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# Model Boltzmann equations

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It is shown that the Boltzmann equation for isotropic scattering of Maxwell molecules, considered by Bobylev, Krook, and Wu (BKW), can be written in the form of an energy-space kinetic equation. A transformation leads to a new class of kinetic models, possessing exact solutions of the BKW type.

## I. INTRODUCTION

Several years ago, Bobylev[1] and Krook and Wu[2] (BKW) independently discovered that the Boltzmann equation (BE) for a spatially uniform system of Maxwell molecules,

$$\frac{\partial f(v,t)}{\partial t} = \frac{1}{4\pi}\int d\Omega\, d\vec{w}\, h(\cos\theta)\,[f(\vec{v}',t)f(\vec{w}',t) - f(\vec{v},t)f(\vec{w},t)], \tag{1}$$

has an exact, nonequilibrium solution (the "BKW mode") given by

$$f(v,t) = \frac{e^{-v^2/2\alpha}}{(2\pi\alpha)^{3/2}}\left(\frac{5\alpha-3}{2\alpha} + \frac{v^2}{2}\,\frac{1-\alpha}{\alpha}\right), \tag{2}$$

where

$$\alpha(t) = 1 - e^{-\lambda(t-t_0)}, \tag{3}$$

$f(v,t)$ is the velocity distribution function (dimensionless units), $\theta$ is the scattering angle for the collision $(v)+(w)\to(v')+(w')$, $d\Omega = \sin\theta\, d\phi\, d\theta$, and $h(\cos\theta)$ represents the collision frequency, equal to the product of the relation velocity $g \equiv |v-w|$ and the differential cross section. For Maxwell molecules $(\phi \sim 1/r^4)$, $h$ is independent of $g$, but depends upon $\theta$. The BKW mode, which displays the approach to equilibrium for a class of initial conditions in the form of (2), was first obtained[1,2] for "pseudo-Maxwell" molecules for which $h$ is equal to unity, in which case $\lambda = \frac{1}{6}$. Although it was later shown[3,4] that (2) and (3) also holds for a general $h(\cos\theta)$ (in which case $\lambda$ depends upon $h$) and thus for the true Maxwell molecules, we will consider only the pseudo-Maxwell molecules in this paper, and by the BKW model we will always mean a system of such molecules. Note that a constant value of $h$ does not follow from any known intermolecular potential.

Following the discovery of (2) and (3), Tjon and Wu (TW) (Ref. 5) devised a soluble kinetic model, described by the simplified kinetic equation

$$\frac{\partial F(x,t)}{\partial t} + F(x,t) = \int_x^\infty \frac{d\xi}{\xi}\int_0^\xi dy\, F(y,t)F(\xi-y,t), \tag{4}$$

where $F(x,t)$ is the energy distribution function, related to $f(v,t)$ by

$$F(x,t) = 4\pi v f(v,t), \quad x = v^2/2. \tag{5}$$

This model is intimately related to the BKW model, as we will discuss below. Equation (4) is apparently of a much simpler form than the BE, (1), and has proven to be advantageous for numerical calculations.[6] Tjon and Wu showed that (4) can be interpreted physically as the kinetic equation for a spatially uniform, two-dimensional system of Maxwell molecules that scatter diffusively, such that momentum is not conserved. Ernst[7] generalized the TW diffusive-scattering argument to $d$ dimensions, to yield a whole class of models, characterized by the kinetic equation

$$\frac{\partial F(x,t)}{\partial t} = \int_x^\infty d\xi \int_0^\xi dy[F(y,t)F(\xi-y,t)K(x,y;\xi) - F(x,t)F(\xi-x,t)K(y,x;\xi)], \tag{6}$$

where

$$K(x,y;\xi) = \frac{\Gamma(d)}{\Gamma(d/2)^2}\,\frac{[x(\xi-x)]^{(d/2)-1}}{\xi^{d-1}}. \tag{7}$$

For all $d$, these models possess a generalized BKW-mode solution. Other kinetic models, also defined by an equation of the form of (6), with the corresponding expressions for $K$, have been considered by Ernst and Hendriks,[8] and Futcher and Hoare.[9] Note that these models are also related to an earlier model considered by Kac.[10]

