\sin\frac{j2\pi}{\lambda}\,, \quad \text{(D3a)}$$

$$-Bd_0+Dg_0=2iD\frac{m_1-m_2}{m_1+m_2}\sum_{j=1}^{\infty}g_j\sin\frac{j2\pi}{\lambda}\,. \quad \text{(D3b)}$$

After solving Eqs. (D3) we obtain the following:

$$g_0=2i\frac{D}{A+D}\;\frac{m_1-m_2}{m_1+m_2}\sum_{j=1}^{\infty}\frac{q^{2j}}{1-q^{2j}}\sin\frac{j2\pi}{\lambda}+\frac{D^2}{A(A+D)}-\frac{D}{A}\,, \quad \text{(D4)}$$

$$-d_0=2i\frac{AD}{B(A+D)}\;\frac{m_1-m_2}{m_1+m_2}\sum_{j=1}^{\infty}\frac{q^{2j}}{1-q^{2j}}\sin\frac{j2\pi}{\lambda}+\frac{D^2}{B(A+D)}\,. \quad \text{(D5)}$$

Now using Eqs. (D1),

$$a_0=2\pi\nu\frac{A}{X}\left(\frac{q^2}{1-q^2}-g_0\right)\frac{(1-q^2)}{q}\,, \quad \text{(D6)}$$

$$b_0=2\pi\nu\frac{B}{Y}\left(\frac{q^2}{1-q^2}-d_0\right)\frac{(1-q^2)}{q}\,. \quad \text{(D7)}$$

APPENDIX E: VERIFICATION OF EQUATIONS (3.5)

We observe that Eqs. (3.2) possess periodic solutions of the form given by Eqs. (3.28) and (3.30) if the coefficients a_j and b_j satisfy the Eqs. (3.5). By our method we get particular solutions where a_j, b_j, a_0, and b_0 are given by Eqs. (3.24), (3.25), (D6), and (D7). In this Appendix we put these equations back in Eqs. (3.5) and show that the coefficients a_j, b_j, a_0, and b_0 satisfy the original equations (3.5) if the frequency is given by the dispersion relation (3.22).

With our expressions (3.24) and (3.25) we see that

$$a_{-j}=-a_j\,, \quad b_{-j}=-b_j\,. \tag{E1}$$

Again, $c_j=d_j=g_j=q^{2j}/(1-q^{2j})$ satisfies the relation

$$g_j+g_{-j}=-1\,. \tag{E2}$$

Also,

$$2\pi\nu A(c_{j-1}-c_j)=2\pi\nu\frac{X^2}{2\pi\nu}\left(\frac{q^{2j-2}}{1-q^{2j-2}}-\frac{q^{2j}}{1-q^{2j}}\right)\frac{q}{(1-q^2)}$$
$$=X^2\left(\frac{q^j}{1-q^{2j}}\right)\left(\frac{q^{j-1}}{1-q^{2j-2}}\right)=a_j a_{j-1}\,. \tag{E3a}$$

Similarly, using the expressions for the coefficients we verify the following relations.

$$2\pi\nu B(d_{j-1}-d_j)=b_j b_{j-1}\,, \tag{E3b}$$

$$2\pi\nu D(g_{j-1}-g_j)=b_j a_{j-1}\,. \tag{E3c}$$

Using expressions (E1)–(E3c) and taking the help of the summations carried over in Appendix A we get

$$M=i(2\pi\nu)^2\left(-(m_1+m_2)(Ac_0+Dg_0+D)+2i(m_1-m_2)D\sum_{j=1}^{\infty}g_j\sin\frac{j2\pi}{\lambda}-2D[e(1)-e(0)]\sum_{j=1}^{\infty}jg_j\cos\frac{j2\pi}{\lambda}\right),$$

$$N=i(2\pi\nu)^2\left((m_1+m_2)(-Bd_0+Dg_0)-2i(m_1-m_2)D\sum_{j=1}^{\infty}g_j\sin\frac{j2\pi}{\lambda}-2D[e(-1)-e(0)]\sum_{j=1}^{\infty}jg_j\cos\frac{j2\pi}{\lambda}\right).$$