As reported in a recent letter,[11] I have found a new class of kinetic models that exhibit BKW-mode solutions. This class can be thought of as a $d$-dimensional generalization of the BKW model, giving the latter for $d=3$ and the TW model for $d=2$. I wrote the kinetic equations for this new class in the following general form

$$\frac{\partial F(x,t)}{\partial t} = \int_0^\infty dy\, F(y,t)\int_0^\infty dz\, F(z,t)P(y,z;x) - F(x,t)\int_0^\infty dy\, F(y,t)\int_0^\infty dz\, P(x,y;z), \tag{8}$$

 

with each model characterized by an explicit expression for $P$. The purpose of this paper is to give a detailed derivation and discussion of this result.

## II. THE KINETIC EQUATION

The first term on the RHS of (8) represents the gain in $F(x,t)$ due to all collisions $(y)+(z)\to(x)+(y+z-x)$, and the second term represents the loss due to all collisions $(x)+(y)\to(z)+(x+y-z)$. $P(y,z;x)$ is the collision rate for $(y)+(z)\to(x)+(y+z-x)$ and characterizes the particular model being considered. Identity of the particles implies that $P$ has the symmetries

$$P(y,z;x)=P(z,y;x), \tag{9a}$$

$$P(y,z;x)=P(y,z;y+z-x), \tag{9b}$$

and conservation of energy implies that $P(y,z;x)=0$ whenever $x>(y+z)$. $P$ also has an inverse collision symmetry, which will be discussed later.

For example, for the class of models of Ernst, $P$ is given by

$$P(y,z,x)=K(x,y,y+z),\quad x<(y+z) \tag{10}$$

and for the TW model, we have simply

$$P^{\mathrm{TW}}(y,z;x)=\begin{cases}\dfrac{1}{y+z}, & 0<x<(y+z)\\ 0, & x>(y+z).\end{cases} \tag{11}$$

Here, $P$ versus $x$ is a step function, which implies that particles come out of collisions so that all allowed energies are equally probable.

It follows from (8) that in general the moments $M_n$, defined by

$$M_n(t)\equiv\int_0^\infty x^nF(x,t)\,dx, \tag{12}$$

satisfy the equation

$$\frac{dM_n}{dt}=\int_0^\infty dy\,F(y,t)\times\int_0^\infty dz\;F(z,t)\int_0^\infty dx\;P(y,z;x)\,[x^n-\tfrac{1}{2}(y^n+z^n)]. \tag{13}$$

Equation (13) implies that $M_0$ (the total mass) is constant, and by virtue of (9b), that $M_1$ (the total energy) is also constant. We assume that the units of $f$ and $x$ are chosen such that $M_0=1$ for all models, while the value of $M_1$ will depend upon the model. Following the conventions in the literature, we take $M_1^{\mathrm{TW}}=1$, $M_1^{\mathrm{BKW}}=\frac{3}{2}$.

Equation (8), much like Eq. (6), is of a very general form, containing most features of the BE of uniform systems, including binary collisions, and the Stossahlansatz. However, since the direction of the velocity does not appear, momentum conservation is not necessarily observed, and thus (8) describes the inelastic models mentioned above as well as models that derive from the usual BE. In Sec. III, we show that the BE for the BKW model can be written in the form of (8).

## III. $P(y,z;x)$ FOR THE BKW MODEL

Krook and Wu[2] showed that the moments in their model, $M_n^{\mathrm{BKW}}$, when renormalized according to

$$M_n^*=\frac{\Gamma(3/2)}{\Gamma(n+3/2)}M_n^{\mathrm{BKW}}, \tag{14}$$

satisfy the equation

$$\frac{dM_n^*}{dt}+M_n^*=\frac{1}{n+1}\sum_{i+j=n}M_i^*M_j^*, \tag{15}$$

where the sum is over all $0\le i\le n$, with $j=n-i$. If we assume that $F(x,t)$ for this model satisfies an equation in the form of (8), it follows that $M_n^{\mathrm{BKW}}$ also satisfy an equation in the form of (13). If we further assume that

$$\int_0^\infty P(y,z;x)\,dx=1, \tag{16}$$

then (13) becomes

$$\frac{dM_n^{\mathrm{BKW}}}{dt}+M_n^{\mathrm{BKW}}=\int_0^\infty dyF(y,t)\times\int_0^\infty dzF(z,t)\int_0^\infty dx\;x^nP(y,z;x). \tag{17}$$

Comparing (15) and (17), and making use of (12) and (14), we deduce

$$\int_0^\infty dy\;F(y,t)\int_0^\infty dzF(z,t)x^nP(y,z;x)=\frac{\Gamma(n+3/2)\Gamma(3/2)}{n+1}\sum_{i+j=n}\frac{1}{\Gamma(i+3/2)\Gamma(j+3/2)}\int_0^\infty dy\;F(y,t)\int_0^\infty dz\;F(z,t)y^iz^j.$$

Since $F(x,t)$ is an arbitrary function of $x$ (at a given $t$), it follows that

$$\int_0^\infty x^n P(y,z;x)\,dx = \frac{\Gamma(n+3/2)\Gamma(3/2)}{n+1}\sum_{i+j=n}\frac{y^i z^j}{\Gamma(i+3/2)\Gamma(j+3/2)}, \quad (18)$$

which is consistent with the assumption (16). In the Appendix, we show that (18) may be inverted to yield the following explicit expression for $P^{\text{BKW}}$,

$$P^{\text{BKW}}(y,z;x)=\frac{1}{\sqrt{yz}}\begin{cases}\arcsin\left(\frac{x}{y+z}\right)^{1/2}, & 0<x<y\\ \arcsin\left(\frac{y}{y+z}\right)^{1/2}, & y<x<z\\ \arcsin\left(1-\frac{x}{y+z}\right)^{1/2}, & z<x<(y+z).\end{cases} \quad (19)$$

Here and in the following expressions of this kind, we assume $y<z$; when $y>z$, $y$ and $z$ should be interchanged in the expressions on the RHS.

Thus, we have cast the BE for the BKW model in the form of (8), which entails only two integrations rather than the BE's five. Note that $P^{\text{BKW}}$ as a function of $x$ is symmetric about $x=(y+z)/2$, as is required by (9b), and has the interesting property that it is constant in the mid-range, $y<x<z$.

## IV. THE GENERATING MODEL

Other kinetic models can conceivably be constructed by choosing alternative expressions for $P$, consistent with (9). As one possibility, suggested by some of the features of $P^{\text{BKW}}$, we consider a model in which $P$ is given by

$$P^*(y,z;x)=\begin{cases}0, & 0<x<y\\ \frac{1}{z-y}, & y<x<z\\ 0, & x>z.\end{cases} \quad (20)$$

We call the kinetic model defined by (20) the "generating model", for reasons that will be apparent later. In this model, the energies of the outgoing particles are restricted to lie in an interval bounded by the energies of the incoming particles. To calculate the moment equation, we note that

$$\int_0^\infty x^n P^*(y,z;x) = \frac{1}{|z-y|}\,\frac{|z^{n+1}-y^{n+1}|}{n+1} = \frac{1}{n+1}\sum_{i+j=n} y^i z^j, \quad (21)$$

and assuming that $M_0^*=1$, it follows that the moments of this model, $M_n^*$, satisfy (15) *without renormalization of the moments*.

Besides being satisfied by $M_n^{\text{BKW}}$, renormalized according to (14), Eq. (15) is also satisfied by the moments of the TW model, $M_n^{\text{TW}}$, when renormalized according to

$$M_n^* = M_n^{\text{TW}}/n!. \quad (22)$$

It is through this link that the TW and BKW models are related; Barnsley and Turchetti[12] have shown that any solution of one model can be transformed into a solution of the others, as follows

$$F^{\text{TW}}(x,t)=\tfrac{1}{2}\int_x^\infty \frac{dz\,F^{\text{BKW}}(z,t)}{(z-x)^{1/2}z^{1/2}}, \quad (23)$$

$$F^{\text{BKW}}(x,t)=-\frac{2}{\pi}\sqrt{x}\,\frac{\partial}{\partial x}\int_x^\infty \frac{dz\,F^{\text{TW}}(z,t)}{(z-x)^{1/2}}. \quad (24)$$

Similarly, using (14) and (22), we can derive relations between solutions of the generating model, $F^*$, and $F^{\text{TW}}$ and $F^{\text{BKW}}$, as follows.