Using expressions for A, B, D, as given by Eqs. (3.27), and c_0, d_0, g_0, as expressed by Eqs. (D2), (D4), and (D5), we see that

$$-(m_1+m_2)(Ac_0+Dg_0+D)+2i(m_1-m_2)D\sum_{j=1}^{\infty}g_j\sin\frac{j2\pi}{\lambda}=0\,,$$

and

$$(-Bd_0+Dg_0)(m_1+m_2)-2i(m_1-m_2)D\sum_{j=1}^{\infty}g_j\sin\frac{j2\pi}{\lambda}=0\,.$$

Hence,

$$M=-i(2\pi\nu)^2 2D[e(1)-e(0)]\sum_{j=1}^{\infty}jg_j\cos\frac{j2\pi}{\lambda}\,.$$

Using Eqs. (3.24b) and (3.27c) as well as the expansion formula (2.14), we write M as follows:

$$M=2(2\pi\nu)^2 4m_1m_2\sin^2\frac{\pi}{\lambda}\;\frac{q}{1-q^2}X\left[\frac{\pi^2}{4K^2}\csc^2\frac{\pi}{\lambda}-\left(\frac{1}{\mathrm{sn}^2 2K/\lambda}+\frac{E}{K}-1\right)\right]\frac{K^2}{2\pi^2}\,. \tag{E4}$$

Similarly,

$$N=2(2\pi\nu)^2 4m_1m_2\sin^2\left(\frac{\pi}{\lambda}\right)\frac{q}{1-q^2}\,Y\left[\frac{\pi^2}{4K^2}\csc^2\frac{\pi}{\lambda}-\left(\frac{1}{\mathrm{sn}^2 2K/\lambda}+\frac{E}{K}-1\right)\right]\frac{K^2}{2\pi^2}\,. \tag{E5}$$

Substituting Eqs. (E4) and (E5) in Eqs. (3.5) and noting that

$$a_1=Xq/(1-q^2)\,,\quad b_1=Yq/(1-q^2)$$

we get the following:

$$[m_1m_2(2\pi\nu)^2-m_1-m_2]a_1+[e(1)]b_1=(2\pi\nu)^2 4m_1m_2\sin^2\left(\frac{\pi}{\lambda}\right)a_1\left[\frac{\pi^2}{4K^2}\csc^2\frac{\pi}{\lambda}-\left(\frac{1}{\mathrm{sn}^2 2K/\lambda}+\frac{E}{K}-1\right)\right]\frac{K^2}{\pi^2}\,. \tag{E6a}$$

Also,

$$[e(-1)]a_1+[m_1m_2(2\pi\nu)^2-m_1-m_2]b_1=(2\pi\nu)^2 4m_1m_2\sin^2\left(\frac{\pi}{\lambda}\right)b_1\left[\frac{\pi^2}{4K^2}\csc^2\frac{\pi}{\lambda}-\left(\frac{1}{\mathrm{sn}^2 2K/\lambda}+\frac{E}{K}-1\right)\right]\frac{K^2}{\pi^2}\,. \tag{E6b}$$

Equations (E6a) and (E6b) will be simultaneously satisfied if

$$\begin{vmatrix} -m_1-m_2+4m_1m_2(2K\nu)^2\left(\frac{1}{\mathrm{sn}^2 2K/\lambda}+\frac{E}{K}-1\right)\sin^2\left(\frac{\pi}{\lambda}\right) & e(1) \\ e(-1) & -m_1-m_2+4m_1m_2(2K\nu)^2\left(\frac{1}{\mathrm{sn}^2 2K/\lambda}+\frac{E}{K}-1\right)\sin^2\frac{\pi}{\lambda} \end{vmatrix}=0\,,$$

or if

$$(2K\nu)^2=\frac{m_1+m_2\pm[(m_1+m_2)^2-4m_1m_2\sin^2 2\pi/\lambda]^{1/2}}{4m_1m_2\left(\frac{1}{\mathrm{sn}^2 2K/\lambda}+\frac{E}{K}-1\right)\sin^2\frac{\pi}{\lambda}} \tag{E7}$$

As Eq. (E7) is identical with the dispersion relation (3.22), our solution satisfies Eq. (3.5).

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