$$\begin{aligned}\int_0^\infty e^{-xs}F^{\text{TW}}(x,t)\,dx &= \sum_{n=0}^\infty \frac{(-s)^n}{n!}M_n^{\text{TW}}\\ &= \sum_{n=0}^\infty (-s)^n M_n^*\\ &= \int_0^\infty \frac{1}{1+zs}F^*(z,t)\,dz\\ &= \int_0^\infty dx\, e^{-xs}\int_0^\infty dz\, F^*(z,t)z^{-1}e^{-x/z}.\end{aligned} \quad (25)$$

This implies that

$$F^{\text{TW}}(x,t)=\int_0^\infty F^*(z,t)z^{-1}e^{-x/z}\,dz. \quad (26)$$

In the same way, we find

$$F^{\text{BKW}}(x,t)=\frac{2\sqrt{x}}{\sqrt{\pi}}\int_0^\infty F^*(z,t)z^{-3/2}e^{-x/z}\,dz. \quad (27)$$

The two transformations above are essentially Laplace transforms, and thus can be formally inverted. Because the model defined by (20) generates solutions to both the TW and BKW models, and because we will find that it lies at the base of a whole class of models, we call it the generating model.

Equation (27) is identical to a transformation considered by Alexanian,[13] in which $F^*(z,t)$ is interpreted as a "temperature" distribution function. This interpretation is motivated by the fact that (27) is formally equivalent to a superposition of equilibrium distributions $2(x/\pi)^{1/2}z^{-3/2}e^{-x/z}$ at temperature $z$. Alexanian derived a kinetic equation for $F^*$, Eqs. (2) and (3) in Ref. 13, which is equivalent to (8) and (20) above, although of a much different form.

The equilibrium distribution for this model, $F^*_{eq}$, can be found from (15). Setting $dM^*_n/dt=0$, and assuming $M^*_0=M^*_1=1$, it follows that $M^*_n=1$ for all $n$, implying that[13]

$$F^*_{eq}(x)=\delta(x-1). \tag{28}$$

Evidently, the restriction that $P^*$ imposes on the outgoing particles causes the distribution function to sharpen and eventually turn into a $\delta$ function, at which time all the particles will have the same energy or speed. From the point of view of the temperature distribution interpretation of $F^*$, Eq. (28) states that a uniform temperature has been achieved.

Using the expression for the moments that correspond to the BKW mode,[2]

$$M^*_n=\alpha^{n-1}[\alpha+n(1-\alpha)], \tag{29}$$

with $\alpha(t)$ given by (3), we find the exact solution (also given by Alexanian[13])

$$F^*(x,t)=\delta(x-\alpha)-(1-\alpha)\frac{\partial}{\partial x}\delta(x-\alpha). \tag{30}$$

Note that (2) can be recovered through the use of (27). Besides being singular, (30) is negative at $x=\alpha^-$ (for all finite $t$) and therefore is unphysical as a distribution function. As $t$ increases, the coincident singular functions in the two terms of (30) move towards $x=1$; at $t=\infty$, the second term vanishes and (28) follows.

Even though this model exhibits singular behavior, it might prove useful for numerical studies, similar to those done on the TW model.[6] The advantage of this model is that the distribution function does not spread in energy space as time increases, thus eliminating the need to impose numerical cutoffs.

## V. A NEW CLASS OF MODELS

The major significance of the generating model is that it can be used to create new models. We observe that (23) and (24) can both be obtained from the transformation

$$F^{(m)}(x,t)=\frac{x^{m-1}}{\Gamma(m)}\int_0^\infty F^*(z,t)z^{-m}e^{-x/z}\,dz, \tag{31}$$

with $m=1$ and $m=\frac{3}{2}$, respectively. Equation (31) defines an infinite class of models for all $m>0$, of which the TW and BKW models are special cases.

From (28) and (31) we find that the general equilibrium distribution is given by

$$F^{(m)}_{eq}(x)=x^{m-1}e^{-x}/\Gamma(m) \tag{32}$$

and using (30) we find the generalization of the BKW mode:

$$F^{(m)}(x,t)=\frac{x^{m-1}e^{-x/\alpha}}{\Gamma(m)\alpha^{m+1}}\left[(\alpha-m+m\alpha)+x\left(\frac{1-\alpha}{\alpha}\right)\right]. \tag{33}$$

The moments of this class are related to those of the generating model by

$$M^{(m)}_n=[\Gamma(n+m)/\Gamma(m)]M^*_n. \tag{34}$$

To complete the description of these models, the corresponding $P^{(m)}$ must be found. Mimicking the derivation of (19), we find

$$\int_0^\infty x^nP^{(m)}(y,z;x)\,dx=\frac{\Gamma(m)\Gamma(m+n)}{n+1}\sum_{i+j=n}\frac{y^iz^j}{\Gamma(m+i)\Gamma(m+j)}. \tag{35}$$

In the Appendix, we show that (35) may be inverted to yield the following expression for $P^{(m)}$:

$$P^{(m)}(y,z;x)=\frac{(y+z)^{2m-3}}{(yz)^{m-1}}\times\begin{cases}q^{(m)}\left(\dfrac{x}{y+z}\right), & 0<x<y\\ q^{(m)}\left(\dfrac{y}{y+z}\right), & y<x<z\\ q^{(m)}\left(1-\dfrac{x}{y+z}\right), & z<x<(y+z)\end{cases} \tag{36}$$

where $q^{(m)}$ is a form of the incomplete beta function given by

$$q^{(m)}(u)=(m-1)\int_0^u[v(1-v)]^{m-2}dv. \tag{37}$$

For $m$ integral, $q^{(m)}$ is a simple polynomial in $u$. For $m$ nonintegral, we have the expansion

$$q^{(m)}(u)=\frac{u^{m-1}\Gamma(m)}{\pi}\sum_{i=0}^\infty\frac{(-u)^i\Gamma(i+m-2)\sin(m-1+i)\pi}{i!(m-1+i)}. \tag{38}$$

Explicit expressions of $q^{(m)}$ for $m=\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, and 3 are given in Table I. Note that the results for $m=1$ and $m=\frac{3}{2}$ agree with (11) and (20). Plots of the corresponding $P^{(m)}$ are given in Fig. 1. For example, $P^{(2)}$ is in the form of a trapezoid. $P^{(1/2)}$ goes to $\infty$ at $x=0$ and $x=(y+z)$, and the resulting increased production of particles at zero energy leads to an equilibrium distribution given by (32), which becomes infinite as $x\to\infty$. Note that for $0<m<1$, the lower limit of the integral in (37) cannot be zero but must be adjusted (or the constant of integration chosen) so that the normalization (16) is satisfied. The expression of $q^{(1/2)}$ given in Table I is consistent with this requirement.

As $m$ increases, it can be seen from Fig. 1 that $P^{(m)}\to 0$ in the shoulder regions, $0<x<y$ and $y<x<(y+z)$. In the mid-range, $y<x<z$, where $P^{(m)}$

TABLE I. Some explicit expression for $q^{(m)}(u)$.

| $m$ | $q^{(m)}(u)$ |
|---|---|
| 1/2 | $\frac{1-2u}{[u(1-u)]^{1/2}}$ |
| 1 | 1 |
| 3/2 | $\arcsin\sqrt{u}$ |
| 2 | $u$ |
| 5/2 | $\frac{3}{8}\{\arcsin\sqrt{u}-(1-2u)[u(1-u)]^{1/2}\}$ |
| 3 | $u^2-\frac{2}{3}u^3$ |

is a constant, $P^{(m)}$ must therefore approach the value $1/(z-y)$ as $m\to\infty$, by virtue of the normalization, (16). Thus, the generating model $P^*$ represents the infinite-$m$ limit of the class $P^{(m)}$ [which can also be proven directly from (36) and (37)] suggesting that the kernel in the integral transformation (31) should become $\delta(z-x)$ when $m\to\infty$. However, the latter does not follow, because the units of $F^{(m)}$, defined such that $M_1^{(m)}=m$, are inconsistent with the units of $F^*$, for which $M_1^*=1$. If we rescale the units so that $M_1^{(m)}=1$, implying that $F^{(m)}(x,t)$ is replaced by $mF^{(m)}(mx,t)$, then indeed the kernel in (31) becomes $\delta(z-x)$. This means that for any solution $F^{(m)}(x,t)$ of (36),

$$F^*(x,t)\equiv\lim_{x\to\infty} mF^{(m)}(mx,t) \tag{39}$$

is a solution of the generating model.

Ernst and Hendrik[14] have shown that a solution of the $P^{(m)}$ model can be written as a series of generalized Laguerre polynomials,

$$F^{(m)}(x,t)=F_{\rm eq}^{(m)}(x)\sum_{n=0}^{\infty}c_n(t)(-1)^nL_n^{(m-1)}(x), \tag{40}$$

where coefficients $c_n(t)$ satisfy an equation that is identical to the Krook-Wu moment equation, (15).

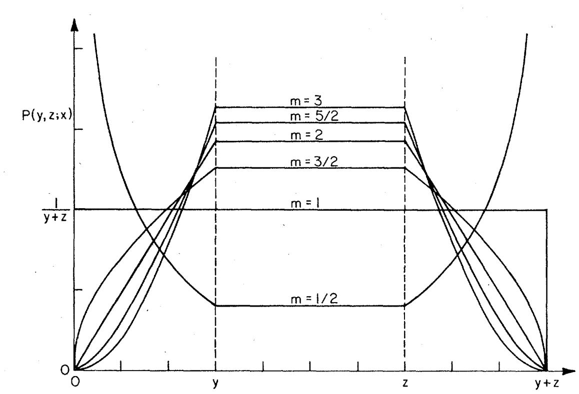


FIG. 1. $P^{(m)}(y,z;\,x)$ plotted as a function of $x$, for $y/(y+z)=0.3$ for the class of models given by (36).

Equation (40), which is a generalization of the series expansions that have been given for the BKW model[1] and the TW model,[15] may be proven in the following way: It follows from (40) and the orthogonality of $L_n^{(\alpha)}(x)$ that

$$c_n=\frac{(-1)^n\Gamma(m)n!}{\Gamma(n+m)}\int_0^\infty dx\,F^{(m)}(x,t)L_n^{(m-1)}(x). \tag{41}$$

By virtue of (31), we may also write $c_n$ as

$$c_n=\int_0^\infty (x-1)^nF^*(x,t)\,dx. \tag{42}$$

Using (42), a direct calculation of $\partial c_n/\partial t$ along the lines of (20) may be made, and (15) follows.

Note that the above relation (42) was introduced by Alexanian[13] in connection with his representation of $F^*(x,t)$,

$$F^*(x,t)=\sum_{n=0}^{\infty}\frac{c_n(t)}{n!}\left(\frac{\partial}{\partial x}\right)^n\delta(1-x). \tag{43}$$

Although it can be verified from (43) that (42) is identically satisfied for each $n$, this clearly does not converge to $F^*(x,t)$ since each term is always zero for $x\neq1$. Thus, the term-by-term applications of (31) to (43), and (39) to (40), which appear to show the equivalence of (40) and (43), are not valid in this case.

Besides being related to $F^*$ through (31), a solution $F^{(m)}$ of the model $P^{(m)}$ is related to a solution $F^{(m+1)}$ of the model $P^{(m+1)}$ through a simple transformation. From (31), it follows directly that the $f^{(m)}(x,t)$, defined by

$$f^{(m)}(x,t)=\Gamma(m)x^{1-m}F^{(m)}(x,t), \tag{44}$$

are interrelated by

$$f^{(m)}(x,t)=\int_x^\infty f^{(m+1)}(y,t)\,dy, \tag{45}$$

$$f^{(m+1)}(x,t)=-\frac{\partial f^{(m)}(x,t)}{\partial x}. \tag{46}$$

More generally, we have

$$f^{(m-\delta)}(x,t)=\frac{1}{\Gamma(\delta)}\int_x^\infty\frac{dz\,f^{(m)}(z,t)}{(z-x)^{1-\delta}}, \tag{47}$$

which includes (23), taking $m=\frac{3}{2}$ and $\delta=\frac{1}{2}$.

It can be shown that these kinetic models satisfy an $H$ theorem. Besides the symmetries of $P$ listed in (9), we need an inverse-collision symmetry of the form

$$\bar{P}^{(m)}(y,\,\xi-y;x)=\bar{P}^{(m)}(x,\,\xi-x;y). \tag{48}$$

The $P^{(m)}$ we have found do not in general satisfy this relation, but by inspection of (36), it can be seen that

$$\tilde{P}^{(m)}(y,z;x)\equiv(yz)^{m-1}P^{(m)}(y,z;x) \quad (49)$$

satisfies (38). Note that Futcher and Hoare[9] introduce this symmetry on the basis of the requirement of detailed balance in equilibrium. Considering an $H$ function given by

$$H(t)=\int_0^\infty F^{(m)}(x,t)\ln\frac{\Gamma(m)F^{(m)}(x,t)}{(2\pi)^m x^{m-1}} \quad (50)$$

and using (8) and the various symmetry properties of $P$, we find

$$\frac{dH}{dt}=-\tfrac{1}{4}\int_0^\infty d\xi\int_0^\xi dy\int_0^\xi dz\,\tilde{P}^{(m)}(y,\xi-y;x)[\tilde{F}(y)\tilde{F}(\xi-y)-\tilde{F}(x)\tilde{F}(\xi-x)][\ln\tilde{F}(y)\tilde{F}(\xi-y)-\ln\tilde{F}(x)\tilde{F}(\xi-x)] \leq 0, \quad (51)$$

where $\tilde{F}(x)\equiv x^{1-m}F^{(m)}(x,t)$. This proves the monotonic approach to the equilibrium distribution.

## VI. CONCLUSIONS

Our new class of soluble kinetic models has been characterized by a kinetic equation in the form of (8) with $P^{(m)}(y,z;x)$ given by (36). These models, which are a kind of generalization of the model of BKW, each possess an exact solution, related to the BKW mode, and thus show explicitly the approach to equilibrium.

The mathematical basis of this new class was the generating model, whose kinetic equation is given by (8) with the particularly simple expression $P^*$ given by (20). The link between the generating model and the new class was provided by the integral transform, (31), and the corresponding moment relation, (34), was used to find an explicit expression for the $P^{(m)}$. Conceivably, it should be possible to generalize this procedure by considering other integral transforms than the one given in (31), and thus generate still further classes of models from $P^*$, although the models that can be generated from $P^*$ are probably restricted to those that are physically related to the pseudo-Maxwell molecule. That is, it is doubtful that more general solutions of the BE—representing more general forms of $h(g, \cos\theta)$—can be generated this way.

The equilibrium solution, (32), is apparently in the form of the general equilibrium distribution of a $(2m)$-dimensional system, suggesting that $P^{(m)}$ represents the kinetic equation for some $(2m)$-dimensional model. However, the physical meaning of these models for dimensionalities other than two and three (when the TW and BKW models are given) has not been explored. One question to be considered is whether the new models follow from the complete, momentum-conserving BE with a well defined $g$, as does the BKW model, or whether, like the TW model, they represent some kind of modified kinetic equation. Note that the solution (33) is identical in form to the BKW-mode solution in Ernst's $(2m)$-dimensional diffusive scattering model, except that $\lambda(t)$ in (3) is given by $m/2(2m+1)$ in Ernst's model, instead of by $\frac{1}{6}$, given here.

In a forthcoming paper, we will present the derivation of an explicit expression for $P^*(y,z;x)$ for a general scattering function $h(\cos\theta)$ and thus for the Maxwell molecule.

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## APPENDIX

*Derivation of (19) and (36).* We prove the general expression (36), which includes (19) as the special case, $m=\frac{3}{2}$. Multiplying (35) by $(-s)^n(n+1)/\Gamma(n+m)$, and summing over all $n$, we find

$$\int_0^\infty dx\,P^{(m)}\sum_{n=0}^\infty\frac{(n+1)(-sx)^n}{\Gamma(n+m)}=\Gamma(m)\sum_{i=0}^\infty\frac{(-sz)^i}{\Gamma(m+i)}\sum_{j=0}^\infty\frac{(-sz)^j}{\Gamma(m+j)}. \quad (A1)$$

By partial integration, the LHS can be written

$$-\int_0^{y+z}dxx\frac{\partial P^{(m)}}{\partial x}\sum_{n=0}^\infty\frac{(-sx)^n}{\Gamma(n+m)} \quad (A2)$$

since $P^{(m)}(y,z;x)=0$ for $x>y+z$. Using the integral representation

$$\sum_{n=0}^\infty\frac{(-sx)^n}{\Gamma(n+m)}=\frac{x^{1-m}}{\Gamma(m-1)}\int_0^x e^{-sx}(x-t)^{m-2}dt, \quad (A3)$$

we can write (A1) and (A2)

$$-\int_0^{y+z} dx \frac{\partial P^{(m)}}{\partial x} x^{2-m} \int_0^x dt\, e^{-st}(x-t)^{m-2}$$
$$= (m-1)(yz)^{1-m} \int_0^y e^{-su}(y-u)^{m-2} du \times \int_0^z e^{-sv}(z-v)^{m-2} dv. \quad \text{(A4)}$$

On the LHS of (A4) we interchange the integrations,

$$\int_0^{y+z} dx \int_0^x dt = \int_0^{y+z} dt \int_t^{y+z} dx, \quad \text{(A5)}$$

and on the RHS of (A4) we make the change of variables $(u,v) \to (u,t)$ where $t \equiv u+v$:

$$\int_0^y du \int_0^z dv = \int_0^{y+z} dt \int_a^b du, \quad \text{(A6)}$$

where $a$ and $b$ depend upon $t$, $y$, and $z$, and are given by

$$(a,b) = \begin{cases} (0,t), & 0<t<y \\ (0,y), & y<t<z \\ (t-z,y), & z<t<(y+z). \end{cases} \quad \text{(A7)}$$

This change of variables is illustrated in Fig. 2. Thus, (A4) can be rewritten

$$-\int_0^{y+z} dt\, e^{-st} \int_t^{y+z} dx (x-t)^{m-2}\left(x^{2-m} \frac{\partial P^{(m)}}{\partial x}\right)$$
$$= (m-1)(yz)^{1-m} \int_0^{y+z} dt\, e^{-st} \times \int_a^b du[(y-u)(z-t+u)]^{m-2}. \quad \text{(A8)}$$

We have, by the uniqueness of the Laplace transform,

$$-\int_t^{y+z} dx (x-t)^{m-2}\left(x^{2-m} \frac{\partial P^{(m)}}{\partial x}\right)$$
$$= (m-1)(yz)^{1-m} \int_{a+z}^{b+z} dx (x-t)^{m-2} \times (y+z-x)^{m-2}, \quad \text{(A9)}$$

where we have also let $x = u+z$ on the RHS.

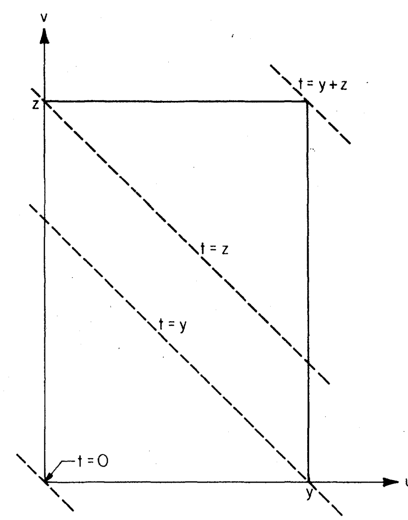


FIG. 2. The change of variables $(u,v) \to (t,u)$, where $t = u+v$.

(i) Say $z<t<(y+z)$. According to (A7), the limits of the integral on the RHS of (A9) are $(t, y+z)$, identical to the limits of the integral on the LHS. Since (A9) is valid for all $t$ in this range, it follows that

$$\frac{\partial P^{(m)}}{\partial x} = -(m-1)(yz)^{1-m}[x(y+z-x)]^{m-2}. \quad \text{(A10)}$$

(ii) $y<t<z$. Now the limits on the RHS of (A9) are $(z, y+z)$. Comparing the two sides of (A9), and using that $P^{(m)}$ satisfies (A10) in the range (i), we deduce that

$$\frac{\partial P^{(m)}}{\partial x} = 0. \quad \text{(A11)}$$

(iii) $0<t<y$. The limits of the RHS of (A9) are $(z, z+t$. This time, replacing $x$ in the RHS of (A9) by $(y+z+t-x)$, and using both (A10) and (A11), we deduce

$$\frac{\partial P^{(m)}}{\partial x} = (m-1)(yz)^{1-m}[x(y+z-x)]^{m-2}. \quad \text{(A12)}$$

Integrating (A10)–(A12), we find (36) and (37).


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